Lagrange Tori and Equation of Hamilton–Jacobi

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It is well-known that the theory of classical equation of Hamilton–Jacobi has being connected with the theory of Lagrange and Kolmogorov tori. The solutions of such equations are the representations of fundemental groups of such manifolds. After that the theory of tori "in general" is shown. It is known that for the entire symplectic manifold there exists a symplectic connection such that its covariant derivative with respect to connection is equal to zero.

It is shown that Lagrange tori have a lots of "non-linear" properties with respect to classical equation of Hamilton–Jacobi but Kolmogorov tori have not such variety with respect to that equation.

The examples from dynamics of a rigid body interacting with the resisting medium are presented. Furthermore, a lots of examples from the different areas of natural sciences are illustrated.

Keywords: multi-dimensional rigid body dynamics, pendulum, nonconservative force field, case of integrability, transcendental first integral

1 Introduction

Earlier (see [1, 2]), the author already proved the complete integrability of the equations of a plane-parallel motion of a fixed rigid body–pendulum in a homogeneous flow of a medium under the jet flow conditions when the system of dynamical equations possesses a first integral, which is a transcendental (in the sense of the theory of functions of a complex variable, i.e., it has essential singularities) function of quasi-velocities. It was assumed that the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a (one-dimensional) plate.

In [2, 3], the planar problem was generalized to the spatial (three-dimensional) case, where the system of dynamical equations has a complete set of transcendental first integrals. It was assumed that the interaction of the homogeneous medium flow with the fixed body (the spherical pendulum) is concentrated on a part of the body surface that has the form of a planar (two-dimensional) disk.

Later on (see [3, 4]), the equations of motion of the fixed dynamically symmetric four-dimensional rigid bodies, where the force field is concentrated on a part of the body surface that has the form of a (three-dimensional) disk, in this case, the force field is concentrated on the one-dimensional straight line perpendicular to this disk.

In this activity, the results relate to the case where all interaction of the homogeneous flow of a medium with the fixed body is concentrated on that part of the surface of the body, which has the form of a $(n-1)$-dimensional disk, and the action of the force is concentrated in a direction perpendicular to this disk. These results are systematized and are presented in invariant form.

2 Model assumptions

Let consider the homogeneous $(n-1)$-dimensional disk $\mathcal{D}^{n-1}$ (with the center in the point D), the hyperplane of which perpendicular to the holder OD in the multi-dimensional Euclidean space $\mathbf{E}^n$. The disk is rigidly fixed perpendicular to the tool holder OD located on the (generalized) spherical hinge O, and it flows about homogeneous fluid flow. In this case, the body is a physical (generalized spherical) pendulum. The medium flow moves from infinity with constant velocity $\mathbf{v} = \mathbf{v}_\infty \neq \mathbf{0}$. Assume that the holder does not create a resistance.

We suppose that the total force $\mathbf{S}$ of medium flow interaction perpendicular to the disk $\mathcal{D}^{n-1}$, and point N of application of this force is determined by at least the angle of attack α, which is made by the velocity vector $\mathbf{v}_D$ of the point D with respect to the flow and the holder OD; the total force is also determined by the angles $\beta_1, \ldots, \beta_{n-2}$, which are made in the hyperplane of the disk $\mathcal{D}^{n-1}$ (thus, $(v, \alpha, \beta_1, \ldots, \beta_{n-2})$ are the (generalized) spherical coordinates of the tip of the vector $\mathbf{v}_D$), and also the reduced angular velocity tensor

$$\tilde{\omega} \cong \frac{l\tilde{\Omega}}{v_D}, \;\; v_D = |\mathbf{v}_D|$$

(l is the length of the holder, $\tilde{\Omega}$ is the angular velocity tensor of the pendulum). Such conditions generalize the model of streamline flow around spatial bodies [4, 5].

The vector

$$\mathbf{e} = \frac{\mathbf{OD}}{l} \tag{1}$$

determines the orientation of the holder. Then

$$\mathbf{S} = s(\alpha) v_D^2 \mathbf{e}, \tag{2}$$

where

$$s(\alpha) = s_1(\alpha) \operatorname{sign} \cos\alpha, \tag{3}$$

and the resistance coefficient $s_1 \geq 0$ depends only on the angle of attack α. By the axe-symmetry properties of the body–pendulum with respect to the axis $Dx_1 = OD$, the function $s(\alpha)$ is (formally) even.

Let $Dx_1 \ldots x_n$ be the coordinate system rigidly attached to the body, herewith, the axis Dx_1 has a direction vector $\mathbf{e}$, and the axes $Dx_2, \ldots, Dx_{n-1}$ and Dx_n lie in the hyperplane of the disk $\mathcal{D}^{n-1}$.

By the angles $(\xi, \eta_1, \ldots, \eta_{n-2})$, we define the position of the holder OD in the multi-dimensional space $\mathbf{E}^n$. In this case, the angle ξ is made by the holder and the direction of the over-running medium flow. In other words, the angles introduced are the (generalized) spherical coordinates of the point D of the center of a disk $\mathcal{D}^{n-1}$ on the $(n-1)$-dimensional sphere of the constant radius OD.

The space of positions of this (generalized) spherical (physical) pendulum is the $(n-1)$-dimensional sphere

$$\mathbf{S}^{n-1}\{(\xi, \eta_1, \ldots, \eta_{n-2}) \in \mathbf{R}^{n-1} : \; 0 \leq \xi, \eta_1, \ldots, \eta_{n-3} \leq \pi, \;\; \eta_{n-2} \bmod 2\pi\}, \tag{4}$$

and its phase space is the tangent bundle of the $(n-1)$-dimensional sphere

$$T_*\mathbf{S}^{n-1}\{(\dot{\xi}, \dot{\eta}_1, \ldots, \dot{\eta}_{n-2}; \xi, \eta_1, \ldots, \eta_{n-2}) \in \mathbf{R}^{2(n-1)} : \; 0 \leq \xi, \eta_1, \ldots, \eta_{n-3} \leq \pi, \;\; \eta_{n-2} \bmod 2\pi\}. \tag{5}$$

The tensor (of the second-rank) $\tilde{\Omega}$ of the angular velocity in the coordinate system $Dx_1 \ldots x_n$, we define through the skew-symmetric matrix. And so, to be specific, in the case

$n = 5$ that matrix has the form

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\omega_{10} & \omega_9 & -\omega_7 & \omega_4 \\ \omega_{10} & 0 & -\omega_8 & \omega_6 & -\omega_3 \\ -\omega_9 & \omega_8 & 0 & -\omega_5 & \omega_2 \\ \omega_7 & -\omega_6 & \omega_5 & 0 & -\omega_1 \\ -\omega_4 & \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}, \ \tilde{\Omega} \in \mathrm{so}(5). \tag{6}$$

The distance from the center D of the disk $\mathcal{D}^{n-1}$ to the center of pressure (the point N) has the form

$$|\mathbf{r}_N| = r_N = DN\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{l\Omega}{v_D}\right), \tag{7}$$

where

$$\mathbf{r}_N = \{0, x_{2N}, \ldots, x_{nN}\}$$

in system $Dx_1 \ldots x_n$ (we omit the wave over Ω).

We note, likely in lower-dimensional cases, that the model used to describe the effects of fluid flow on fixed pendulum is similar to the model constructed for free body and, in further, takes into account of the rotational derivative of the moment of the forces of medium influence with respect to the pendulum angular velocity tensor. An analysis of the problem of the (generalized) spherical (physical) pendulum in a flow will allow to find the qualitative analogies in the dynamics of partially fixed bodies and free multi-dimensional ones.

3 Some general discourses

3.1 Cases of dynamical symmetries of multi-dimensional rigid body

Let a n-dimensional rigid body Θ of mass m with smooth $(n-1)$-dimensional boundary $\partial\Theta$ be under the influence of a nonconservative force field; this can be interpreted as a motion of the body in a resisting medium that fills up the multi-dimensional domain of Euclidean space $\mathbf{E}^n$.

We assume that the body is dynamically symmetric. In this case, for instance, for the four-dimensional body, there are two logical possibilities of the representation of its inertia tensor in the case of existence of *two* independent equations on the principal moments of inertia; i.e., either in some coordinate system $Dx_1x_2x_3x_4$ attached to the body, the operator of inertia has the form

$$\mathrm{diag}\{I_1, I_2, I_2, I_2\} \tag{8}$$

(the so called case (1—3)), or the form

$$\mathrm{diag}\{I_1, I_1, I_3, I_3\} \tag{9}$$

(the case (2—2)). In the first case, the body is dynamically symmetric in the hyperplane $Dx_2x_3x_4$ (in other words, Dx_1 is the axis of dynamical symmetry) and, in the second case, the two-dimensional planes Dx_1x_2 and Dx_3x_4 are the planes of dynamical symmetry of the body.

For the five-dimensional body, it could be logically to study the cases of existence of *three* independent equations on the principal moments of inertia; i.e., either in some coordinate system $Dx_1x_2x_3x_4x_5$ attached to the body, the operator of inertia has the form

$$\mathrm{diag}\{I_1, I_2, I_2, I_2, I_2\} \tag{10}$$

(the case (1—4)), or the form

$$\mathrm{diag}\{I_1, I_1, I_3, I_3, I_3\} \qquad (11)$$

(the case (2—3)). In the first case, the body is dynamically symmetric in the hyperplane $Dx_2x_3x_4x_5$ (in other words, Dx_1 is the axis of dynamical symmetry) and, in the second case, the two-dimensional plane Dx_1x_2 and three-dimensional plane Dx_3x_4 are the planes of dynamical symmetry of the body.

Respectively, for the n-dimensional body, it could also be logically to study the cases of existence of $n-1$ independent equations on the principal moments of inertia. In this case, $[n/2]$ variants of the forms (8), (9) (or (10), (11)) are possible (here, [...] is the integral part). For instance, for the five-dimensional body, three cases (1—5), (2—4), and (3—3) are possible.

For the case of n-dimensional rigid body, primarily, we shall be interested of the case (1—$(n-1)$), i.e., when, in the certain coordinate system $Dx_1 \ldots x_n$ attached to the body, the operator of inertia has the form

$$\mathrm{diag}\{I_1, \underbrace{I_2, \ldots, I_2}_{n-1}\}, \qquad (12)$$

precisely, in the hyperplane $Dx_2 \ldots x_n$, a body is dynamically symmetric (in other words, Dx_1 is the axis of dynamical symmetry).

3.2 Dynamics on so(n) and $\mathbf{R}^n$

The configuration space of a free, n-dimensional rigid body is the direct product

$$\mathbf{R}^n \times \mathrm{SO}(n) \qquad (13)$$

of the space $\mathbf{R}^n$, which defines the coordinates of the center of mass of the body, and the rotation group SO(n), which defines the rotations of the body about its center of mass and has dimension

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

Respectively, the dimension of the phase space is equal to

$$n(n+1).$$

In particular, if Ω is the tensor of angular velocity of a n-dimensional rigid body (it is a second-rank tensor), $\Omega \in \mathrm{so}(n)$, then *the part of the dynamical equations of motion corresponding to the Lie algebra* so(n) has the following form:

$$\dot{\Omega}\Lambda + \Lambda\dot{\Omega} + [\Omega,\ \Omega\Lambda + \Lambda\Omega] = M, \qquad (14)$$

where

$$\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}, \qquad (15)$$

$$\lambda_1 = \frac{-I_1 + I_2 + \ldots + I_n}{2},\ \lambda_2 = \frac{I_1 - I_2 + I_3 + \ldots + I_n}{2}, \ldots,$$

$$\lambda_{n-1} = \frac{I_1 + \ldots + I_{n-2} - I_{n-1} + I_n}{2},\ \lambda_n = \frac{I_1 + \ldots + I_{n-1} - I_n}{2},$$

$M = M_F$ is the natural projection of the moment of external forces $\mathbf{F}$ acting on the body in $\mathbf{R}^n$ on the natural coordinates of the Lie algebra so(n) and $[.,.]$ is the commutator in so(n).

The skew-symmetric matrix corresponding to this second-rank tensor $\Omega \in \mathrm{so}(5)$ we represent in the form

$$\begin{pmatrix} 0 & -\omega_{10} & \omega_9 & -\omega_7 & \omega_4 \\ \omega_{10} & 0 & -\omega_8 & \omega_6 & -\omega_3 \\ -\omega_9 & \omega_8 & 0 & -\omega_5 & \omega_2 \\ \omega_7 & -\omega_6 & \omega_5 & 0 & -\omega_1 \\ -\omega_4 & \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}, \tag{16}$$

(see also [5, 6]), where $\omega_1, \ \omega_2, \ \ldots, \ \omega_{10}$ are the components of the tensor of angular velocity corresponding to the projections on the coordinates of the Lie algebra so(5).

In this case, obviously, the following relations hold:

$$\lambda_i - \lambda_j = I_j - I_i \tag{17}$$

for any $i, j = 1, \ldots, n$.

For the calculation of the moment of an external force acting on the body, we need to construct the mapping

$$\mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathrm{so}(n), \tag{18}$$

than maps a pair of vectors

$$(\mathbf{DN}, \mathbf{F}) \in \mathbf{R}^n \times \mathbf{R}^n \tag{19}$$

from $\mathbf{R}^n \times \mathbf{R}^n$ to an element of the Lie algebra $\mathrm{so}(n)$, where

$$\mathbf{DN} = \{\delta_1, \delta_2, \ldots, \delta_n\}, \ \mathbf{F} = \{F_1, F_2, \ldots, F_n\}, \tag{20}$$

and $\mathbf{F}$ is an external force acting on the body (here, $\mathbf{DN}$ is the vector passing through the origin D of the coordinate system $Dx_1 \ldots x_n$ to the point N of application of the force). For this end, we construct the following auxiliary matrix

$$\begin{pmatrix} \delta_1 & \delta_2 & \ldots & \delta_n \\ F_1 & F_2 & \ldots & F_n \end{pmatrix}. \tag{21}$$

All kinds of second-order minors with the sign (and they are exactly $n(n-1)/2$ pieces $n(n-1)/2$) of this auxiliary matrix are the coordinates of the moment $(\mathbf{DN}, \mathbf{F})$ of the force $\mathbf{F}$, and this moment currently identified with an element of the Lie algebra $\mathrm{so}(n)$.

Since the ordering the coordinates $\omega_1, \ \omega_2, \ \ldots, \ \omega_f, f = 1, \ldots, n(n-1)/2$, has been introduced on the Lie algebra $\mathrm{so}(n)$, then we also introduce the same ordering for the calculating of the moment $M_F = (\mathbf{DN}, \mathbf{F})$ of the force $\mathbf{F}$. Indeed, the first set G_1 of coordinates of the desired moment consists of $n-1$ alternating minors

$$+\begin{vmatrix} \delta_{n-1} & \delta_n \\ F_{n-1} & F_n \end{vmatrix}, \ -\begin{vmatrix} \delta_{n-2} & \delta_n \\ F_{n-2} & F_n \end{vmatrix}, \ +\begin{vmatrix} \delta_{n-3} & \delta_n \\ F_{n-3} & F_n \end{vmatrix}, \ \ldots, (-1)^n \begin{vmatrix} \delta_1 & \delta_n \\ F_1 & F_n \end{vmatrix}.$$

The second set G_2 of coordinates consists of $n-2$ alternating minors

$$+\begin{vmatrix} \delta_{n-2} & \delta_{n-1} \\ F_{n-2} & F_{n-1} \end{vmatrix}, \ -\begin{vmatrix} \delta_{n-3} & \delta_{n-1} \\ F_{n-3} & F_{n-1} \end{vmatrix}, \ +\begin{vmatrix} \delta_{n-4} & \delta_{n-1} \\ F_{n-4} & F_{n-1} \end{vmatrix}, \ \ldots, (-1)^{n+1} \begin{vmatrix} \delta_1 & \delta_{n-1} \\ F_1 & F_{n-1} \end{vmatrix}.$$

Continuing, the final set G_{n-1} of coordinates consists of one minor

$$+\begin{vmatrix} \delta_1 & \delta_2 \\ F_1 & F_2 \end{vmatrix}.$$

As seen, the first minors in any set begin from the sign "+".

The resulting ordered set from $n(n-1)/2$ values, we call *the coordinates of moment* $(\mathbf{DN}, \mathbf{F})$ *of the force* $\mathbf{F}$.

Dynamical systems studied in the following sections, generally speaking, are not conservative; they are dynamical systems with variable dissipation with zero mean (see [7, 8]). We need to examine by direct methods a part of the main system of dynamical equations, namely, the Newton equation, which plays the role of the equation of motion of the center of mass, i.e., *the part of the dynamical equations corresponding to the space* $\mathbf{R}^n$:

$$m\mathbf{w}_C = \mathbf{F}, \tag{22}$$

where $\mathbf{w}_C$ is the acceleration of the center of mass C of the body and m is its mass. Moreover, due to the higher-dimensional Rivals formula (in this case, it can be obtained by the operator method not difficultly) we have the following relations:

$$\mathbf{w}_C = \mathbf{w}_D + \Omega^2\mathbf{DC} + E\mathbf{DC},\ \ \mathbf{w}_D = \dot{\mathbf{v}}_D + \Omega\mathbf{v}_D,\ \ E = \dot{\Omega}, \tag{23}$$

where $\mathbf{w}_D$ is the acceleration of the point D, $\mathbf{F}$ is the external force acting on the body, and E is the tensor of angular acceleration (second-rank tensor).

