To the light memory of Professor Vladimir Aleksandrovich Kondratiev

"The single that my long life has taught me: that all our science is look like primitive and naive on-baby behind the face of reality, and even so this is the most valuable that we have."

Albert Einstein

Introduction

The research presents the review of cases of integrability obtained earlier and also new in two-, three-, and four-dimensional rigid body dynamics in a nonconservative force field. The problems studied are described in terms of so-called zero mean variable dissipation dynamical systems.

Therefore, we study nonconservative systems for which the usual methods, for example, Hamiltonian systems, are not applicable in general. Therefore, for such systems, it is necessary, in some sense, to "directly" integrate the main equation of the dynamics. We generalize old cases and also obtain new cases of a complete integrability in transcendental functions in the two-, three-, and four-dimensional rigid body dynamics in a nonconservative force field.

Of course, in the general case, it is sufficiently difficult to construct some theory of integrating nonconservative systems (even of low dimension). But in a number of cases where the systems considered have additional symmetries, we succeed in finding first integrals through finite combinations of elementary functions [72–74].

We obtain a whole spectrum of complete integrability cases for nonconservative dynamical systems having nontrivial symmetries. Moreover, in almost all cases, each of the first integrals is expressed through a finite combination of elementary functions, being one transcendental function of its variables. In this case, the transcendence is understood in the sense of the complex analysis, when, after continuation of given functions to the complex domain, they have essentially singular points. The latter fact is stipulated by the existence of attracting and repelling sets in the system (for example, attracting and repelling foci).

We discover new integrable cases of motion of a rigid body, in particular, in the classical problem of motion of a spherical pendulum in an over-run medium flow.

The first chapter is devoted to general aspects of integrability of so-called variable dissipation dynamical systems. In the beginning, we give the visual characteristic of these systems. Therefore, in this case, we will speak about systems with the variable dissipation, where the term "variable" refers not to the value of the dissipation coefficient, but to the possible alternation of its sign (therefore, it is more reasonable to use the term "sign-alternating").

We will give later one of the possible definitions of zero (nonzero) mean variable dissipation systems through such a characteristic of the system vector field as divergence, which, as a well-known, is characterized by the phase volume change in the phase space of the system considered [20, 21, 24, 25, 28, 93, 97, 106].

We introduce the class of autonomous dynamical systems having one periodic phase coordinate and, therefore, possessing the certain symmetries which are typical for the pendulum-like systems. We show that offered class of systems is embedded to the class of zero mean variable dissipation systems in a natural way, i.e., on the average, for the period of the existing periodic coordinate, the sop and diffusing to energy balance to each other in certain sense. We offer the examples of pendulum-like systems on lower-dimension manifolds from the dynamics of a rigid body in a nonconservative force field.

Then we study certain general conditions of the integrability in elementary functions for the systems both on two-dimensional planes and tangent bundle of a one-dimensional sphere (i.e., a two-dimensional cylinder), and two-dimensional sphere (a four-dimensional manifold). Therefore, we offer the interesting example of a three-dimensional phase pattern of pendulum-like system which describes the motion of the spherical pendulum, placed in an over-run medium flow [365–367].

We present sufficient conditions of existence of the first integrals expressed through a finite combination of elementary functions for the multi-parameter third-order systems.

Since we present the cases of a complete integrability in the spatial rigid body dynamics of the motion in a nonconservative field, we deal with three (at first thought) independent properties:

(i) the distinguished class of systems with the symmetries above;
(ii) the fact that this class of systems consists of systems with the zero mean variable dissipation (in the periodic variable), which allows us to consider them as "almost" conservative systems;
(iii) in certain (although lower-dimensional) cases, these systems have the complete tuple of first integrals, which are transcendental in general (from the viewpoint of the complex analysis).