If the position of a body Θ in the Euclidean space $\mathbf{E}^n$ is determined by the functions which are the cyclic in the following sense, i.e., the generalized force $\mathbf{F}$ and its moment $M_F = (\mathbf{DN}, \mathbf{F})$ depend on the generalized velocities only (quasi-velocities) and do not depend on the position of a body in the space, then the system of equations (14) and (22) on the manifold $\mathbf{R}^n \times \mathrm{so}(n)$ is a *closed* system of dynamical equations of the motion of a free multi-dimensional rigid body under the action of an external force $\mathbf{F}$. This system has been separated from the kinematic part of the equations of motion on the manifold (13) and can be examined independently.

In particular, the right-hand side of the system (14) for $n = 5$ has the form

$$\begin{gathered} M = \{M_1, M_2, \ldots, M_{10}\} = \\ = \{\delta_4F_5 - \delta_5F_4, \delta_5F_3 - \delta_3F_5, \delta_2F_5 - \delta_5F_2, \delta_5F_1 - \delta_1F_5, \delta_3F_4 - \delta_4F_3, \\ \delta_4F_2 - \delta_2F_4, \delta_1F_4 - \delta_4F_1, \delta_2F_3 - \delta_3F_2, \delta_3F_1 - \delta_1F_3, \delta_1F_2 - \delta_2F_1\}, \end{gathered} \tag{24}$$

where $M_1,\ M_2,\ \ldots,\ M_{10}$ are the components of the tensor of moment of the external forces in the projections on the coordinates in the Lie algebra so(5),

$$M = \begin{pmatrix} 0 & -M_{10} & M_9 & -M_7 & M_4 \\ M_{10} & 0 & -M_8 & M_6 & -M_3 \\ -M_9 & M_8 & 0 & -M_5 & M_2 \\ M_7 & -M_6 & M_5 & 0 & -M_1 \\ -M_4 & M_3 & -M_2 & M_1 & 0 \end{pmatrix}. \tag{25}$$

4 Set of dynamical equations in Lie algebra so(n)

In our case of a fixed pendulum, the case (12) is realized. Then the dynamical part of the equations of the motion corresponding to the Lie algebra so(n), has the following form:

$$
\begin{aligned}
&(I_1+(n-3)I_2)\dot\omega_1 = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&(I_1+(n-3)I_2)\dot\omega_{r_1-1} = 0,\\
&(n-2)I_2\dot\omega_{r_1} + (-1)^{n+1}(I_1-I_2)W_{n-1}(\Omega) = (-1)^n x_{nN}\left(\alpha,\beta_1,\dots,\beta_{n-2},\frac{\Omega}{v}\right)s(\alpha)v^2,\\
&(I_1+(n-3)I_2)\dot\omega_{r_1+1} = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&(I_1+(n-3)I_2)\dot\omega_{r_2-1} = 0,\\
&(n-2)I_2\dot\omega_{r_2} + (-1)^{n}(I_1-I_2)W_{n-2}(\Omega) = (-1)^{n-1} x_{n-1,N}\left(\alpha,\beta_1,\dots,\beta_{n-2},\frac{\Omega}{v}\right)s(\alpha)v^2,\\
&(I_1+(n-3)I_2)\dot\omega_{r_2+1} = 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&(I_1+(n-3)I_2)\dot\omega_{r_{n-2}-1} = 0,\\
&(n-2)I_2\dot\omega_{r_{n-2}} + (I_1-I_2)W_2(\Omega) = -x_{3N}\left(\alpha,\beta_1,\dots,\beta_{n-2},\frac{\Omega}{v}\right)s(\alpha)v^2,\\
&(n-2)I_2\dot\omega_{r_{n-1}} + (I_2-I_1)W_1(\Omega) = x_{2N}\left(\alpha,\beta_1,\dots,\beta_{n-2},\frac{\Omega}{v}\right)s(\alpha)v^2,
\end{aligned}
\tag{26}
$$

where $r_{n-2}+1 = r_{n-1}$, and the functions $W_t(\Omega)$, $t=1,\dots,n-1$, are the quadratic forms on the components $\omega_1,\dots,\omega_f$, $f=n(n-1)/2$, of tensor Ω, herewith,

$$W_t(\Omega)|_{\omega_{k_1}=\ldots=\omega_{k_s}=0} = 0,\ \ s=(n-1)(n-2)/2,\ \ k_j\neq r_i,\ \ j=1,\dots,s,\ \ i=1,\dots,n-1. \tag{27}$$

Let us explain the formula (27). The tensor $\Omega\in$ so(n) has

$$f = n(n-1)/2$$

components totally. Respectively, the moment of the forces $M_F = (\mathbf{DN},\mathbf{F})$ has as many components. Since the auxiliary matrix (21) has the following form

$$\begin{pmatrix} 0 & x_{2N} & \dots & x_{nN} \\ -s(\alpha)v_D^2 & 0 & \dots & 0 \end{pmatrix}, \tag{28}$$

in the right-hand side of the system (26)

$$s = (n-1)(n-2)/2$$

equations contain the identical zero. We denote the numbers of those equations as follows:

$$k_1,\dots,k_s.$$

In this case, the corresponding components $\omega_{k_j}, j=1,\dots,s$, of the tensor Ω of the angular velocity, we call *the cyclic*.

The rest of the numbers of equations in which the following values with the sign

$$x_{lN}\left(\alpha,\beta_1,\ldots,\beta_{n-2},\frac{\Omega}{v}\right)s(\alpha)v^2,\ l=2,\ldots,n,$$

present, we denote through

$$r_1,\ldots,r_{n-1},$$

since

$$f-s=\frac{n(n-1)}{2}-\frac{(n-1)(n-2)}{2}=n-1.$$

Obviously that

$$W_t(0)\equiv 0$$

for any $t=1,\ldots,n-1$, i.e., the quadratic forms $W_t(\Omega)$ are equal to zero identically, when all the components of the tensor Ω are equal to zero. In this case, the formula (27) means that for the vanishing of quadratic forms $W_t(\Omega)$, $t=1,\ldots,n-1$, to zero, it is sufficient that all the cyclic components of the tensor Ω could be zero.

In particular, in the case $n=5$ this system has the from:

$$\begin{aligned}
&(I_1+2I_2)\dot\omega_1=0,\\
&(I_1+2I_2)\dot\omega_2=0,\\
&(I_1+2I_2)\dot\omega_3=0,\\
&3I_2\dot\omega_4+(I_1-I_2)(\omega_3\omega_{10}+\omega_2\omega_9+\omega_1\omega_7)=-x_{5N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)s(\alpha)v^2,\\
&(I_1+2I_2)\dot\omega_5=0,\\
&(I_1+2I_2)\dot\omega_6=0,\\
&3I_2\dot\omega_7+(I_2-I_1)(\omega_1\omega_4-\omega_6\omega_{10}-\omega_5\omega_9)=x_{4N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)s(\alpha)v^2,\\
&(I_1+2I_2)\dot\omega_8=0,\\
&3I_2\dot\omega_9+(I_1-I_2)(\omega_8\omega_{10}-\omega_5\omega_7-\omega_2\omega_4)=-x_{3N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)s(\alpha)v^2,\\
&3I_2\dot\omega_{10}+(I_2-I_1)(\omega_8\omega_9+\omega_6\omega_7+\omega_3\omega_4)=x_{2N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)s(\alpha)v^2,
\end{aligned}\tag{29}$$

since the moment of the medium interaction force for $n=5$ is determined through the following auxiliary matrix:

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} & x_{4N} & x_{5N}\\ -s(\alpha)v_D^2 & 0 & 0 & 0 & 0\end{pmatrix},\tag{30}$$

where

$$\{-s(\alpha)v_D^2,0,0,0,0\}$$

is the decomposition of the force $\mathbf{S}$ of the medium interaction in the coordinate system $Dx_1x_2x_3x_4x_5$. In this case,

$$r_1=4,\ r_2=7,\ r_3=9,\ r_4=10.$$

Since the dimension of the Lie algebra so(n) is equal to $f=n(n-1)/2$, the system of equations (26) represents the set of the dynamical equations on so(n).

We see, that in the right-hand side of Eq. (26), first of all, it includes the angles $\alpha,\beta_1,\ldots,\beta_{n-2}$, therefore, this system of equations is not closed. In order to obtain a complete system of equations of motion of the pendulum, it is necessary to attach several sets of kinematic equations to the dynamic equations on the Lie algebra so(n).

4.1 Cyclic first integrals

We immediately note that the system (26) obtained from (14), by the existing dynamic symmetry

$$I_2 = \ldots = I_n, \tag{31}$$

possesses $s = (n-1)(n-2)/2$ cyclic first integrals

$$\omega_{k_1} \equiv \omega^0_{k_1} = \text{const}, \ \ldots, \ \omega_{k_s} \equiv \omega^0_{k_s} = \text{const}, \ s = \frac{(n-1)(n-2)}{2}. \tag{32}$$

In this case, further, we consider the dynamics of our system at zero levels:

$$\omega^0_{k_1} = \ldots = \omega^0_{k_s} = 0. \tag{33}$$

In particular, the system (29) possesses the first integrals

$$\omega_1 \equiv \omega^0_1, \ \omega_2 \equiv \omega^0_2, \ \omega_3 \equiv \omega^0_3, \ \omega_5 \equiv \omega^0_5, \ \omega_6 \equiv \omega^0_6, \ \omega_8 \equiv \omega^0_8, \tag{34}$$

which are considered at zero levels:

$$\omega^0_1 = \omega^0_2 = \omega^0_3 = \omega^0_5 = \omega^0_6 = \omega^0_8 = 0. \tag{35}$$

The nonzero components $\omega_{r_1}, \ldots, \omega_{r_p}$ of the tensor Ω, it has $p = f - s = n - 1$ pieces (here $r_1, \ldots, r_p$ the rest p of numbers from the set $Q_1 = \{1, 2, \ldots, n(n-1)/2\}$, which are not equal to $k_1, \ldots, k_s$).

Under conditions (31)–(33) the system (26) has the form of unclosed system of $n-1$ equations:

$$\begin{gathered}
(n-2)I_2\dot{\omega}_{r_1} = (-1)^n x_{nN}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right) s(\alpha)v^2, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(n-2)I_2\dot{\omega}_{r_2} = (-1)^{n-1} x_{n-1,N}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right) s(\alpha)v^2, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(n-2)I_2\dot{\omega}_{r_{n-2}} = -x_{3N}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right) s(\alpha)v^2, \\
(n-2)I_2\dot{\omega}_{r_{n-1}} = x_{2N}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right) s(\alpha)v^2.
\end{gathered} \tag{36}$$

In particular, under conditions (34)–(35) the system (29) has the form of unclosed system of four equations:

$$\begin{gathered}
3I_2\dot{\omega}_4 = -x_{5N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) s(\alpha)v^2, \\
3I_2\dot{\omega}_7 = x_{4N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) s(\alpha)v^2, \\
3I_2\dot{\omega}_9 = -x_{3N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) s(\alpha)v^2, \\
3I_2\dot{\omega}_{10} = x_{2N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) s(\alpha)v^2.
\end{gathered} \tag{37}$$

5 First set of kinematic equations

In order to obtain a complete system of equations of motion, it needs the set of kinematic equations which relate the velocities of the point D (i.e., the center of the disk $\mathcal{D}^{n-1}$) and the over-running medium flow:

$$\mathbf{v}_D = v_D \cdot \mathbf{i}_v(\alpha, \beta_1, \ldots, \beta_{n-2}) = \tilde{\Omega} \begin{pmatrix} l \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (-v_\infty)\mathbf{i}_v(-\xi, \eta_1, \ldots, \eta_{n-2}), \tag{38}$$

where

$$\mathbf{i}_v(\alpha, \beta_1, \ldots, \beta_{n-2}) = \begin{pmatrix} \cos\alpha \\ \sin\alpha\cos\beta_1 \\ \sin\alpha\sin\beta_1\cos\beta_2 \\ \cdots\cdots\cdots \\ \sin\alpha\sin\beta_1\ldots\sin\beta_{n-3}\cos\beta_{n-2} \\ \sin\alpha\sin\beta_1\ldots\sin\beta_{n-2} \end{pmatrix}. \tag{39}$$

The equation (38) expresses the theorem of addition of velocities in projections on the related coordinate system $Dx_1 \ldots x_n$.

Indeed, the left-hand side of Eq. (38) is the velocity of the point D of the pendulum with respect to the flow in the projections on the related with the pendulum coordinate system $Dx_1 \ldots x_n$. Herewith, the vector $\mathbf{i}_v(\alpha, \beta_1, \ldots, \beta_{n-2})$ is the unit vector along the axis of the vector $\mathbf{v}_D$. The vector $\mathbf{i}_v(\alpha, \beta_1, \ldots, \beta_{n-2})$ has the (generalized) spherical coordinates $(1, \alpha, \beta_1, \ldots, \beta_{n-2})$ which determines the decomposition (39).

The right-hand side of the Eq. (38) is the sum of the velocities of the point D when you rotate the pendulum (the first term), and the motion of the flow (the second term). In this case, in the first term, we have the coordinates of the vector вектора $\mathbf{OD} = \{l, 0, \ldots, 0\}$ in the coordinate system $Dx_1 \ldots x_n$.

We explain the second term of the right-hand side of Eq. (38) in more detail. We have in it the coordinates of the vector $(-\mathbf{v}_\infty) = \{-v_\infty, 0, \ldots, 0\}$ in the immovable space. In order to describe it in the projections on the related coordinate system $Dx_1 \ldots x_n$, we need to make a (reverse) rotation of the pendulum at the angle $(-\xi)$ that is algebraically equivalent to multiplying the value $(-v_\infty)$ on the vector $\mathbf{i}_v(-\xi, \eta_1, \ldots, \eta_{n-2})$.

Thus, the first set of kinematic equations (38) has the following form in our case:

$$\begin{aligned}
&v_D\cos\alpha = -v_\infty\cos\xi, \\
&v_D\sin\alpha\cos\beta_1 = l\omega_{r_{n-1}} + v_\infty\sin\xi\cos\eta_1, \\
&v_D\sin\alpha\sin\beta_1\cos\beta_2 = -l\omega_{r_{n-2}} + v_\infty\sin\xi\sin\eta_1\cos\eta_2, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&v_D\sin\alpha\sin\beta_1\ldots\sin\beta_{n-3}\cos\beta_{n-2} = (-1)^{n+1}l\omega_{r_2} + v_\infty\sin\xi\sin\eta_1\ldots\sin\eta_{n-3}\cos\eta_{n-2}, \\
&v_D\sin\alpha\sin\beta_1\ldots\sin\beta_{n-2} = (-1)^n l\omega_{r_1} + v_\infty\sin\xi\sin\eta_1\ldots\sin\eta_{n-2}.
\end{aligned} \tag{40}$$

In particular, in the case $n = 5$ this set of equations has the form:

$$\begin{aligned}
&v_D\cos\alpha = -v_\infty\cos\xi, \\
&v_D\sin\alpha\cos\beta_1 = l\omega_{10} + v_\infty\sin\xi\cos\eta_1, \\
&v_D\sin\alpha\sin\beta_1\cos\beta_2 = -l\omega_9 + v_\infty\sin\xi\sin\eta_1\cos\eta_2, \\
&v_D\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3 = l\omega_7 + v_\infty\sin\xi\sin\eta_1\sin\eta_2\cos\eta_3, \\
&v_D\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3 = -l\omega_4 + v_\infty\sin\xi\sin\eta_1\sin\eta_2\sin\eta_3.
\end{aligned} \tag{41}$$

6 Second set of kinematic equations

We also need a set of kinematic equations which relate the angular velocity tensor $\tilde{\Omega}$ and coordinates $\dot{\xi}, \dot{\eta}_1, \ldots, \dot{\eta}_{n-2}, \xi, \eta_1, \ldots, \eta_{n-2}$ of the phase space (5) of pendulum studied, i.e., the tangent bundle $T_*\mathbf{S}^n\{\dot{\xi}, \dot{\eta}_1, \ldots, \dot{\eta}_{n-2}; \xi, \eta_1, \ldots, \eta_{n-2}\}$.

We draw the reasoning style allowing arbitrary dimension. The desired equations are obtained from the following two sets of relations. Since the motion of the body takes place in a Euclidean space $\mathbf{E}^n$ formally, at the beginning, we express the tuple consisting of a phase variables $\omega_{r_1}, \omega_{r_2}, \ldots, \omega_{r_{n-1}}$, through new variable $z_1, \ldots, z_{n-1}$ (from the tuple z). For this, we draw the following turns by the angles $\eta_1, \ldots, \eta_{n-2}$:

$$\begin{pmatrix} \omega_{r_1} \\ \omega_{r_2} \\ \cdots \\ \omega_{r_{n-1}} \end{pmatrix} = T_{1,2}(\eta_{n-2}) \circ T_{2,3}(\eta_{n-3}) \circ \ldots \circ T_{n-2,n-1}(\eta_1) \begin{pmatrix} z_1 \\ z_2 \\ \cdots \\ z_{n-1} \end{pmatrix}, \tag{42}$$

where the matrix $T_{k,k+1}(\eta),\ k = 1, \ldots, n-2$, is obtained from the unit one by the existence of the second-order minor $M_{k,k+1}$:

$$T_{k,k+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & M_{k,k+1} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{43}$$

$$M_{k,k+1} = \begin{pmatrix} m_{k,k} & m_{k,k+1} \\ m_{k+1,k} & m_{k+1,k+1} \end{pmatrix},\ m_{k,k} = m_{k+1,k+1} = \cos\eta,\ m_{k+1,k} = -m_{k,k+1} = \sin\eta.$$

In other words, the following relations hold:

$$\begin{pmatrix} z_1 \\ z_2 \\ \cdots \\ z_{n-1} \end{pmatrix} = T_{n-2,n-1}(-\eta_1) \circ T_{n-3,n-2}(-\eta_2) \circ \ldots \circ T_{1,2}(-\eta_{n-2}) \begin{pmatrix} \omega_{r_1} \\ \omega_{r_2} \\ \cdots \\ \omega_{r_{n-1}} \end{pmatrix}. \tag{44}$$

In particular, for $n = 5$ the values $\omega_4, \omega_7, \omega_9, \omega_{10}$ are transformed through the composition of the following three turns:

$$\begin{pmatrix} \omega_4 \\ \omega_7 \\ \omega_9 \\ \omega_{10} \end{pmatrix} = T_{1,2}(\eta_3) \circ T_{2,3}(\eta_2) \circ T_{3,4}(\eta_1) \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}, \tag{45}$$

where

$$T_{3,4}(\eta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\eta & -\sin\eta \\ 0 & 0 & \sin\eta & \cos\eta \end{pmatrix},$$

$$T_{2,3}(\eta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\eta & -\sin\eta & 0 \\ 0 & \sin\eta & \cos\eta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_{1,2}(\eta) = \begin{pmatrix} \cos\eta & -\sin\eta & 0 & 0 \\ \sin\eta & \cos\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In other words, the following relations hold:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = T_{3,4}(-\eta_1) \circ T_{2,3}(-\eta_2) \circ T_{1,2}(-\eta_3) \begin{pmatrix} \omega_4 \\ \omega_7 \\ \omega_9 \\ \omega_{10} \end{pmatrix}, \tag{46}$$

i.e.,

$$\begin{aligned} z_1 &= \omega_4 \cos\eta_3 + \omega_7 \sin\eta_3, \\ z_2 &= (\omega_7 \cos\eta_3 - \omega_4 \sin\eta_3) \cos\eta_2 + \omega_9 \sin\eta_2, \\ z_3 &= [(-\omega_7 \cos\eta_3 + \omega_4 \sin\eta_3) \sin\eta_2 + \omega_9 \cos\eta_2] \cos\eta_1 + \omega_{10} \sin\eta_1, \\ z_4 &= [(\omega_7 \cos\eta_3 - \omega_4 \sin\eta_3) \sin\eta_2 - \omega_9 \cos\eta_2] \sin\eta_1 + \omega_{10} \cos\eta_1. \end{aligned} \tag{47}$$

Then we substitute the following relations instead of the variables z:

$$\begin{aligned} z_{n-1} &= \dot{\xi}, \\ z_{n-2} &= -\dot{\eta}_1 \frac{\sin\xi}{\cos\xi}, \\ z_{n-3} &= \dot{\eta}_2 \frac{\sin\xi}{\cos\xi} \sin\eta_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ z_2 &= (-1)^{n+1} \dot{\eta}_{n-3} \frac{\sin\xi}{\cos\xi} \sin\eta_1 \ldots \sin\eta_{n-4}, \\ z_1 &= (-1)^n \dot{\eta}_{n-2} \frac{\sin\xi}{\cos\xi} \sin\eta_1 \ldots \sin\eta_{n-3}. \end{aligned} \tag{48}$$

In particular, for $n = 5$ we have the following formula:

$$\begin{aligned} z_4 &= \dot{\xi}, \\ z_3 &= -\dot{\eta}_1 \frac{\sin\xi}{\cos\xi}, \\ z_2 &= \dot{\eta}_2 \frac{\sin\xi}{\cos\xi} \sin\eta_1, \\ z_1 &= -\dot{\eta}_3 \frac{\sin\xi}{\cos\xi} \sin\eta_1 \sin\eta_2. \end{aligned} \tag{49}$$

Thus, two sets of Eqs. (42) and (48) give the second set of kinematic equations:

$$\begin{pmatrix} \omega_{r_1} \\ \omega_{r_2} \\ \cdots \\ \omega_{r_{n-1}} \end{pmatrix} =$$

$$= T_{1,2}(\eta_{n-2}) \circ T_{2,3}(\eta_{n-3}) \circ \ldots \circ$$

$$
\circ T_{n-3,n-2}(\eta_2)T_{n-2,n-1}(\eta_1)\begin{pmatrix} (-1)^n\dot{\eta}_{n-2}\frac{\sin\xi}{\cos\xi}\sin\eta_1\ldots\sin\eta_{n-3} \\ (-1)^{n+1}\dot{\eta}_{n-3}\frac{\sin\xi}{\cos\xi}\sin\eta_1\ldots\sin\eta_{n-4} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{\eta}_2\frac{\sin\xi}{\cos\xi}\sin\eta_1 \\ -\dot{\eta}_1\frac{\sin\xi}{\cos\xi} \\ \dot{\xi} \end{pmatrix}. \tag{50}
$$

In particular, for $n=5$ we have:

$$
\begin{aligned}
&\omega_4 = -\dot{\xi}\sin\eta_1\sin\eta_2\sin\eta_3 - \dot{\eta}_1\frac{\sin\xi}{\cos\xi}\cos\eta_1\sin\eta_2\sin\eta_3 - \\
&-\dot{\eta}_2\frac{\sin\xi}{\cos\xi}\sin\eta_1\cos\eta_2\sin\eta_3 - \dot{\eta}_3\frac{\sin\xi}{\cos\xi}\sin\eta_1\sin\eta_2\cos\eta_3, \\
&\omega_7 = \dot{\xi}\sin\eta_1\sin\eta_2\cos\eta_3 + \dot{\eta}_1\frac{\sin\xi}{\cos\xi}\cos\eta_1\sin\eta_2\cos\eta_3 + \\
&+\dot{\eta}_2\frac{\sin\xi}{\cos\xi}\sin\eta_1\cos\eta_2\cos\eta_3 - \dot{\eta}_3\frac{\sin\xi}{\cos\xi}\sin\eta_1\sin\eta_2\sin\eta_3, \\
&\omega_9 = -\dot{\xi}\sin\eta_1\cos\eta_2 - \dot{\eta}_1\frac{\sin\xi}{\cos\xi}\cos\eta_1\cos\eta_2 + \dot{\eta}_2\frac{\sin\xi}{\cos\xi}\sin\eta_1\sin\eta_2, \\
&\omega_{10} = \dot{\xi}\cos\eta_1 - \dot{\eta}_1\frac{\sin\xi}{\cos\xi}\sin\eta_1.
\end{aligned} \tag{51}
$$

We see that three sets of the relations (36), (40) and (50) form the closed system of equations. These three sets of equations include the following functions:

$$
x_{2N}\left(\alpha,\beta_1,\ldots,\beta_{n-2},\frac{\Omega}{v_D}\right),\ \ldots,\ x_{nN}\left(\alpha,\beta_1,\ldots,\beta_{n-2},\frac{\Omega}{v_D}\right),\ s(\alpha). \tag{52}
$$

In this case, the function s is considered to be dependent only on α, and the functions $x_{2N},\ldots,x_{nN}$ may depend on, along with the angles α, β_1, …, β_{n-2}, generally speaking, the reduced angular velocity tensor $l\Omega/v_D$.

7 Case where the moment of nonconservative forces is independent of the angular velocity

We take the function $\mathbf{r}_N$ as follows (the disk $\mathcal{D}^{n-1}$ is given by the equation $x_{1N}\equiv 0$):

$$
\mathbf{r}_N = \begin{pmatrix} 0 \\ x_{2N} \\ \vdots \\ x_{nN} \end{pmatrix} = R(\alpha)\mathbf{i}_N, \tag{53}
$$

where

$$
\mathbf{i}_N = \mathbf{i}_v\left(\frac{\pi}{2},\beta_1,\ldots,\beta_{n-2}\right) \tag{54}
$$

(see (39)).

In our case

$$\mathbf{i}_N = \begin{pmatrix} 0 \\ \cos\beta_1 \\ \sin\beta_1\cos\beta_2 \\ \dots \\ \sin\beta_1\dots\sin\beta_{n-3}\cos\beta_{n-2} \\ \sin\beta_1\dots\sin\beta_{n-2} \end{pmatrix}. \tag{55}$$

Thus, the equalities

$$\begin{gathered} x_{2N} = R(\alpha)\cos\beta_1,\ x_{3N} = R(\alpha)\sin\beta_1\cos\beta_2,\ \dots, \\ x_{n-1,N} = R(\alpha)\sin\beta_1\dots\sin\beta_{n-3}\cos\beta_{n-2},\ x_{nN} = R(\alpha)\sin\beta_1\dots\sin\beta_{n-2}, \end{gathered} \tag{56}$$

hold and show that for the considered system, the moment of the nonconservative forces is independent of the angular velocity tensor (it depends only on the angles $\alpha, \beta_1, \dots, \beta_{n-2}$).

And so, for the construction of the force field, we use the pair of dynamical functions $R(\alpha), s(\alpha)$; the information about them is of a qualitative nature. Similarly to the choice of the Chaplygin analytical functions (see [9, 10]), we take the dynamical functions s and R as follows:

$$R(\alpha) = A\sin\alpha,\ s(\alpha) = B\cos\alpha,\ A, B > 0. \tag{57}$$

7.1 Reduced systems

Theorem 7.1. *The simultaneous equations (26), (40), (50) under conditions (31)–(33), (53), (57) can be reduced to the dynamical system on the tangent bundle (5) of the $(n-1)$-dimensional sphere (4).*

Indeed, if we introduce the dimensionless parameter and the differentiation by the formulas

$$b_* = ln_0,\ n_0^2 = \frac{AB}{(n-2)I_2},\ <\cdot>= n_0 v_\infty <'>, \tag{58}$$

then the obtained equations have the following form:

$$
\begin{gathered}
\xi'' + b_*\xi'\cos\xi + \sin\xi\cos\xi - \\
- \left[\eta_1'^2 + \eta_2'^2\sin^2\eta_1 + \eta_3'^2\sin^2\eta_1\sin^2\eta_2 + \ldots + \eta_{n-2}'^2\sin^2\eta_1\ldots\sin^2\eta_{n-3}\right]\frac{\sin\xi}{\cos\xi} = 0, \\
\eta_1'' + b_*\eta_1'\cos\xi + \xi'\eta_1'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} - \\
- \left[\eta_2'^2 + \eta_3'^2\sin^2\eta_2 + \eta_4'^2\sin^2\eta_2\sin^2\eta_3 + \ldots + \eta_{n-2}'^2\sin^2\eta_2\ldots\sin^2\eta_{n-3}\right]\sin\eta_1\cos\eta_1 = 0, \\
\eta_2'' + b_*\eta_2'\cos\xi + \xi'\eta_2'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\
+ 2\eta_1'\eta_2'\frac{\cos\eta_1}{\sin\eta_1} - \\
- \left[\eta_3'^2 + \eta_4'^2\sin^2\eta_3 + \eta_5'^2\sin^2\eta_3\sin^2\eta_4 + \ldots + \eta_{n-2}'^2\sin^2\eta_3\ldots\sin^2\eta_{n-3}\right]\sin\eta_2\cos\eta_2 = 0, \\
\eta_3'' + b_*\eta_3'\cos\xi + \xi'\eta_3'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\
+ 2\eta_1'\eta_3'\frac{\cos\eta_1}{\sin\eta_1} + 2\eta_2'\eta_3'\frac{\cos\eta_2}{\sin\eta_2} - \\
- \left[\eta_4'^2 + \eta_5'^2\sin^2\eta_4 + \eta_6'^2\sin^2\eta_4\sin^2\eta_5 + \ldots + \eta_{n-2}'^2\sin^2\eta_4\ldots\sin^2\eta_{n-3}\right]\sin\eta_3\cos\eta_3 = 0, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\eta_{n-4}'' + b_*\eta_{n-4}'\cos\xi + \xi'\eta_{n-4}'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\
+ 2\eta_1'\eta_{n-4}'\frac{\cos\eta_1}{\sin\eta_1} + \ldots + 2\eta_{n-5}'\eta_{n-4}'\frac{\cos\eta_{n-5}}{\sin\eta_{n-5}} - \\
- \left[\eta_{n-3}'^2 + \eta_{n-2}'^2\sin^2\eta_{n-3}\right]\sin\eta_{n-4}\cos\eta_{n-4} = 0, \\
\eta_{n-3}'' + b_*\eta_{n-3}'\cos\xi + \xi'\eta_{n-3}'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\
+ 2\eta_1'\eta_{n-3}'\frac{\cos\eta_1}{\sin\eta_1} + \ldots + 2\eta_{n-4}'\eta_{n-3}'\frac{\cos\eta_{n-4}}{\sin\eta_{n-4}} - \\
- \eta_{n-2}'^2\sin\eta_{n-3}\cos\eta_{n-3} = 0, \\
\eta_{n-2}'' + b_*\eta_{n-2}'\cos\xi + \xi'\eta_{n-2}'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\
+ 2\eta_1'\eta_{n-2}'\frac{\cos\eta_1}{\sin\eta_1} + \ldots + 2\eta_{n-3}'\eta_{n-2}'\frac{\cos\eta_{n-3}}{\sin\eta_{n-3}} = 0,\ b_* > 0.
\end{gathered} \tag{59}
$$

In particular, for $n = 5$ we have:

$$
\begin{gathered}
\xi'' + b_*\xi'\cos\xi + \sin\xi\cos\xi - \left[\eta_1'^2 + \eta_2'^2\sin^2\eta_1 + \eta_3'^2\sin^2\eta_1\sin^2\eta_2\right]\frac{\sin\xi}{\cos\xi} = 0, \\
\eta_1'' + b_*\eta_1'\cos\xi + \xi'\eta_1'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} - \left[\eta_2'^2 + \eta_3'^2\sin^2\eta_2\right]\sin\eta_1\cos\eta_1 = 0, \\
\eta_2'' + b_*\eta_2'\cos\xi + \xi'\eta_2'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + 2\eta_1'\eta_2'\frac{\cos\eta_1}{\sin\eta_1} - \eta_3'^2\sin\eta_2\cos\eta_2 = 0, \\
\eta_3'' + b_*\eta_3'\cos\xi + \xi'\eta_3'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + 2\eta_1'\eta_3'\frac{\cos\eta_1}{\sin\eta_1} + 2\eta_2'\eta_3'\frac{\cos\eta_2}{\sin\eta_2} = 0,\ b_* > 0.
\end{gathered} \tag{60}
$$

After the transition from the variables z (about the variables z see (48)) to the intermediate dimensionless variables w

$$
z_k = n_0 v_\infty Z_k,\ k = 1, \ldots, n-2,\ z_{n-1} = n_0 v_\infty Z_{n-1} - n_0 v_\infty b_* \sin\xi, \tag{61}
$$

system (59) is equivalent to the system

$$\xi' = Z_{n-1} - b_* \sin\xi, \tag{62}$$

$$Z'_{n-1} = -\sin\xi\cos\xi + (Z_1^2 + \ldots + Z_{n-2}^2)\frac{\cos\xi}{\sin\xi}, \tag{63}$$

$$Z'_{n-2} = -Z_{n-2}Z_{n-1}\frac{\cos\xi}{\sin\xi} - (Z_1^2 + \ldots + Z_{n-3}^2)\frac{\cos\xi}{\sin\xi}\frac{\cos\eta_1}{\sin\eta_1}, \tag{64}$$

$$Z'_{n-3} = -Z_{n-3}Z_{n-1}\frac{\cos\xi}{\sin\xi} + Z_{n-3}Z_{n-2}\frac{\cos\xi}{\sin\xi}\frac{\cos\eta_1}{\sin\eta_1} +$$

$$+(Z_1^2 + \ldots + Z_{n-4}^2)\frac{\cos\xi}{\sin\xi}\frac{1}{\sin\eta_1}\frac{\cos\eta_2}{\sin\eta_2}, \tag{65}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$Z'_1 = -Z_1\frac{\cos\xi}{\sin\xi}\left\{\sum_{s=1}^{n-2}(-1)^{s+1}Z_{n-s}\frac{\cos\eta_{s-1}}{\sin\eta_1\ldots\sin\eta_{s-1}}\right\}, \tag{66}$$

$$\eta'_1 = -Z_{n-2}\frac{\cos\xi}{\sin\xi}, \tag{67}$$

$$\eta'_2 = Z_{n-3}\frac{\cos\xi}{\sin\xi\sin\eta_1}, \tag{68}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\eta'_{n-3} = (-1)^{n+1}Z_2\frac{\cos\xi}{\sin\xi\sin\eta_1\ldots\sin\eta_{n-4}}, \tag{69}$$

$$\eta'_{n-2} = (-1)^{n}Z_1\frac{\cos\xi}{\sin\xi\sin\eta_1\ldots\sin\eta_{n-3}}, \tag{70}$$

on the tangent bundle

$$T_*\mathbf{S}^{n-1}\{(Z_{n-1},\ldots,Z_1;\xi,\eta_1,\ldots,\eta_{n-2}) \in \mathbf{R}^{2(n-1)}:\ 0 \le \xi,\eta_1,\ldots,\eta_{n-3} \le \pi,\ \eta_{n-2} \bmod 2\pi\} \tag{71}$$

of the $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}\{(\xi,\eta_1,\ldots,\eta_{n-2}) \in \mathbf{R}^{n-1} :\ 0 \le \xi,\eta_1,\ldots,\eta_{n-3} \le \pi,\ \eta_{n-2} \bmod 2\pi\}$.