In the second and third chapters, the obtained results are systematized on the study of the dynamic equations of the motion of symmetrical two-dimensional ($2D$-) rigid body residing in a certain nonconservative field of the forces. Its type is originated from the dynamics of real rigid bodies interacting with a resisting medium on the laws of a jet flow, under which the nonconservative tracing force acts on the body, and it either forces the value of the velocity of a certain typical point of the rigid body to remain constant at all time of the motion, which means the presence in the system of a nonintegrable servo-constraint (the second chapter), or forces the center of mass of the body to move rectilinearly and uniformly at all time of the motion, which means the presence in the system of a nonconservative pair of forces (the third chapter) (see also [1, 49, 51, 88, 111–113, 139, 146–149, 184–186, 188, 195, 198, 202, 216, 235, 241, 246, 262, 295, 350, 373–376, 396, 412, 436]).

Therefore, in the second chapter, the additional transcendental first integral expressed through a finite combination of elementary functions is found to having an analytical nonintegrable constraint, and, in the third chapter, the same was made to having an analytical first integral (the square of the center of mass) only.

New obtained results are systematized and given in an invariant form. Moreover, the additional dependence of the moment of the nonconservative force on the angular velocity is introduced. The given dependence can also be wide-spread on the cases of the motions in the spaces of higher dimensions.

In the fourth and fifth chapters, the obtained results are systematized on the study of the dynamic equations of the motion of symmetrical three-dimensional ($3D$-) rigid body residing in a certain nonconservative field of forces. Its type is also originated from the dynamics of real rigid bodies interacting with a resisting medium on the laws of a jet flow, under which the nonconservative tracing force acts on the body, and it either forces the value of the velocity of a certain typical point of the rigid body to remain constant at all time of the motion, which means the presence in the system of a nonintegrable servo-constraint (the fourth chapter), or forces the center of mass of the body to move rectilinearly and uniformly at all time of the motion, which means the presence in the system of a nonconservative pair of forces (the fifth chapter) (see also [1, 27, 49, 51, 69–71, 88, 111–114, 130, 131, 139–141, 146–149, 152, 162–164, 181, 184–186, 188, 195, 198, 202, 216, 235, 241, 246, 256, 262, 263, 281, 347, 350, 353, 373–376, 396, 412, 416, 420–423, 431–433, 436–438, 440–442]).

Therefore, in the fourth chapter, three additional transcendental first integrals expressed through a finite combination of elementary functions are found to having analytical invariant relations (nonintegrable constraint and the integral on the equality to zero of one of the components of the angular velocity), and, in the fifth chapter, the same was made to having analytical first integrals (the square of the center of mass and integral on the equality to zero of one of the components of the angular velocity) only.

New obtained results are also systematized and given in an invariant form. Moreover, the additional dependence of the moment of the nonconservative force on the angular velocity is introduced. This dependence can also be wide spread on the cases of the motions in the spaces of higher dimensions.

The sixth chapter begins by the statement of general aspects of dynamics of a multi-dimensional rigid body, i.e., the notion of the angular velocity tensor, joint dynamic equations of the motion on the direct product $\mathbf{R}^n \times \mathrm{so}(n)$, Euler and Rivals formulas in multi-dimensional case.

The question on the tensor of inertia of a four-dimensional ($4D$-) rigid body is considered. We propose to study two possible cases logically on principal moments of inertia, i.e., when there exists *two* relations on the principal moments of inertia:

(i) there exist *three* equal principal moments of inertia ($I_2 = I_3 = I_4$);
(ii) there exist *two pairs* of equal moments of inetria ($I_1 = I_2, I_3 = I_4$).

In the sixth and seventh chapters, the obtained results are systematized on studying the dynamic equations of the motion of a symmetrical four-dimensional ($4D$-) rigid body residing in a certain nonconservative field of forces for the case (i). Its type is also originated from dynamics of lower-dimensional real rigid bodies interacting with a resisting medium on the laws of a jet flow, under which the nonconservative tracing force acts onto the body, and it either forces the value of the velocity of a certain typical point of the rigid body to remain constant at all time of the motion, which means the presence in the system of a nonintegrable servo-constraint (the sixth chapter), or forces the center of mass of the body to move rectilinearly and uniformly at all time of the motion, which means the presence in system of a nonconservative pair of the forces (the seventh chapter) (see also [1, 27, 33–36, 40, 44, 49, 51, 88, 111–113, 139, 146–149, 184–188, 195, 198, 202, 216, 225–227, 233, 235, 237, 241, 246, 251, 252, 255, 261, 262, 282, 284, 295, 350, 373–376, 396, 412, 424, 425, 436]).

Therefore, in the sixth chapter, four additional transcendental first integrals expressed through a finite combination of elementary functions are found to have four analytical invariant relations (a nonintegrable constraint and three integrals on the equalities to zero of some of the components of the angular velocity tensor), and in the seventh chapter, the same was made to have four analytical first integrals (the square of the center of mass and three integrals on the equalities to zero of some of the components of the angular velocity tensor) only.

The results are pertained to the case, where all the interaction of a medium with the body is concentrated on a part of the body surface that has the form of a three-dimensional disk, and, the force interaction is concentrated in the direction, which is perpendicular to this disk. These results are systematized and given in an invariant form. Herewith, the additional dependence of the moment of the nonconservative force on the angular velocity is introduced. This dependence can be wide spread on the cases of motions in spaces of higher dimensions.

In the eighth chapter, the obtained results are systematized on studying the dynamic equations of the motion of a symmetrical four-dimensional ($4D$-) rigid body residing in a certain nonconservative field of forces for the case (ii). Its type is also originated from the dynamics of lower-dimensional real rigid bodies interacting with a resisting medium on the laws of a jet flow, under which the nonconservative tracing force acts onto the body, and it forces both the value of the velocity of a certain typical point of the rigid body and the certain phase variable to remain constant at all time, which means the presence in the system of a nonintegrable servo-constraints (see also [1, 49, 51, 88, 111–113, 128, 139, 146–149, 186, 188, 195, 202, 241, 246, 298, 314, 322, 325, 330, 332, 333, 339, 345, 350, 373–374, 412]).

Therefore, in the eighth chapter, two additional transcendental and three analytical first integrals expressed through a finite combination of elementary functions are found to have four analytical invariant relations (two nonintegrable constraints and two integrals on the equalities to zero of some of the components of the angular velocity tensor).

The results which are obtained now are pertained to the case where all interaction of a medium with the body is concentrated on the part of the body surface that has the form of a two-dimensional disk, and the force interaction is concentrated on the two-dimensional plane which is perpendicular to this disk. These results are systematized and given in an invariant form. Herewith, the additional dependence of the moment of the nonconservative force on the angular velocity is introduced. The given dependence can also be wide spread on the cases of motions in spaces of higher dimensions.

And, therefore, in the chapters 2–8, the cases of integrability in lower- and multi-dimensional dynamics of a rigid body placed in a nonconservative force field are presented. To systemize, we place all of them to the following table.

The denotation $h = 0$ (or $h \neq 0$) means that the dependence of the force field on the components of the angular velocity tensor is present (or is absent) in the system.

The sign $\oplus$ means that the case is placed to this review.

Two signs $\ominus$ in the right lower corner of the table mean that these two cases are not placed to this review (indeed, the eighth chapter is devoted to the case $I_1 = I_2,\ I_3 = I_4$ only).

Nevertheless, the corresponding results have been already obtained for the case $I_2 = \cdots = I_n$ of a symmetric n-dimensional rigid body, and these results are also not placed to this review.

Many results of this work were regularly reported at numerous workshops, including the workshop "Actual Problems of Geometry and Mechanics" named after Professor V. V. Trofimov [57] led by D. V. Georgievskii and M. V. Shamolin (see also [1, 2, 49–52, 54, 58–63, 65, 178, 179, 234, 235, 265–268, 270, 273, 274, 276, 278, 279, 299, 300, 307, 308, 310, 315, 344, 348, 356, 357]).

Chapter 1

INTEGRABILITY OF CERTAIN CLASSES OF NONCONSERVATIVE SYSTEMS IN ELEMENTARY FUNCTIONS

If in the first part of the work, we consider conservative systems for which the technique of the study based on topological invariants was elaborated.

We study non-conservative systems, for which the methods of the study, for example, Hamiltonian systems, are not applicable in general. Therefore, for such systems, it is necessary, in some sense, to "directly" integrate the main equation of the dynamics. Herewith, we offer more universal interpretation of both obtained cases and new ones of complete the integrability in transcendental functions in the two-, three- and four-dimensional rigid body dynamics in a non-conservative force field.