We see that the independent subsystem (62)–(69) of the order $2(n-1)-1$ (due to cyclicity of the variable η_{n-2}) can be substituted into the system (62)–(70) of the order $2(n-1)$ and can be considered separately on its own $(2n-3)$-dimensional manifold.

In particular, for $n = 5$ we obtain the following eighth-order system:

$$\xi' = Z_4 - b_* \sin\xi, \tag{72}$$

$$Z'_4 = -\sin\xi\cos\xi + (Z_1^2 + Z_2^2 + Z_3^2)\frac{\cos\xi}{\sin\xi}, \tag{73}$$

$$Z'_3 = -Z_3Z_4\frac{\cos\xi}{\sin\xi} - (Z_1^2 + Z_2^2)\frac{\cos\xi}{\sin\xi}\frac{\cos\eta_1}{\sin\eta_1}, \tag{74}$$

$$Z'_2 = -Z_2Z_4\frac{\cos\xi}{\sin\xi} + Z_2Z_3\frac{\cos\xi}{\sin\xi}\frac{\cos\eta_1}{\sin\eta_1} + Z_1^2\frac{\cos\xi}{\sin\xi}\frac{1}{\sin\eta_1}\frac{\cos\eta_2}{\sin\eta_2}, \tag{75}$$

$$Z'_1 = -Z_1Z_4\frac{\cos\xi}{\sin\xi} + Z_1Z_3\frac{\cos\xi}{\sin\xi}\frac{\cos\eta_1}{\sin\eta_1} - Z_1Z_2\frac{\cos\xi}{\sin\xi}\frac{1}{\sin\eta_1}\frac{\cos\eta_2}{\sin\eta_2}, \tag{76}$$

$$\eta'_1 = -Z_3\frac{\cos\xi}{\sin\xi}, \tag{77}$$

$$\eta_2' = Z_2 \frac{\cos\xi}{\sin\xi \sin\eta_1}, \tag{78}$$

$$\eta_3' = -Z_1 \frac{\cos\xi}{\sin\xi \sin\eta_1 \sin\eta_2}, \tag{79}$$

on the tangent bundle

$$T_*\mathbf{S}^4\{(Z_4, Z_3, Z_2, Z_1; \xi, \eta_1, \eta_2, \eta_3,) \in \mathbf{R}^8 : \ 0 \le \xi, \eta_1, \eta_2 \le \pi, \ \eta_3 \bmod 2\pi\} \tag{80}$$

of the four-dimensional sphere $\mathbf{S}^4\{(\xi, \eta_1, \eta_2, \eta_3) \in \mathbf{R}^4 : \ 0 \le \xi, \eta_1, \eta_2 \le \pi, \ \eta_3 \bmod 2\pi\}$.

We see that the independent seventh-order subsystem (72)–(78) (due to cyclicity of the variable η_3) can be substituted into the eighth-order system (72)–(79) and can be considered separately on its own seven-dimensional manifold.

7.2 General remarks on integrability of system for any finite n

As already mentioned, in order to integrate completely the system (62)–(70) of the order $2(n-1)$, we have to obtain, generally speaking, $2n-3$ independent first integrals. But the systems considered have such symmetries that allow to reduce a sufficient number of the first integrals down to n, in order to integrate the system.

7.2.1 The system under the absence of a force field

Let study the system (62)–(70) on the tangent bundle $T_*\mathbf{S}^{n-1}\{Z_{n-1}, \ldots, Z_1; \xi, \eta_1, \ldots, \eta_{n-2}\}$ of the $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}\{\xi, \eta_1, \ldots, \eta_{n-2}\}$. At the same time, we get out of this system the conservative one. Furthermore, we assume that the function (81) is identically equal to zero (in particular, $b_* = 0$, and also the coefficient $\sin\xi\cos\xi$ in Eq. (63) is absent):

$$\Gamma_v\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right) = |\mathbf{r}_N| = (\mathbf{r}_N, \mathbf{i}_N(\beta_1, \ldots, \beta_{n-2})) =$$

$$= 0 \cdot \cos\frac{\pi}{2} + \sum_{s=2}^{n} x_{sN}\left(\alpha, \beta_1, \ldots, \beta_{n-2}, \frac{\Omega}{v}\right) i_{sN}(\beta_1, \ldots, \beta_{n-2}) \equiv 0. \tag{81}$$

Here $i_{sN}(\beta_1, \ldots, \beta_{n-2})$, $s = 1, \ldots, n$, $(i_{1N}(\beta_1, \ldots, \beta_{n-2}) \equiv 0)$ are the components of the unit vector on the axis of the vector $\mathbf{r}_N = \{0, x_{2N}, \ldots, x_{nN}\}$ on $(n-2)$-dimensional sphere $\mathbf{S}^{n-2}\{\beta_1, \ldots, \beta_{n-2}\}$, defining the equation $\alpha = \pi/2$ as the equatorial section of corresponding $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}\{\alpha, \beta_1, \ldots, \beta_{n-2}\}$.

The system studied has the form

$$\xi' = Z_{n-1}, \tag{82}$$

$$Z_{n-1}' = (Z_1^2 + \ldots + Z_{n-2}^2)\frac{\cos\xi}{\sin\xi}, \tag{83}$$

$$Z_{n-2}' = -Z_{n-2}Z_{n-1}\frac{\cos\xi}{\sin\xi} - (Z_1^2 + \ldots + Z_{n-3}^2)\frac{\cos\xi}{\sin\xi}\frac{\cos\eta_1}{\sin\eta_1}, \tag{84}$$

$$Z_{n-3}' = -Z_{n-3}Z_{n-1}\frac{\cos\xi}{\sin\xi} + Z_{n-3}Z_{n-2}\frac{\cos\xi}{\sin\xi}\frac{\cos\eta_1}{\sin\eta_1} +$$

$$+(Z_1^2 + \ldots + Z_{n-4}^2)\frac{\cos\xi}{\sin\xi}\frac{1}{\sin\eta_1}\frac{\cos\eta_2}{\sin\eta_2}, \tag{85}$$

. . . .

$$Z_1' = -Z_1 \frac{\cos\xi}{\sin\xi}\left\{\sum_{s=1}^{n-2}(-1)^{s+1} Z_{n-s}\frac{\cos\eta_{s-1}}{\sin\eta_1\ldots\sin\eta_{s-1}}\right\}, \tag{86}$$

$$\eta_1' = -Z_{n-2}\frac{\cos\xi}{\sin\xi}, \tag{87}$$

$$\eta_2' = Z_{n-3}\frac{\cos\xi}{\sin\xi\sin\eta_1}, \tag{88}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\eta_{n-3}' = (-1)^{n+1} Z_2\frac{\cos\xi}{\sin\xi\sin\eta_1\ldots\sin\eta_{n-4}}, \tag{89}$$

$$\eta_{n-2}' = (-1)^{n} Z_1\frac{\cos\xi}{\sin\xi\sin\eta_1\ldots\sin\eta_{n-3}}. \tag{90}$$

The system (82)–(90) describes the motion of a rigid body in the absence of an external force field.

Theorem 7.2. *System (82)–(90) has n analytical independent first integrals as follows:*

$$\Phi_1(Z_{n-1},\ldots,Z_1;\xi,\eta_1,\ldots,\eta_{n-2}) = \sqrt{Z_1^2+\ldots+Z_{n-1}^2} = C_1 = const, \tag{91}$$

$$\Phi_2(Z_{n-1},\ldots,Z_1;\xi,\eta_1,\ldots,\eta_{n-2}) = \sqrt{Z_1^2+\ldots+Z_{n-2}^2}\sin\xi = C_2 = const, \tag{92}$$

$$\Phi_3(Z_{n-1},\ldots,Z_1;\xi,\eta_1,\ldots,\eta_{n-2}) = \sqrt{Z_1^2+\ldots+Z_{n-3}^2}\sin\xi\sin\eta_1 = C_3 = const, \tag{93}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Phi_{n-2}(Z_{n-1},\ldots,Z_1;\xi,\eta_1,\ldots,\eta_{n-2}) = \sqrt{Z_1^2+Z_2^2}\sin\xi\sin\eta_1\ldots\sin\eta_{n-4} = C_{n-2} = const, \tag{94}$$

$$\Phi_{n-1}(Z_{n-1},\ldots,Z_1;\xi,\eta_1,\ldots,\eta_{n-2}) = Z_1\sin\xi\sin\eta_1\ldots\sin\eta_{n-3} = C_{n-1} = const, \tag{95}$$

$$\Phi_{n}(Z_{n-1},\ldots,Z_1;\xi,\eta_1,\ldots,\eta_{n-2}) = C_n = const. \tag{96}$$

These first integrals (91)–(95) states that as the external force field is not present, it is preserved (in general, nonzero) $n-1$ components of the angular velocity tensor of a ("n-dimensional") rigid body, precisely

$$\omega_{r_1} \equiv \omega_{r_1}^0 = \text{const},\ \ldots,\ \omega_{r_{n-1}} \equiv \omega_{r_{n-1}}^0 = \text{const}. \tag{97}$$

In particular, the existence of the first integral (91) is explained by the equation

$$Z_1^2+\ldots+Z_{n-1}^2 = \frac{1}{n_0^2 v_\infty^2}[\omega_{r_1}^2+\ldots+\omega_{r_{n-1}}^2] \equiv C_1^2 = \text{const}. \tag{98}$$

The first integral (96) has the kinematic sense, "attaches" the equation on η_{n-2} and can be found from the following quadrature:

$$\frac{d\eta_{n-2}}{d\eta_{n-3}} = -\frac{Z_1}{Z_2}\frac{1}{\sin\eta_{n-3}}, \tag{99}$$

in this case, if we use the levels of the first integrals (94), (95) and obtain the equality

$$\frac{Z_1}{Z_2} = \pm\sqrt{\frac{C_{n-2}^2}{C_{n-1}^2}\sin^2\eta_{n-3}-1}, \tag{100}$$

then the quadrature (99) has the form

$$\eta_{n-2} = \pm \int \frac{du}{(1-u^2)\sqrt{\left(\frac{C_{n-2}^2}{C_{n-1}^2} - 1\right) - \frac{C_{n-2}^2}{C_{n-1}^2} u^2}}, \quad u = \cos\eta_{n-3}. \tag{101}$$

The calculation of its quadrature implies to the following form:

$$\eta_{n-2} + C_n = \pm \mathrm{arctg} \frac{\cos\eta_{n-3}}{\sqrt{\frac{C_{n-2}^2}{C_{n-1}^2}\sin^2\eta_{n-3} - 1}}, \quad C_n = \mathrm{const}, \tag{102}$$

that allows to obtain the first integral (96). Transforming the last equality, we have the following invariant relation:

$$\mathrm{tg}^2(\eta_{n-2} + C_n) = \frac{C_{n-1}^2}{(C_{n-2}^2 - C_{n-1}^2)\mathrm{tg}^2\eta_{n-3} - C_{n-1}^2}. \tag{103}$$

Now we rephrase the Theorem 7.2.

Theorem 7.3. *System (82)–(90) possesses n independent first integrals of the following form:*

$$\Psi_1(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = \frac{\Phi_1^2}{\Phi_2} = \frac{Z_1^2 + \ldots + Z_{n-1}^2}{\sqrt{Z_1^2 + \ldots + Z_{n-2}^2}\sin\xi} = C_1' = const, \tag{104}$$

$$\Psi_2(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = C_2' = const, \tag{105}$$

$$\Psi_3(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = \frac{\Phi_{n-2}}{\Phi_{n-1}} = \frac{\sqrt{Z_1^2 + Z_2^2}}{Z_1 \sin\eta_{n-3}} = C_3' = const, \tag{106}$$

. . . .

$$\Psi_{n-2}(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = \frac{\Phi_3}{\Phi_4} = \frac{\sqrt{Z_1^2 + \ldots + Z_{n-3}^2}}{\sqrt{Z_1^2 + \ldots + Z_{n-4}^2}\sin\eta_2} = C_{n-2}' = const, \tag{107}$$

$$\Psi_{n-1}(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = \frac{\Phi_2}{\Phi_3} = \frac{\sqrt{Z_1^2 + \ldots + Z_{n-2}^2}}{\sqrt{Z_1^2 + \ldots + Z_{n-3}^2}\sin\eta_1} = C_{n-1}' = const, \tag{108}$$

$$\Psi_n(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = C_n' = const. \tag{109}$$

The first integral (109) has also the kinematic sense and “attaches” the equation on η_{n-2}, and the functions Ψ_2, Ψ_n can be selected equal to Φ_2, Φ_n, respectively.

In the formulation of the Theorem 7.3 (unlike Theorem 7.2), the characteristics of smooth of the first integrals is absent. Precisely, where the denominators (or the numerators and denominators simultaneously) of the first integrals (104)–(109) are equal to zero, the integrals considered, as functions, are the singularities. Furthermore, its are not often, generally speaking, even the continuous functions.

By Theorem 7.3, the transformed set of the first integrals (104)–(109) of the system (82)–(90) (i.e., the system under the absence of a force field) still remains as the set of the first integrals of the system studied.

For the complete integration of system (82)–(90) of the order $2(n-1)$, in general, we need $2n-3$ independent first integrals. However, after the following change of variables

$$\begin{pmatrix} Z_{n-1} \\ Z_{n-2} \\ \dots \\ Z_2 \\ Z_1 \end{pmatrix} \to \begin{pmatrix} w_{n-1} \\ w_{n-2} \\ \dots \\ w_2 \\ w_1 \end{pmatrix},$$

$$w_{n-1} = -Z_{n-1},\ w_{n-2} = \sqrt{Z_1^2 + \ldots + Z_{n-2}^2},\ w_{n-3} = \frac{Z_2}{Z_1},\ w_{n-4} = -\frac{Z_3}{\sqrt{Z_1^2 + Z_2^2}},\ \ldots, \tag{110}$$
$$w_2 = -\frac{Z_{n-3}}{\sqrt{Z_1^2 + \ldots + Z_{n-4}^2}},\ w_1 = -\frac{Z_{n-2}}{\sqrt{Z_1^2 + \ldots + Z_{n-3}^2}},$$

the system (82)–(90) splits as follows:

$$\left.\begin{aligned} \xi' &= -w_{n-1}, \\ w'_{n-1} &= -w_{n-2}^2 \frac{\cos\xi}{\sin\xi}, \\ w'_{n-2} &= w_{n-2} w_{n-1} \frac{\cos\xi}{\sin\xi}, \end{aligned}\right\} \tag{111}$$

$$\left.\begin{aligned} w'_s &= d_s(w_{n-1}, \ldots, w_1; \xi, \eta_1, \ldots, \eta_{n-2}) \frac{1 + w_s^2}{w_s} \frac{\cos\eta_s}{\sin\eta_s}, \\ \eta'_s &= d_s(w_{n-1}, \ldots, w_1; \xi, \eta_1, \ldots, \eta_{n-2}),\ s = 1, \ldots, n-3, \end{aligned}\right\} \tag{112}$$

$$\eta'_{n-2} = d_{n-2}(w_{n-1}, \ldots, w_1; \xi, \eta_1, \ldots, \eta_{n-2}), \tag{113}$$

where

$$\begin{aligned} & d_1(w_{n-1}, \ldots, w_1; \xi, \eta_1, \ldots, \eta_{n-2}) = -Z_{n-2}(w_{n-1}, \ldots, w_1) \frac{\cos\xi}{\sin\xi}, \\ & d_2(w_{n-1}, \ldots, w_1; \xi, \eta_1, \ldots, \eta_{n-2}) = Z_{n-3}(w_{n-1}, \ldots, w_1) \frac{\cos\xi}{\sin\xi \sin\eta_1}, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & d_{n-2}(w_{n-1}, \ldots, w_1; \xi, \eta_1, \ldots, \eta_{n-2}) = (-1)^n Z_1(w_{n-1}, \ldots, w_1) \frac{\cos\xi}{\sin\xi \sin\eta_1 \ldots \sin\eta_{n-3}}, \end{aligned} \tag{114}$$

in our case

$$Z_k = Z_k(w_{n-1}, \ldots, w_1),\ k = 1, \ldots, n-2, \tag{115}$$

are the functions, by virtue of change (110).

The system (111)–(113) is studied on the tangent bundle

$$T_*\mathbf{S}^{n-1}\{(w_{n-1}, \ldots, w_1; \xi, \eta_1, \ldots, \eta_{n-2}) \in \mathbf{R}^{2(n-1)} :\ 0 \le \xi, \eta_1, \ldots, \eta_{n-3} \le \pi,\ \eta_{n-2} \bmod 2\pi\} \tag{116}$$

of the $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}\{(\xi, \eta_1, \ldots, \eta_{n-2}) \in \mathbf{R}^{n-1} :\ 0 \le \xi, \eta_1, \ldots, \eta_{n-3} \le \pi,\ \eta_{n-2} \bmod 2\pi\}$.

We see that the independent third-order subsystem (111) (which can be considered separately on its own three-dimensional manifold), $n-3$ independent second-order subsystems (112) (after the change of independent variable) can be substituted into the system (111)–(113) of the order $3+2(n-3)+1 = 2(n-1)$, and also Eq. (113) on η_{n-2} is separated (due to cyclicity of the variable η_{n-2}).

Thus, for the complete integration of the system (111)–(113), it suffices to specify two independent first integrals of system (111), one by one first integral of systems (112) (all $n-3$ pieces), and an additional first integral that "attaches" Eq. (113) (*i.e., only n*).

Remark 7.1. *We write the first integrals (104)–(109) in the variables $w_1, \ldots, w_{n-1}$ by virtue of (110). We have:*

$$\Theta_1(w_{n-1}, \ldots, w_1; \xi, \eta_1, \ldots, \eta_{n-2}) = \frac{w_{n-2}^2 + w_{n-1}^2}{w_{n-2}\sin\xi} = C_1'' = const, \tag{117}$$

$$\Theta_2(w_{n-1},\ldots,w_1;\xi,\eta_1,\ldots,\eta_{n-2}) = w_{n-2}\sin\xi = C_2'' = const, \tag{118}$$

$$\Theta_{s+2}(w_{n-1},\ldots,w_1;\xi,\eta_1,\ldots,\eta_{n-2}) = \frac{\sqrt{1+w_s^2}}{\sin\eta_s} = C_{s+2}'' = const,\ \ s=1,\ldots,n-3, \tag{119}$$

$$\Theta_n(w_{n-1},\ldots,w_1;\xi,\eta_1,\ldots,\eta_{n-2}) = C_n'' = const. \tag{120}$$

Thus, two independent first integrals (117), (118) are sufficient to integrate the system (111), the first integrals (119) (all $n-3$ pieces) are sufficient to integrate the independent first-order equations

$$\frac{dw_s}{d\eta_s} = \frac{1+w_s^2}{w_s}\frac{\cos\eta_s}{\sin\eta_s},\ \ s=1,\ldots,n-3, \tag{121}$$

that are equivalent to the systems (112) after the change of independent variable, and, finally, the first integral (120) is sufficient "to attach" Eq. (113). We have proved the following Theorem.

Theorem 7.4. *The system (82)–(90) of the order* $2(n-1)$ *possesses the sufficient number (n) of the independent first integrals.*

7.2.2 The system under the presence of a conservative force field

Now let us study the system (62)–(70) under assumption $b_* = 0$. In this case, we obtain the conservative system. Precisely, the coefficient $\sin\xi\cos\xi$ in Eq. (63) (unlike the system (82)–(90)) characterizes the presence of the force field. The system studied has the form

$$\xi' = Z_{n-1}, \tag{122}$$

$$Z_{n-1}' = -\sin\xi\cos\xi + (Z_1^2+\ldots+Z_{n-2}^2)\frac{\cos\xi}{\sin\xi}, \tag{123}$$

$$Z_{n-2}' = -Z_{n-2}Z_{n-1}\frac{\cos\xi}{\sin\xi} - (Z_1^2+\ldots+Z_{n-3}^2)\frac{\cos\xi}{\sin\xi}\frac{\cos\eta_1}{\sin\eta_1}, \tag{124}$$

$$Z_{n-3}' = -Z_{n-3}Z_{n-1}\frac{\cos\xi}{\sin\xi} + Z_{n-3}Z_{n-2}\frac{\cos\xi}{\sin\xi}\frac{\cos\eta_1}{\sin\eta_1} +$$
$$+(Z_1^2+\ldots+Z_{n-4}^2)\frac{\cos\xi}{\sin\xi}\frac{1}{\sin\eta_1}\frac{\cos\eta_2}{\sin\eta_2}, \tag{125}$$

....

$$Z_1' = -Z_1\frac{\cos\xi}{\sin\xi}\left\{\sum_{s=1}^{n-2}(-1)^{s+1}Z_{n-s}\frac{\cos\eta_{s-1}}{\sin\eta_1\ldots\sin\eta_{s-1}}\right\}, \tag{126}$$

$$\eta_1' = -Z_{n-2}\frac{\cos\xi}{\sin\xi}, \tag{127}$$

$$\eta_2' = Z_{n-3}\frac{\cos\xi}{\sin\xi\sin\eta_1}, \tag{128}$$

....

$$\eta_{n-3}' = (-1)^{n+1}Z_2\frac{\cos\xi}{\sin\xi\sin\eta_1\ldots\sin\eta_{n-4}}, \tag{129}$$

$$\eta_{n-2}' = (-1)^nZ_1\frac{\cos\xi}{\sin\xi\sin\eta_1\ldots\sin\eta_{n-3}}. \tag{130}$$

Thus, the system (122)–(130) describes the motion of a rigid body in a conservative field of external forces.

Theorem 7.5. *System (122)–(130) has n independent analytical first integrals as follows:*

$$\Phi_1(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = Z_1^2+\dots+Z_{n-1}^2+\sin^2\xi = C_1 = const, \tag{131}$$

$$\Phi_2(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = \sqrt{Z_1^2+\dots+Z_{n-2}^2}\sin\xi = C_2 = const, \tag{132}$$

$$\Phi_3(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = \sqrt{Z_1^2+\dots+Z_{n-3}^2}\sin\xi\sin\eta_1 = C_3 = const, \tag{133}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\Phi_{n-2}(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = \sqrt{Z_1^2+Z_2^2}\sin\xi\sin\eta_1\dots\sin\eta_{n-4} = C_{n-2} = const, \tag{134}$$

$$\Phi_{n-1}(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = Z_1\sin\xi\sin\eta_1\dots\sin\eta_{n-3} = C_{n-1} = const, \tag{135}$$

$$\Phi_n(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = C_n = const. \tag{136}$$

The first integral (131) is an integral of the total energy. The first integral (136) has the kinematic sense, "attaches" the equation on β_{n-2}, and was found above.

Now we rephrase the Theorem 7.5.

Theorem 7.6. *System (122)–(130) possesses n independent first integrals of the following form::*

$$\Psi_1(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = \frac{\Phi_1}{\Phi_2} = \frac{Z_1^2+\dots+Z_{n-1}^2+\sin^2\xi}{\sqrt{Z_1^2+\dots+Z_{n-2}^2}\sin\xi} = C_1' = const, \tag{137}$$

$$\Psi_2(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = C_2' = const, \tag{138}$$

$$\Psi_3(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = \frac{\Phi_{n-2}}{\Phi_{n-1}} = \frac{\sqrt{Z_1^2+Z_2^2}}{Z_1\sin\eta_{n-3}} = C_3' = const, \tag{139}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\Psi_{n-2}(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = \frac{\Phi_3}{\Phi_4} = \frac{\sqrt{Z_1^2+\dots+Z_{n-3}^2}}{\sqrt{Z_1^2+\dots+Z_{n-4}^2}\sin\eta_2} = C_{n-2}' = const, \tag{140}$$

$$\Psi_{n-1}(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = \frac{\Phi_2}{\Phi_3} = \frac{\sqrt{Z_1^2+\dots+Z_{n-2}^2}}{\sqrt{Z_1^2+\dots+Z_{n-3}^2}\sin\eta_1} = C_{n-1}' = const, \tag{141}$$

$$\Psi_n(Z_{n-1},\dots,Z_1;\xi,\eta_1,\dots,\eta_{n-2}) = C_n' = const. \tag{142}$$

The functions Ψ_2, Ψ_n can be selected equal to Φ_2, Φ_n, respectively.

In the formulation of the Theorem 7.6 (unlike Theorem 7.5), the characteristics of smooth of the first integrals is absent. Precisely, where the denominators (or the numerators and denominators simultaneously) of the first integrals (137)–(142) are equal to zero, the integrals considered, as functions, are the singularities. Furthermore, its are not often, generally speaking, even the continuous functions.

By Theorem 7.6, the transformed set of the first integrals (137)–(142) of the system (122)–(130) (i.e., the system under the presence of a conservative force field) still remains as the set of the first integrals of the system studied.

For the complete integration of system (122)–(130) of the order $2(n-1)$, in general, we need $2n-3$ independent first integrals. However, after the change of variables (110) the system (122)–(130) splits as follows:

$$\left.\begin{aligned} &\xi' = -w_{n-1},\\ &w_{n-1}' = \sin\xi\cos\xi - w_{n-2}^2\frac{\cos\xi}{\sin\xi},\\ &w_{n-2}' = w_{n-2}w_{n-1}\frac{\cos\xi}{\sin\xi}, \end{aligned}\right\} \tag{143}$$

$$\left.\begin{aligned} w_s' &= d_s(w_{n-1},\dots,w_1;\xi,\eta_1,\dots,\eta_{n-2})\frac{1+w_s^2}{w_s}\frac{\cos\eta_s}{\sin\eta_s},\\ \eta_s' &= d_s(w_{n-1},\dots,w_1;\xi,\eta_1,\dots,\eta_{n-2}),\ s=1,\dots,n-3,\end{aligned}\right\}\tag{144}$$

$$\eta_{n-2}' = d_{n-2}(w_{n-1},\dots,w_1;\xi,\eta_1,\dots,\eta_{n-2}),\tag{145}$$

where the conditions (114) hold.

The system (143)–(145) is studied on the tangent bundle (116) of the $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}\{(\xi,\eta_1,\dots,\eta_{n-2})\in\mathbf{R}^{n-1}:\ 0\le\xi,\eta_1,\dots,\eta_{n-3}\le\pi,\ \eta_{n-2} \bmod 2\pi\}$.

We see that the independent third-order subsystem (143) (which can be considered separately on its own three-dimensional manifold), $n-3$ independent second-order subsystems (144) (after the change of independent variable) can be substituted into the system (143)–(145) of the order $3+2(n-3)+1=2(n-1)$, and also Eq. (145) on η_{n-2} is separated (due to cyclicity of the variable η_{n-2}).

Thus, for the complete integration of the system (143)–(145), it suffices to specify two independent first integrals of system (143), one by one first integral of systems (144) (all $n-3$ pieces), and an additional first integral that "attaches" Eq. (145) (*i.e., only* n).

Remark 7.2. *We write the first integrals (137)–(142) in the variables* $w_1,\dots,w_{n-1}$ *by virtue of (110). We have:*

$$\Theta_1(w_{n-1},\dots,w_1;\xi,\eta_1,\dots,\eta_{n-2}) = \frac{w_{n-2}^2+w_{n-1}^2+\sin^2\xi}{w_{n-2}\sin\xi} = C_1'' = const,\tag{146}$$

$$\Theta_2(w_{n-1},\dots,w_1;\xi,\eta_1,\dots,\eta_{n-2}) = w_{n-2}\sin\xi = C_2'' = const,\tag{147}$$

$$\Theta_{s+2}(w_{n-1},\dots,w_1;\xi,\eta_1,\dots,\eta_{n-2}) = \frac{\sqrt{1+w_s^2}}{\sin\eta_s} = C_{s+2}'' = const,\ s=1,\dots,n-3,\tag{148}$$

$$\Theta_n(w_{n-1},\dots,w_1;\xi,\eta_1,\dots,\eta_{n-2}) = C_n'' = const.\tag{149}$$

Thus, two independent first integrals (146), (147) are sufficient to integrate the system (143), the first integrals (148) (all $n-3$ pieces) are sufficient to integrate the independent first-order equations

$$\frac{dw_s}{d\eta_s} = \frac{1+w_s^2}{w_s}\frac{\cos\eta_s}{\sin\eta_s},\ s=1,\dots,n-3,\tag{150}$$

that is equivalent to the systems (144) after the change of independent variable, and, finally, the first integral (149) is sufficient "to attach" Eq. (145). We have proved the following Theorem.

Theorem 7.7. *The system (122)–(130) of the order* $2(n-1)$ *possesses the sufficient number (n) of the independent first integrals.*

7.3 Complete list of the first integrals for any finite n

We turn now to the integration of the desired system (62)–(70) of the order $2(n-1)$ (without any simplifications, i.e., in the presence of all coefficients).

Similarly, for the complete integration of system (62)–(70) of the order $2(n-1)$, in general, we need $2n-3$ independent first integrals. However, after the change of variables (110) the system (62)–(70) splits as follows:

$$\left.\begin{aligned} \xi' &= -w_{n-1}-b_*\sin\xi,\\ w_{n-1}' &= \sin\xi\cos\xi - w_{n-2}^2\frac{\cos\xi}{\sin\xi},\\ w_{n-2}' &= w_{n-2}w_{n-1}\frac{\cos\xi}{\sin\xi},\end{aligned}\right\}\tag{151}$$

$$\left.\begin{aligned} w_s' &= d_s(w_{n-1},\dots,w_1;\xi,\eta_1,\dots,\eta_{n-2})\frac{1+w_s^2}{w_s}\frac{\cos\eta_s}{\sin\eta_s},\\ \eta_s' &= d_s(w_{n-1},\dots,w_1;\xi,\eta_1,\dots,\eta_{n-2}),\ \ s=1,\dots,n-3,\end{aligned}\right\} \tag{152}$$

$$\eta_{n-2}' = d_{n-2}(w_{n-1},\dots,w_1;\xi,\eta_1,\dots,\eta_{n-2}), \tag{153}$$

where the conditions (114) hold.

The system (151)–(153) is studied on the tangent bundle (116) of the $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}\{(\xi,\eta_1,\dots,\eta_{n-2})\in\mathbf{R}^{n-1}:\ 0\le\xi,\eta_1,\dots,\eta_{n-3}\le\pi,\ \eta_{n-2} \bmod 2\pi\}$.

We see that the independent third-order subsystem (151) (which can be considered separately on its own three-dimensional manifold), $n-3$ independent second-order subsystems (152) (after the change of independent variable) can be substituted into the system (151)–(153) of the order $3+2(n-3)+1=2(n-1)$, and also Eq. (153) on η_{n-2} is separated (due to cyclicity of the variable η_{n-2}).

Thus, for the complete integration of the system (151)–(153), it suffices to specify two independent first integrals of system (151), one by one first integral of systems (152) (all $n-3$ pieces), and an additional first integral that "attaches" Eq. (153) (*i.e., only n*).

First, we compare the third-order system (151) with the nonautonomous second-order system

$$\begin{aligned} \frac{dw_{n-1}}{d\xi} &= \frac{\sin\xi\cos\xi - w_{n-2}^2\cos\xi/\sin\xi}{-w_{n-1}-b_*\sin\xi},\\ \frac{dw_{n-2}}{d\xi} &= \frac{w_{n-2}w_{n-1}\cos\xi/\sin\xi}{-w_{n-1}-b_*\sin\xi}. \end{aligned} \tag{154}$$

Using the substitution $\tau=\sin\xi$, we rewrite system (154) in the algebraic form:

$$\begin{aligned} \frac{dw_{n-1}}{d\tau} &= \frac{\tau - w_{n-2}^2/\tau}{-w_{n-1}-b_*\tau},\\ \frac{dw_{n-2}}{d\tau} &= \frac{w_{n-2}w_{n-1}/\tau}{-w_{n-1}-b_*\tau}. \end{aligned} \tag{155}$$

Further, if we introduce the uniform variables by the formulas

$$w_{n-1}=u_2\tau,\ \ w_{n-2}=u_1\tau, \tag{156}$$

we reduce system (155) to the following form:

$$\begin{aligned} \tau\frac{du_2}{d\tau}+u_2 &= \frac{1-u_1^2}{-u_2-b_*},\\ \tau\frac{du_1}{d\tau}+u_1 &= \frac{u_1u_2}{-u_2-b_*}, \end{aligned} \tag{157}$$

which is equivalent to

$$\begin{aligned} \tau\frac{du_2}{d\tau} &= \frac{1-u_1^2+u_2^2-bu_2}{-u_2-b_*},\\ \tau\frac{du_1}{d\tau} &= \frac{2u_1u_2-bu_1}{-u_2-b_*}. \end{aligned} \tag{158}$$

We compare the second-order system (158) with the nonautonomous first-order equation

$$\frac{du_2}{du_1} = \frac{1-u_1^2+u_2^2+b_*u_2}{2u_1u_2+b_*u_1}, \tag{159}$$

which can be easily reduced to the exact differential equation

$$d\left(\frac{u_2^2+u_1^2+b_*u_2+1}{u_1}\right)=0. \tag{160}$$

Therefore, Eq. (159) has the following first integral:

$$\frac{u_2^2+u_1^2+b_*u_2+1}{u_1}=C_1=\text{const}, \tag{161}$$

which in the old variables has the form

$$\Theta_1(w_{n-1},w_{n-2};\xi)=\frac{w_{n-1}^2+w_{n-2}^2+b_*w_{n-1}\sin\xi+\sin^2\xi}{w_{n-2}\sin\xi}=C_1=\text{const}. \tag{162}$$

Remark 7.3. *We consider system (151) with variable dissipation with zero mean (see [11, 12]), which becomes conservative for $b_*=0$:*

$$\begin{aligned}
&\xi'=-w_{n-1},\\
&w_{n-1}'=\sin\xi\cos\xi-w_{n-2}^2\frac{\cos\xi}{\sin\xi},\\
&w_{n-2}'=w_{n-2}w_{n-1}\frac{\cos\xi}{\sin\xi}.
\end{aligned} \tag{163}$$

It has two analytical first integrals of the form

$$w_{n-1}^2+w_{n-2}^2+\sin^2\xi=C_1^*=\mathit{const}, \tag{164}$$

$$w_{n-2}\sin\xi=C_2^*=\mathit{const}. \tag{165}$$

It is obvious that the ratio of the first integrals (164), (165) is also a first integral of system (163). However, for $b_*\neq 0$ both functions

$$w_{n-1}^2+w_{n-2}^2+b_*w_{n-1}\sin\xi+\sin^2\xi \tag{166}$$

and (165) are not first integrals of system (151), but their ratio (i.e., the ratio of the functions (166) and (165)) is a first integral of system (151) for any b_*.

Later on, we find the obvious form of the additional first integral of the third-order system (151). For this, at the beginning, we transform the invariant relation (161) for $u_1\neq 0$ as follows:

$$\left(u_2+\frac{b_*}{2}\right)^2+\left(u_1-\frac{C_1}{2}\right)^2=\frac{b_*^2+C_1^2}{4}-1. \tag{167}$$

We see that the parameters of the given invariant relation must satisfy the condition

$$b_*^2+C_1^2-4\geq 0, \tag{168}$$

and the phase space of system (151) is stratified into a family of surfaces defined by Eq. (167) in the coordinates u_1,u_2.

Thus, by virtue of relation (161), the first equation of system (158) has the form

$$\tau\frac{du_2}{d\tau}=\frac{2(1+b_*u_2+u_2^2)-C_1U_1(C_1,u_2)}{-u_2-b_*}, \tag{169}$$

where

$$U_1(C_1,u_2)=\frac{1}{2}\{C_1\pm\sqrt{C_1^2-4(u_2^2+b_*u_2+1)}\},\tag{170}$$

and the integration constant C_1 is chosen from condition (168).

Therefore, the quadrature for the search of an additional first integral of system (151) has the form

$$\int\frac{d\tau}{\tau}=\int\frac{(-b_*-u_2)du_2}{2(1+b_*u_2+u_2^2)-C_1\{C_1\pm\sqrt{C_1^2-4(u_2^2+b_*u_2+1)}\}/2}.\tag{171}$$

Obviously, the left-hand side up to an additive constant is equal to

$$\ln|\sin\xi|.\tag{172}$$

If

$$u_2+\frac{b_*}{2}=r_1,\ \ b_1^2=b_*^2+C_1^2-4,\tag{173}$$

then the right-hand side of Eq. (171) has the form

$$-\frac{1}{4}\int\frac{d(b_1^2-4r_1^2)}{(b_1^2-4r_1^2)\pm C_1\sqrt{b_1^2-4r_1^2}}-b\int\frac{dr_1}{(b_1^2-4r_1^2)\pm C_1\sqrt{b_1^2-4r_1^2}}=$$

$$=-\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2-4r_1^2}}{C_1}\pm 1\right|\pm\frac{b}{2}I_1,\tag{174}$$

where

$$I_1=\int\frac{dr_3}{\sqrt{b_1^2-r_3^2}(r_3\pm C_1)},\ \ r_3=\sqrt{b_1^2-4r_1^2}.\tag{175}$$

In the calculation of integral (175), the following three cases are possible.

I. $b_*>2$.

$$I_1=-\frac{1}{2\sqrt{b_*^2-4}}\ln\left|\frac{\sqrt{b_*^2-4}+\sqrt{b_1^2-r_3^2}}{r_3\pm C_1}\pm\frac{C_1}{\sqrt{b_*^2-4}}\right|+$$

$$+\frac{1}{2\sqrt{b_*^2-4}}\ln\left|\frac{\sqrt{b_*^2-4}-\sqrt{b_1^2-r_3^2}}{r_3\pm C_1}\mp\frac{C_1}{\sqrt{b_*^2-4}}\right|+\text{const}.\tag{176}$$

II. $b_*<2$.

$$I_1=\frac{1}{\sqrt{4-b_*^2}}\arcsin\frac{\pm C_1r_3+b_1^2}{b_1(r_3\pm C_1)}+\text{const}.\tag{177}$$

III. $b_*=2$.

$$I_1=\mp\frac{\sqrt{b_1^2-r_3^2}}{C_1(r_3\pm C_1)}+\text{const}.\tag{178}$$

When we return to the variable

$$r_1=\frac{w_{n-1}}{\sin\xi}+\frac{b_*}{2},\tag{179}$$

we obtain the final form for the value I_1:

I. $b_*>2$.

$$I_1=-\frac{1}{2\sqrt{b_*^2-4}}\ln\left|\frac{\sqrt{b_*^2-4}\pm 2r_1}{\sqrt{b_1^2-4r_1^2}\pm C_1}\pm\frac{C_1}{\sqrt{b_*^2-4}}\right|+$$

$$+\frac{1}{2\sqrt{b_*^2-4}}\ln\left|\frac{\sqrt{b_*^2-4}\mp 2r_1}{\sqrt{b_1^2-4r_1^2}\pm C_1}\mp\frac{C_1}{\sqrt{b_*^2-4}}\right|+\text{const}. \tag{180}$$

II. $b_* < 2$.

$$I_1=\frac{1}{\sqrt{4-b_*^2}}\arcsin\frac{\pm C_1\sqrt{b_1^2-4r_1^2}+b_1^2}{b_1(\sqrt{b_1^2-4r_1^2}\pm C_1)}+\text{const}. \tag{181}$$

III. $b_* = 2$.

$$I_1=\mp\frac{2r_1}{C_1(\sqrt{b_1^2-4r_1^2}\pm C_1)}+\text{const}. \tag{182}$$

Thus, we have found an additional first integral for the third-order system (151), i.e., we have a complete set of first integrals that are transcendental functions of the phase variables.

Remark 7.4. *In the expression of the found first integral, we must formally substitute the left-hand side of the first integral (161) instead of* C_1*.*

Then the obtained additional first integral has the following structure:

$$\Theta_2(w_{n-1},w_{n-2};\xi)=G_2\left(\sin\xi,\frac{w_{n-1}}{\sin\xi},\frac{w_{n-2}}{\sin\xi}\right)=C_2=\text{const}. \tag{183}$$

Thus, we have found two first integrals (162), (183) of the independent third-order system (151). For its complete integrability, it suffices to find one by one first integral for the systems (152) (all $n-3$ pieces), and an additional first integral that "attaches" Eq. (153).

Indeed, the desired first integrals coincide with the first integrals (148), (149), precisely:

$$\Theta_{s+2}(w_s;\eta_s)=\frac{\sqrt{1+w_s^2}}{\sin\eta_s}=C''_{s+2}=\text{const},\ \ s=1,\dots,n-3, \tag{184}$$

$$\Theta_n(w_{n-3},w_{n-4};\eta_{n-4},\eta_{n-3},\eta_{n-2})=\eta_{n-2}\pm\operatorname{arctg}\frac{C_{n-1}\cos\eta_{n-3}}{\sqrt{C_{n-2}^2\sin^2\eta_{n-3}-C_{n-1}^2}}=C''_n=\text{const}, \tag{185}$$

in this case, in the left-hand side of Eq. (185), we must substitute instead of C_{n-2}, C_{n-1} the first integrals (184) for $s=n-4, n-3$.

Theorem 7.8. *The system (151)–(153) of the order* $2(n-1)$ *possesses the sufficient number (n) of the independent first integrals (162), (183), (184), (185).*

Therefore, in the considered case, the system of dynamical equations (62)–(70) has n first integrals expressing by relations (162), (183), (184), (185), which are the transcendental functions of its phase variables (in the sense of the complex analysis) and are expressed as a finite combination of elementary functions (in this case, we use the expressions (171)–(182)).

Theorem 7.9. *Three sets of relations (26), (40), (50) under conditions (31)–(33), (53), (57) possess n the first integrals (the complete set), which are the transcendental function (in the sense of complex analysis) and are expressed as a finite combination of elementary functions.*

8 Case where the moment of nonconservative forces depends on the angular velocity

8.1 Dependence on the angular velocity

This section is devoted to dynamics of the multi-dimensional rigid body in the multi-dimensional space $\mathbf{E}^n$. Since this subsection is devoted to the study of the case of the motion where the moment of forces depends on the angular velocity tensor, we introduce this dependence in the general case.

Let $x = (x_{1N}, \ldots, x_{nN})$ be the coordinates of the point N of application of a nonconservative force (interaction with a medium) on the $(n-1)$-dimensional disk $\mathcal{D}^{n-1}$, and $Q = (Q_1, \ldots, Q_n)$ be the components independent of the angular velocity. We introduce only the linear dependence of the functions $(x_{1N}, \ldots, x_{nN})$ on the angular velocity tensor Ω since the introduction of this dependence itself is not a priori obvious.

Thus, we accept the following dependence:

$$x = Q + R, \tag{186}$$

where $R = (R_1, \ldots, R_n)$ is a vector-valued function containing the angular velocity tensor Ω. Here, the dependence of the function R on the angular velocity tensor is gyroscopic:

$$R = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} = -\frac{1}{v_D}\Omega \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}, \tag{187}$$

where $(h_1, \ldots, h_n)$ are certain positive parameters.

Now, for our problem, since $x_{1N} = x_N \equiv 0$, we have

$$x_{2N} = Q_2 - h_1 \frac{\omega_{r_{n-1}}}{v_D}, \ x_{3N} = Q_3 + h_1 \frac{\omega_{r_{n-2}}}{v_D}, \ \ldots, \ x_{nN} = Q_n + (-1)^{n+1} h_1 \frac{\omega_{r_1}}{v}. \tag{188}$$

Thus, the function $\mathbf{r}_N$ is selected in the following form (the disk $\mathcal{D}^{n-1}$ is defined by the equation $x_{1N} \equiv 0$):

$$\mathbf{r}_N = \begin{pmatrix} 0 \\ x_{2N} \\ \vdots \\ x_{nN} \end{pmatrix} = R(\alpha)\mathbf{i}_N - \frac{1}{v_D}\Omega h, \tag{189}$$

where

$$\mathbf{i}_N = \mathbf{i}_v\left(\frac{\pi}{2}, \beta_1, \ldots, \beta_{n-2}\right), \ h = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}, \ \Omega \in \mathrm{so}(n) \tag{190}$$

(see (6), (39)).

In our case

$$\mathbf{i}_N = \begin{pmatrix} 0 \\ \cos\beta_1 \\ \sin\beta_1 \cos\beta_2 \\ \ldots \\ \sin\beta_1 \ldots \sin\beta_{n-3}\cos\beta_{n-2} \\ \sin\beta_1 \ldots \sin\beta_{n-2} \end{pmatrix}. \tag{191}$$

Thus, the following relations

$$x_{2N} = R(\alpha)\cos\beta_1 - h_1\frac{\omega_{r_{n-1}}}{v_D},\ x_{3N} = R(\alpha)\sin\beta_1\cos\beta_2 + h_1\frac{\omega_{r_{n-2}}}{v_D},\ \ldots,$$
$$x_{n-1,N} = R(\alpha)\sin\beta_1\ldots\sin\beta_{n-3}\cos\beta_{n-2} + (-1)^n h_1\frac{\omega_{r_2}}{v}, \tag{192}$$
$$x_{nN} = R(\alpha)\sin\beta_1\ldots\sin\beta_{n-2} + (-1)^{n+1} h_1\frac{\omega_{r_1}}{v},$$

hold, which show that an additional dependence of the damping (or accelerating in some domains of the phase space) moment of the nonconservative forces is also present in the system considered (i.e., the moment depends on the angular velocity tensor).

And so, for the construction of the force field, we use the pair of dynamical functions $R(\alpha), s(\alpha)$; the information about them is of a qualitative nature. Similarly to the choice of the Chaplygin analytical functions, we take the dynamical functions s and R as follows:

$$R(\alpha) = A\sin\alpha,\ s(\alpha) = B\cos\alpha,\ A, B > 0. \tag{193}$$

8.2 Reduced systems

Theorem 8.1. *The simultaneous equations (26), (40), (50) under conditions (31)–(33), (189), (193) can be reduced to the dynamical system on the tangent bundle (5) of the $(n-1)$-dimensional sphere (4).*

Indeed, if we introduce the dimensionless parameters and the differentiation by the formulas

$$b_* = ln_0,\ n_0^2 = \frac{AB}{(n-2)I_2},\ H_{1*} = \frac{h_1 B}{(n-2)I_2 n_0},\ <\cdot>= n_0 v_\infty <'>, \tag{194}$$

then the obtained equations have the following form:

$$
\begin{aligned}
&\xi'' + (b_* - H_{1*})\xi'\cos\xi + \sin\xi\cos\xi - \\
&- \left[\eta_1'^2 + \eta_2'^2\sin^2\eta_1 + \eta_3'^2\sin^2\eta_1\sin^2\eta_2 + \ldots + \eta_{n-2}'^2\sin^2\eta_1\ldots\sin^2\eta_{n-3}\right]\frac{\sin\xi}{\cos\xi} = 0,\\
&\eta_1'' + (b_* - H_{1*})\eta_1'\cos\xi + \xi'\eta_1'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} - \\
&- \left[\eta_2'^2 + \eta_3'^2\sin^2\eta_2 + \eta_4'^2\sin^2\eta_2\sin^2\eta_3 + \ldots + \eta_{n-2}'^2\sin^2\eta_2\ldots\sin^2\eta_{n-3}\right]\sin\eta_1\cos\eta_1 = 0,\\
&\eta_2'' + (b_* - H_{1*})\eta_2'\cos\xi + \xi'\eta_2'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\
&+ 2\eta_1'\eta_2'\frac{\cos\eta_1}{\sin\eta_1} - \\
&- \left[\eta_3'^2 + \eta_4'^2\sin^2\eta_3 + \eta_5'^2\sin^2\eta_3\sin^2\eta_4 + \ldots + \eta_{n-2}'^2\sin^2\eta_3\ldots\sin^2\eta_{n-3}\right]\sin\eta_2\cos\eta_2 = 0,\\
&\eta_3'' + (b_* - H_{1*})\eta_3'\cos\xi + \xi'\eta_3'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\
&+ 2\eta_1'\eta_3'\frac{\cos\eta_1}{\sin\eta_1} + 2\eta_2'\eta_3'\frac{\cos\eta_2}{\sin\eta_2} - \\
&- \left[\eta_4'^2 + \eta_5'^2\sin^2\eta_4 + \eta_6'^2\sin^2\eta_4\sin^2\eta_5 + \ldots + \eta_{n-2}'^2\sin^2\eta_4\ldots\sin^2\eta_{n-3}\right]\sin\eta_3\cos\eta_3 = 0, \qquad (195)\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\eta_{n-4}'' + (b_* - H_{1*})\eta_{n-4}'\cos\xi + \xi'\eta_{n-4}'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\
&+ 2\eta_1'\eta_{n-4}'\frac{\cos\eta_1}{\sin\eta_1} + \ldots + 2\eta_{n-5}'\eta_{n-4}'\frac{\cos\eta_{n-5}}{\sin\eta_{n-5}} - \\
&- \left[\eta_{n-3}'^2 + \eta_{n-2}'^2\sin^2\eta_{n-3}\right]\sin\eta_{n-4}\cos\eta_{n-4} = 0,\\
&\eta_{n-3}'' + (b_* - H_{1*})\eta_{n-3}'\cos\xi + \xi'\eta_{n-3}'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\
&+ 2\eta_1'\eta_{n-3}'\frac{\cos\eta_1}{\sin\eta_1} + \ldots + 2\eta_{n-4}'\eta_{n-3}'\frac{\cos\eta_{n-4}}{\sin\eta_{n-4}} - \\
&- \eta_{n-2}'^2\sin\eta_{n-3}\cos\eta_{n-3} = 0,\\
&\eta_{n-2}'' + (b_* - H_{1*})\eta_{n-2}'\cos\xi + \xi'\eta_{n-2}'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\
&+ 2\eta_1'\eta_{n-2}'\frac{\cos\eta_1}{\sin\eta_1} + \ldots + 2\eta_{n-3}'\eta_{n-2}'\frac{\cos\eta_{n-3}}{\sin\eta_{n-3}} = 0,\ b_* > 0,\ H_{1*} > 0.
\end{aligned}
$$

In particular, for $n = 5$ we have:

$$
\begin{aligned}
&\xi'' + (b_* - H_{1*})\xi'\cos\xi + \sin\xi\cos\xi - \left[\eta_1'^2 + \eta_2'^2\sin^2\eta_1 + \eta_3'^2\sin^2\eta_1\sin^2\eta_2\right]\frac{\sin\xi}{\cos\xi} = 0,\\
&\eta_1'' + (b_* - H_{1*})\eta_1'\cos\xi + \xi'\eta_1'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} - \left[\eta_2'^2 + \eta_3'^2\sin^2\eta_2\right]\sin\eta_1\cos\eta_1 = 0,\\
&\eta_2'' + (b_* - H_{1*})\eta_2'\cos\xi + \xi'\eta_2'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + 2\eta_1'\eta_2'\frac{\cos\eta_1}{\sin\eta_1} - \eta_3'^2\sin\eta_2\cos\eta_2 = 0, \qquad (196)\\
&\eta_3'' + (b_* - H_{1*})\eta_3'\cos\xi + \xi'\eta_3'\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + 2\eta_1'\eta_3'\frac{\cos\eta_1}{\sin\eta_1} + 2\eta_2'\eta_3'\frac{\cos\eta_2}{\sin\eta_2} = 0,\\
&b_* > 0,\ H_{1*} > 0.
\end{aligned}
$$

After the transition from the variables z (about the variables z see (48)) to the intermediate dimensionless variables w

$$z_k = n_0 v_\infty (1 + b_* H_{1*}) Z_k,\ \ k = 1, \dots, n-2,\ \ z_{n-1} = n_0 v_\infty (1 + b_* H_{1*}) Z_{n-1} - n_0 v_\infty b_* \sin\xi, \quad (197)$$

system (195) is equivalent to the system

$$\xi' = (1 + b_* H_{1*})\, Z_{n-1} - b_* \sin\xi, \quad (198)$$

$$Z'_{n-1} = -\sin\xi\cos\xi +$$

$$+ (1 + b_* H_{1*})\, (Z_1^2 + \ldots + Z_{n-2}^2) \frac{\cos\xi}{\sin\xi} + H_{1*} Z_{n-1} \cos\xi, \quad (199)$$

$$Z'_{n-2} = -(1 + b_* H_{1*})\, Z_{n-2} Z_{n-1} \frac{\cos\xi}{\sin\xi} -$$

$$-(1 + b_* H_{1*})\, (Z_1^2 + \ldots + Z_{n-3}^2) \frac{\cos\xi}{\sin\xi} \frac{\cos\eta_1}{\sin\eta_1} + H_{1*} Z_{n-2} \cos\xi, \quad (200)$$

$$Z'_{n-3} = -(1 + b_* H_{1*})\, Z_{n-3} Z_{n-1} \frac{\cos\xi}{\sin\xi} + (1 + b_* H_{1*})\, Z_{n-3} Z_{n-2} \frac{\cos\xi}{\sin\xi} \frac{\cos\eta_1}{\sin\eta_1} +$$

$$+ (1 + b_* H_{1*})\, (Z_1^2 + \ldots + Z_{n-4}^2) \frac{\cos\xi}{\sin\xi} \frac{1}{\sin\eta_1} \frac{\cos\eta_2}{\sin\eta_2} + H_{1*} Z_{n-3} \cos\xi, \quad (201)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$Z'_1 = -(1 + b_* H_{1*})\, Z_1 \frac{\cos\xi}{\sin\xi} \left\{ \sum_{s=1}^{n-2} (-1)^{s+1} Z_{n-s} \frac{\cos\eta_{s-1}}{\sin\eta_1 \ldots \sin\eta_{s-1}} \right\} +$$

$$+ H_{1*} Z_1 \cos\xi, \quad (202)$$

$$\eta'_1 = -(1 + b_* H_{1*})\, Z_{n-2} \frac{\cos\xi}{\sin\xi}, \quad (203)$$

$$\eta'_2 = (1 + b_* H_{1*})\, Z_{n-3} \frac{\cos\xi}{\sin\xi \sin\eta_1}, \quad (204)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\eta'_{n-3} = (-1)^{n+1} (1 + b_* H_{1*})\, Z_2 \frac{\cos\xi}{\sin\xi \sin\eta_1 \ldots \sin\eta_{n-4}}, \quad (205)$$

$$\eta'_{n-2} = (-1)^{n} (1 + b_* H_{1*})\, Z_1 \frac{\cos\xi}{\sin\xi \sin\eta_1 \ldots \sin\eta_{n-3}}, \quad (206)$$

on the tangent bundle

$$T_* \mathbf{S}^{n-1} \{ (Z_{n-1}, \ldots, Z_1; \xi, \eta_1, \ldots, \eta_{n-2}) \in \mathbf{R}^{2(n-1)} :\ 0 \le \xi, \eta_1, \ldots, \eta_{n-3} \le \pi,\ \eta_{n-2} \bmod 2\pi \} \quad (207)$$

of the $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}\{(\xi, \eta_1, \ldots, \eta_{n-2}) \in \mathbf{R}^{n-1} :\ 0 \le \xi, \eta_1, \ldots, \eta_{n-3} \le \pi,\ \eta_{n-2} \bmod 2\pi\}$.

We see that the independent subsystem (198)–(206) of the order $2(n-1)$ (due to cyclicity of the variable η_{n-2}) can be substituted into the system (198)–(205) of the order $2(n-1)-1$ and can be considered separately on its own $(2n-3)$-dimensional manifold.

In particular, for $n = 5$ we obtain the following eighth-order system:

$$\xi' = (1 + b_* H_{1*})\, Z_4 - b_* \sin\xi, \quad (208)$$

$$Z'_4 = -\sin\xi\cos\xi +$$

$$+ (1 + b_* H_{1*}) (Z_1^2 + Z_2^2 + Z_3^2) \frac{\cos\xi}{\sin\xi} + H_{1*} Z_4 \cos\xi, \tag{209}$$

$$Z_3' = -(1 + b_* H_{1*}) Z_3 Z_4 \frac{\cos\xi}{\sin\xi} -$$

$$-(1 + b_* H_{1*}) (Z_1^2 + Z_2^2) \frac{\cos\xi}{\sin\xi} \frac{\cos\eta_1}{\sin\eta_1} + H_{1*} Z_3 \cos\xi, \tag{210}$$

$$Z_2' = -(1 + b_* H_{1*}) Z_2 Z_4 \frac{\cos\xi}{\sin\xi} + (1 + b_* H_{1*}) Z_2 Z_3 \frac{\cos\xi}{\sin\xi} \frac{\cos\eta_1}{\sin\eta_1} +$$

$$+ (1 + b_* H_{1*}) Z_1^2 \frac{\cos\xi}{\sin\xi} \frac{1}{\sin\eta_1} \frac{\cos\eta_2}{\sin\eta_2} + H_{1*} Z_2 \cos\xi, \tag{211}$$

$$Z_1' = -(1 + b_* H_{1*}) Z_1 Z_4 \frac{\cos\xi}{\sin\xi} + (1 + b_* H_{1*}) Z_1 Z_3 \frac{\cos\xi}{\sin\xi} \frac{\cos\eta_1}{\sin\eta_1} -$$

$$-(1 + b_* H_{1*}) Z_1 Z_2 \frac{\cos\xi}{\sin\xi} \frac{1}{\sin\eta_1} \frac{\cos\eta_2}{\sin\eta_2} + H_{1*} Z_1 \cos\xi, \tag{212}$$

$$\eta_1' = -(1 + b_* H_{1*}) Z_3 \frac{\cos\xi}{\sin\xi}, \tag{213}$$

$$\eta_2' = (1 + b_* H_{1*}) Z_2 \frac{\cos\xi}{\sin\xi \sin\eta_1}, \tag{214}$$

$$\eta_3' = -(1 + b_* H_{1*}) Z_1 \frac{\cos\xi}{\sin\xi \sin\eta_1 \sin\eta_2}, \tag{215}$$

on the tangent bundle

$$T_* \mathbf{S}^4 \{(Z_4, Z_3, Z_2, Z_1; \xi, \eta_1, \eta_2, \eta_3) \in \mathbf{R}^8 : \ 0 \le \xi, \eta_1, \eta_2 \le \pi, \ \eta_3 \bmod 2\pi\} \tag{216}$$

of the four-dimensional sphere $\mathbf{S}^4\{(\xi, \eta_1, \eta_2, \eta_3) \in \mathbf{R}^4 : \ 0 \le \xi, \eta_1, \eta_2 \le \pi, \ \eta_3 \bmod 2\pi\}$.

We see that the independent eighth-order subsystem (208)–(215) (due to cyclicity of the variable η_3) can be substituted into the seventh-order system (208)–(214) and can be considered separately on its own seven-dimensional manifold.

8.3 Complete list of the first integrals for any finite n

We turn now to the integration of the desired system (198)–(206) of the order $2(n-1)$ (without any simplifications, i.e., in the presence of all coefficients).

Similarly, for the complete integration of system (198)–(206) of the order $2(n-1)$, in general, we need $2n-3$ independent first integrals. However, after the change of variables

$$\begin{pmatrix} Z_{n-1} \\ Z_{n-2} \\ \dots \\ Z_2 \\ Z_1 \end{pmatrix} \to \begin{pmatrix} w_{n-1} \\ w_{n-2} \\ \dots \\ w_2 \\ w_1 \end{pmatrix},$$

$$w_{n-1} = -Z_{n-1}, \ w_{n-2} = \sqrt{Z_1^2 + \ldots + Z_{n-2}^2}, \ w_{n-3} = \frac{Z_2}{Z_1}, \ w_{n-4} = -\frac{Z_3}{\sqrt{Z_1^2 + Z_2^2}}, \ \ldots, \tag{217}$$

$$w_2 = -\frac{Z_{n-3}}{\sqrt{Z_1^2 + \ldots + Z_{n-4}^2}}, \ w_1 = -\frac{Z_{n-2}}{\sqrt{Z_1^2 + \ldots + Z_{n-3}^2}},$$

the system (198)–(206) splits as follows:

$$\left.\begin{aligned}
&\xi' = -(1+b_*H_{1*})w_{n-1} - b_*\sin\xi,\\
&w'_{n-1} = \sin\xi\cos\xi - (1+b_*H_{1*})w_{n-2}^2\frac{\cos\xi}{\sin\xi} + H_{1*}w_{n-1}\cos\xi,\\
&w'_{n-2} = (1+b_*H_{1*})w_{n-2}w_{n-1}\frac{\cos\xi}{\sin\xi} + H_{1*}w_{n-2}\cos\xi,
\end{aligned}\right\} \tag{218}$$

$$\left.\begin{aligned}
&w'_s = d_s(w_{n-1},\ldots,w_1;\xi,\eta_1,\ldots,\eta_{n-2})\frac{1+w_s^2}{w_s}\frac{\cos\eta_s}{\sin\eta_s},\\
&\eta'_s = d_s(w_{n-1},\ldots,w_1;\xi,\eta_1,\ldots,\eta_{n-2}),\ \ s=1,\ldots,n-3,
\end{aligned}\right\} \tag{219}$$

$$\eta'_{n-2} = d_{n-2}(w_{n-1},\ldots,w_1;\xi,\eta_1,\ldots,\eta_{n-2}), \tag{220}$$

where

$$\begin{aligned}
&d_1(w_{n-1},\ldots,w_1;\xi,\eta_1,\ldots,\eta_{n-2}) = -(1+b_*H_{1*})Z_{n-2}(w_{n-1},\ldots,w_1)\frac{\cos\xi}{\sin\xi},\\
&d_2(w_{n-1},\ldots,w_1;\xi,\eta_1,\ldots,\eta_{n-2}) = (1+b_*H_{1*})Z_{n-3}(w_{n-1},\ldots,w_1)\frac{\cos\xi}{\sin\xi\sin\eta_1},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&d_{n-2}(w_{n-1},\ldots,w_1;\xi,\eta_1,\ldots,\eta_{n-2}) = (-1)^n(1+b_*H_{1*})Z_1(w_{n-1},\ldots,w_1)\frac{\cos\xi}{\sin\xi\sin\eta_1\ldots\sin\eta_{n-3}},
\end{aligned} \tag{221}$$

in this case

$$Z_k = Z_k(w_{n-1},\ldots,w_1),\ \ k=1,\ldots,n-2, \tag{222}$$

are the functions by virtue of change (217).

In particular, for $n=5$ we obtain the following eighth-order system:

$$\left.\begin{aligned}
&\xi' = -(1+b_*H_{1*})w_4 - b_*\sin\xi,\\
&w'_4 = \sin\xi\cos\xi - (1+b_*H_{1*})w_3^2\frac{\cos\xi}{\sin\xi} + H_{1*}w_4\cos\xi,\\
&w'_3 = (1+b_*H_{1*})w_3w_4\frac{\cos\xi}{\sin\xi} + H_{1*}w_3\cos\xi,
\end{aligned}\right\} \tag{223}$$

$$\left.\begin{aligned}
&w'_2 = d_2(w_4,w_3,w_2,w_1;\xi,\eta_1,\eta_2,\eta_3)\frac{1+w_2^2}{w_2}\frac{\cos\eta_2}{\sin\eta_2},\\
&\eta'_2 = d_2(w_4,w_3,w_2,w_1;\xi,\eta_1,\eta_2,\eta_3),
\end{aligned}\right\} \tag{224}$$

$$\left.\begin{aligned}
&w'_1 = d_1(w_4,w_3,w_2,w_1;\xi,\eta_1,\eta_2,\eta_3)\frac{1+w_1^2}{w_1}\frac{\cos\eta_1}{\sin\eta_1},\\
&\eta'_1 = d_1(w_4,w_3,w_2,w_1;\xi,\eta_1,\eta_2,\eta_3),
\end{aligned}\right\} \tag{225}$$

$$\eta'_3 = d_3(w_4,w_3,w_2,w_1;\xi,\eta_1,\eta_2,\eta_3), \tag{226}$$

where

$$
\begin{aligned}
&d_1(w_4,w_3,w_2,w_1;\xi,\eta_1,\eta_2,\eta_3) = -Z_3(w_4,w_3,w_2,w_1)\frac{\cos\xi}{\sin\xi} = \\
&= \mp\frac{w_1w_3}{\sqrt{1+w_1^2}}\frac{\cos\xi}{\sin\xi},\\
&d_2(w_4,w_3,w_2,w_1;\xi,\eta_1,\eta_2,\eta_3) = Z_2(w_4,w_3,w_2,w_1)\frac{\cos\xi}{\sin\xi\sin\eta_1} = \\
&= \pm\frac{w_2w_3}{\sqrt{1+w_1^2}\sqrt{1+w_2^2}}\frac{\cos\xi}{\sin\xi\sin\eta_1},\\
&d_3(w_4,w_3,w_2,w_1;\xi,\eta_1,\eta_2,\eta_3) = -Z_1(w_4,w_3,w_2,w_1)\frac{\cos\xi}{\sin\xi\sin\eta_1\sin\eta_2} = \\
&= \mp\frac{w_3}{\sqrt{1+w_1^2}\sqrt{1+w_2^2}}\frac{\cos\xi}{\sin\xi\sin\eta_1\sin\eta_2},
\end{aligned}
\tag{227}
$$

in this case

$$
Z_k = Z_k(w_4,w_3,w_2,w_1),\ \ k=1,2,3, \tag{228}
$$

are the functions by virtue of change (217).

The system (218)–(220) is studied on the tangent bundle

$$
T_*\mathbf{S}^{n-1}\{(w_{n-1},\ldots,w_1;\xi,\eta_1,\ldots,\eta_{n-2})\in\mathbf{R}^{2(n-1)}:\ 0\le\xi,\eta_1,\ldots,\eta_{n-3}\le\pi,\ \eta_{n-2} \bmod 2\pi\} \tag{229}
$$

of the $(n-1)$-dimensional sphere $\mathbf{S}^{n-1}\{(\xi,\eta_1,\ldots,\eta_{n-2})\in\mathbf{R}^{n-1}:\ 0\le\xi,\eta_1,\ldots,\eta_{n-3}\le\pi,\ \eta_{n-2} \bmod 2\pi\}$.

In particular, the system (223)–(226) is studied on the tangent bundle

$$
T_*\mathbf{S}^4\{(w_4,w_3,w_2,w_1;\xi,\eta_1,\eta_2,\eta_3)\in\mathbf{R}^8:\ 0\le\xi,\eta_1,\eta_2\le\pi,\ \eta_3 \bmod 2\pi\} \tag{230}
$$

of the four-dimensional sphere $\mathbf{S}^4\{(\xi,\eta_1,\eta_2,\eta_3)\in\mathbf{R}^4:\ 0\le\xi,\eta_1,\eta_2\le\pi,\ \eta_3 \bmod 2\pi\}$.

We see that the independent subsystem (218) (which can be considered separately on its own three-dimensional manifold), $n-3$ independent second-order subsystems (219) (after the change of independent variable) can be substituted into the system (218)–(220) of the order $2(n-1)$, and also Eq. (220) on η_{n-2} is separated (due to cyclicity of the variable η_{n-2}).

In particular, we see that the independent third-order subsystem (223) (which can be considered separately on its own three-dimensional manifold), two independent second-order subsystems (224), (225) (after the change of independent variable) can be substituted into the eighth-order system (223)–(225), and also Eq. (226) on η_3 is separated (due to cyclicity of the variable η_3).

Thus, for the complete integration of the system (218)–(220), it suffices to specify two independent first integrals of system (218), one by one first integral of systems (219) (all $n-3$ pieces), and an additional first integral that "attaches" Eq. (220) (*i.e., only n*).

In particular, for the complete integration of the system (223)–(226), it suffices to specify two independent first integrals of system (223), one by one first integral of systems (224), (225), and an additional first integral that "attaches" Eq. (226) (*i.e., only five*).

First, we compare the third-order system (218) with the nonautonomous second-order system

$$
\begin{aligned}
\frac{dw_{n-1}}{d\xi} &= \frac{\sin\xi\cos\xi-(1+b_*H_{1*})w_{n-2}^2\cos\xi/\sin\xi+H_{1*}w_{n-1}\cos\xi}{-(1+b_*H_{1*})w_{n-1}-b_*\sin\xi},\\
\frac{dw_{n-2}}{d\xi} &= \frac{(1+b_*H_{1*})w_{n-2}w_{n-1}\cos\xi/\sin\xi+H_{1*}w_{n-2}\cos\xi}{-(1+b_*H_{1*})w_{n-1}-b_*\sin\xi}.
\end{aligned}
\tag{231}
$$

Using the substitution $\tau = \sin\xi$, we rewrite system (231) in the algebraic form:

$$\begin{aligned}\frac{dw_{n-1}}{d\tau} &= \frac{\tau - (1 + b_* H_{1*}) w_{n-2}^2/\tau + H_{1*} w_{n-1}}{-(1 + b_* H_{1*}) w_{n-1} - b_*\tau},\\ \frac{dw_{n-2}}{d\tau} &= \frac{(1 + b_* H_{1*}) w_{n-2} w_{n-1}/\tau + H_{1*} w_{n-2}}{-(1 + b_* H_{1*}) w_{n-1} - b_*\tau}.\end{aligned} \tag{232}$$

Further, if we introduce the uniform variables by the formulas

$$w_{n-1} = u_2\tau, \ \ w_{n-2} = u_1\tau, \tag{233}$$

we reduce system (232) to the following form:

$$\begin{aligned}\tau\frac{du_2}{d\tau} + u_2 &= \frac{1 - (1 + b_* H_{1*}) u_1^2 + H_{1*} u_2}{-(1 + b_* H_{1*}) u_2 - b_*},\\ \tau\frac{du_1}{d\tau} + u_1 &= \frac{(1 + b_* H_{1*}) u_1 u_2 + H_{1*} u_1}{-(1 + b_* H_{1*}) u_2 - b_*},\end{aligned} \tag{234}$$

which is equivalent to

$$\begin{aligned}\tau\frac{du_2}{d\tau} &= \frac{(1 + b_* H_{1*})(u_2^2 - u_1^2) + (b_* + H_{1*}) u_2 + 1}{-(1 + b_* H_{1*}) u_2 - b_*},\\ \tau\frac{du_1}{d\tau} &= \frac{2(1 + b_* H_{1*}) u_1 u_2 + (b_* + H_{1*}) u_1}{-(1 + b_* H_{1*}) u_2 - b_*}.\end{aligned} \tag{235}$$

We compare the second-order system (235) with the nonautonomous first-order equation

$$\frac{du_2}{du_1} = \frac{1 - (1 + b_* H_{1*})(u_1^2 - u_2^2) + (b_* + H_{1*}) u_2}{2(1 + b_* H_{1*}) u_1 u_2 + (b_* + H_{1*}) u_1}, \tag{236}$$

which can be easily reduced to the exact differential equation

$$d\left(\frac{(1 + b_* H_{1*})(u_2^2 + u_1^2) + (b_* + H_{1*}) u_2 + 1}{u_1}\right) = 0. \tag{237}$$

Therefore, Eq. (236) has the following first integral:

$$\frac{(1 + b_* H_{1*})(u_2^2 + u_1^2) + (b_* + H_{1*}) u_2 + 1}{u_1} = C_1 = \text{const}, \tag{238}$$

which in the old variables has the form

$$\Theta_1(w_{n-1}, w_{n-2}; \xi) = \frac{(1 + b_* H_{1*})(w_{n-1}^2 + w_{n-2}^2) + (b_* + H_{1*}) w_{n-1}\sin\xi + \sin^2\xi}{w_{n-2}\sin\xi} = C_1 = \text{const}. \tag{239}$$

Remark 8.1. *We consider system (218) with variable dissipation with zero mean, which becomes conservative for $b_* = H_{1*}$:*

$$\begin{aligned}\xi' &= -(1 + b_*^2) w_{n-1} - b_*\sin\xi,\\ w_{n-1}' &= \sin\xi\cos\xi - (1 + b_*^2) w_{n-2}^2\frac{\cos\xi}{\sin\xi} + b_* w_{n-1}\cos\xi,\\ w_{n-2}' &= (1 + b_*^2) w_{n-2} w_{n-1}\frac{\cos\xi}{\sin\xi} + b_* w_{n-2}\cos\xi.\end{aligned} \tag{240}$$

It has two analytical first integrals of the form

$$(1+b_*^2)(w_{n-1}^2+w_{n-2}^2)+2b_*w_{n-1}\sin\xi+\sin^2\xi=C_1^*=const, \tag{241}$$

$$w_{n-2}\sin\xi=C_2^*=const. \tag{242}$$

It is obvious that the ratio of the first integrals (241), (242) is also a first integral of system (240). However, for $b_*\neq H_{1*}$ both functions

$$(1+b_*H_{1*})(w_{n-1}^2+w_{n-2}^2)+(b_*+H_{1*})w_{n-1}\sin\xi+\sin^2\xi \tag{243}$$

and (242) are not first integrals of system (218), but their ratio (i.e., the ratio of the functions (243) and (242)) is a first integral of system (218) for any b_*, H_{1*}.

Later on, we find the obvious form of the additional first integral of the third-order system (218). For this, at the beginning, we transform the invariant relation (238) for $u_1\neq 0$ as follows:

$$\left(u_2+\frac{b_*+H_{1*}}{2(1+b_*H_{1*})}\right)^2+\left(u_1-\frac{C_1}{2(1+b_*H_{1*})}\right)^2=\frac{(b_*-H_{1*})^2+C_1^2-4}{4(1+b_*H_{1*})^2}. \tag{244}$$

We see that the parameters of the given invariant relation must satisfy the condition

$$(b_*-H_{1*})^2+C_1^2-4\geq 0, \tag{245}$$

and the phase space of system (218) is stratified into a family of surfaces defined by Eq. (244).

Thus, by virtue of relation (238) the first equation of system (235) has the form

$$\tau\frac{du_2}{d\tau}=\frac{2(1+b_*H_{1*})u_2^2+2(b_*+H_{1*})u_2+2-C_1U_1(C_1,u_2)}{-b_*-(1+b_*H_{1*})u_2}, \tag{246}$$

where

$$U_1(C_1,u_2)=\frac{1}{2(1+b_*H_{1*})}\{C_1\pm U_2(C_1,u_2)\}, \tag{247}$$

$$U_2(C_1,u_2)=\sqrt{C_1^2-4(1+b_*H_{1*})(1+(b_*+H_{1*})u_2+(1+b_*H_{1*})u_2^2)},$$

and the integration constant C_1 is chosen from condition (245).

Therefore, the quadrature for the search of an additional first integral of system (218) has the form

$$\int\frac{d\tau}{\tau}=$$

$$=\int\frac{(-b_*-(1+b_*H_{1*})u_2)du_2}{2(1+(b_*+H_{1*})u_2+(1+b_*H_{1*})u_2^2)-C_1\{C_1\pm U_2(C_1,u_2)\}/(2(1+b_*H_{1*}))}. \tag{248}$$

Obviously, the left-hand side up to an additive constant is equal to

$$\ln|\sin\xi|. \tag{249}$$

If

$$u_2+\frac{b_*+H_{1*}}{2(1+b_*H_{1*})}=r_1,\ b_1^2=(b_*-H_{1*})^2+C_1^2-4, \tag{250}$$

then the right-hand side of Eq. (248) has the form

$$-\frac{1}{4}\int\frac{d(b_1^2-4(1+b_*H_{1*})r_1^2)}{(b_1^2-4(1+b_*H_{1*})r_1^2)\pm C_1\sqrt{b_1^2-4(1+b_*H_{1*})r_1^2}}+$$

$$+(b_* - H_{1*})(1 + b_* H_{1*}) \int \frac{dr_1}{(b_1^2 - 4(1 + b_* H_{1*})r_1^2) \pm C_1\sqrt{b_1^2 - 4(1 + b_* H_{1*})r_1^2}} =$$

$$= -\frac{1}{2} \ln \left| \frac{\sqrt{b_1^2 - 4(1 + b_* H_{1*})r_1^2}}{C_1} \pm 1 \right| \pm \frac{-b_* + H_{1*}}{2} I_1, \tag{251}$$

where

$$I_1 = \int \frac{dr_3}{\sqrt{b_1^2 - r_3^2}(r_3 \pm C_1)}, \; r_3 = \sqrt{b_1^2 - 4(1 + b_* H_{1*})r_1^2}. \tag{252}$$

In the calculation of integral (252), the following three cases are possible.

I. $|b_* - H_{1*}| > 2$.

$$I_1 = -\frac{1}{2\sqrt{(b_* - H_{1*})^2 - 4}} \ln \left| \frac{\sqrt{(b_* - H_{1*})^2 - 4} + \sqrt{b_1^2 - r_3^2}}{r_3 \pm C_1} \pm \frac{C_1}{\sqrt{(b_* - H_{1*})^2 - 4}} \right| +$$

$$+ \frac{1}{2\sqrt{(b_* - H_{1*})^2 - 4}} \ln \left| \frac{\sqrt{(b_* - H_{1*})^2 - 4} - \sqrt{b_1^2 - r_3^2}}{r_3 \pm C_1} \mp \frac{C_1}{\sqrt{(b_* - H_{1*})^2 - 4}} \right| + \text{const.} \tag{253}$$

II. $|b_* - H_{1*}| < 2$.

$$I_1 = \frac{1}{\sqrt{4 - (b_* - H_{1*})^2}} \arcsin \frac{\pm C_1 r_3 + b_1^2}{b_1(r_3 \pm C_1)} + \text{const.} \tag{254}$$

III. $|b_* - H_{1*}| = 2$.

$$I_1 = \mp \frac{\sqrt{b_1^2 - r_3^2}}{C_1(r_3 \pm C_1)} + \text{const.} \tag{255}$$

When we return to the variable

$$r_1 = \frac{w_{n-1}}{\sin \xi} + \frac{b_* + H_{1*}}{2(1 + b_* H_{1*})}, \tag{256}$$

we obtain the final form for the value I_1:

I. $|b_* - H_{1*}| > 2$.

$$I_1 = -\frac{1}{2\sqrt{(b_* - H_{1*})^2 - 4}} \ln \left| \frac{\sqrt{(b_* - H_{1*})^2 - 4} \pm 2(1 + b_* H_{1*})r_1}{\sqrt{b_1^2 - 4(1 + b_* H_{1*})^2 r_1^2} \pm C_1} \pm \frac{C_1}{\sqrt{(b_* - H_{1*})^2 - 4}} \right| +$$

$$+ \frac{1}{2\sqrt{(b_* - H_{1*})^2 - 4}} \ln \left| \frac{\sqrt{(b_* - H_{1*})^2 - 4} \mp 2(1 + b_* H_{1*})r_1}{\sqrt{b_1^2 - 4(1 + b_* H_{1*})^2 r_1^2} \pm C_1} \mp \frac{C_1}{\sqrt{(b_* - H_{1*})^2 - 4}} \right| + \text{const.} \tag{257}$$

II. $|b_* - H_{1*}| < 2$.

$$I_1 = \frac{1}{\sqrt{4 - (b_* - H_{1*})^2}} \arcsin \frac{\pm C_1 \sqrt{b_1^2 - 4(1 + b_* H_{1*})^2 r_1^2} + b_1^2}{b_1(\sqrt{b_1^2 - 4(1 + b_* H_{1*})^2 r_1^2} \pm C_1)} + \text{const.} \tag{258}$$

III. $|b_* - H_{1*}| = 2$.

$$I_1 = \mp \frac{2(1 + b_* H_{1*})r_1}{C_1(\sqrt{b_1^2 - 4(1 + b_* H_{1*})^2 r_1^2} \pm C_1)} + \text{const.} \tag{259}$$

Thus, we have found an additional first integral for the third-order system (218), i.e., we have a complete set of first integrals that are transcendental functions of the phase variables.

Remark 8.2. *In the expression of the found first integral, we must formally substitute the left-hand side of the first integral (238) instead of* C_1.

Then the obtained additional first integral has the following structure:

$$\Theta_2(w_{n-1}, w_{n-2}; \xi) = G\left(\sin\xi, \frac{w_{n-1}}{\sin\xi}, \frac{w_{n-2}}{\sin\xi}\right) = C_2 = \text{const}. \tag{260}$$

Thus, we have found two first integrals (239), (260) of the independent third-order system (218). For its complete integrability, it suffices to find one by one first integral for the systems (219) (all $n-3$ pieces), and an additional first integral that "attaches" Eq. (220).

Indeed, the desired first integrals coincide with the previous first integrals, precisely:

$$\Theta''_{s+2}(w_s; \eta_s) = \frac{\sqrt{1+w_s^2}}{\sin\eta_s} = C_{s+2} = \text{const}, \ \ s = 1, \ldots, n-3, \tag{261}$$

$$\Theta''_n(w_{n-3}, w_{n-4}; \eta_{n-4}, \eta_{n-3}, \eta_{n-2}) = \eta_{n-2} \pm \operatorname{arctg}\frac{C_{n-1}\cos\eta_{n-3}}{\sqrt{C_{n-2}^2\sin^2\eta_{n-3} - C_{n-1}^2}} = C_n = \text{const}, \tag{262}$$

in this case, in the left-hand side of Eq. (262), we must substitute instead of C_{n-2}, C_{n-1} the first integrals (261) for $s = n-4, n-3$.

Theorem 8.2. *The system (218)–(220) of the order* $2(n-1)$ *possesses the sufficient number (n) of the independent first integrals (239), (260), (261), (262).*

Therefore, in the considered case, the system of dynamical equations (218)–(220) has n first integrals expressing by relations (239), (260), (261), (262), which are the transcendental functions of its phase variables (in the sense of the complex analysis) and are expressed as a finite combination of elementary functions (in this case, we use the expressions (248)–(259)).

Theorem 8.3. *Three sets of relations (26), (40), (50) under conditions (31)–(33), (189), (193) possess n the first integrals (the complete set), which are the transcendental function (in the sense of complex analysis) and are expressed as a finite combination of elementary functions.*

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