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# THE VARIOUS CASES OF COMPLETE INTEGRABILITY IN DYNAMICS OF A RIGID BODY INTERACTING WITH A MEDIUM

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**Abstract.** *This paper contains some the development of qualitative methods in the theory of nonconservative systems that arise, e.g., in such fields of science as the dynamics of a rigid body interacting with a resisting medium, oscillation theory, etc. This material can call the interest of specialists in the qualitative theory of ordinary differential equations, in rigid body dynamics, as well as in fluid and gas dynamics since the work uses the properties of motion of a rigid body in a medium under the streamline flow around conditions.*

## 1 INTRODUCTION

The author obtains some cases of complete integrability for nonconservative dynamical systems having nontrivial symmetries. Moreover, in almost all cases of integrability, each of the first integrals is expressed through a finite linear combination of elementary functions and is a transcendental function of its variables, simultaneously. In this case, the transcendence is meant in the complex analysis case, i.e., after the continuation of the functions considered to the complex domain, they have essentially singular points. The latter fact is stipulated by the existence of attracting and repelling limit sets in the system considered (for example, attracting and repelling foci).

It was obtained new families of phase portraits of systems with variable dissipation on lower- and higher-dimensional manifolds. He discusses the problems of their absolute or relative roughness and discover new integrable cases of the rigid body motion too.

## 2 A CERTAIN PROBLEM OF THE DYNAMICS OF A RIGID BODY INTERACTING WITH A MEDIUM

### 2.1 From important historical past

The problem of the motion of a rigid body in a resisting medium (for example, the problem of the body fall in air) calls the interest of investigators during several tens years: as far as in Middle Edges, there arose the necessity to study the dependence of the shutting distance on the slope of the gun barrel.

The trials of studying the motion of a body in air and fluid allowed Ch. Huygens to discover the empirical law saying that the resistance is proportional to the square of the body velocity in the air (1669). Basing on the trials (of Ph. Hoxby, J. Desaguliers, and his own), Isaac Newton created the mathematical theory of resistance in the air; in XVIII century, its elaboration was continued by Varingnon, D. Bernoulli, J, D'Alembert, L.Euler, and others. The ballistic pendulum was invented that time.

As a result of a deep analysis of the trial material belonging to Englishman B. Robbins, in 1745, L. Euler replaced the quadratic resistance law by the following binomial law: the first summand is proportional to the velocity square, and the send to the velocity fourth power. In what follows, Euler elaborated numerical methods for integrating the differential equation of the shell motion, in particular, using slow convergent series. For the aiming shutting, he suggested another methodology according to which the the shell motion is partitioned into components one of which is related to the resistance.

The efforts of scientists were aimed at not only tho the finding the shell trajectory and the shell motion law but also at the maximally possible account for additional phenomena that lead to the important corrections to the basic theory. In XVIII century, Robbins observed that the center of masses of a rotating shell describes a spatial curve. Later on, in XIX century, S. Poisson and then M. V. Ostrogradskii tried to give a mathematical treatment of this phenomenon. Basing on the general theory of the rigid body motion, it was found that a longitudinal rotating shell obeys a proper rapid rotation around the longitudinal axis of dynamical and dynamical symmetry, the precession near the shell velocity vector, and a nutational motion near the tilting momentum vector.

### 2.2 Studies of professor N. E. Zhukovskii and professor S. A. Chaplygin

It was professor N. E. Zhukovskii who analyzed various problems of the point dynamics in a medium, more precisely, the body fall, the motion of a body moving at an angle with respect to the horizon, the pendulum motion, etc. Along with integrating the equations of motion, he improved the model of the body interaction with a resisting medium and considered that the kinetic energy of a falling body is spent to the formation of vortex air motions and, in addition, to the overcoming the molecular forces of adhering the air to the moving body. The resistance depends not only on the velocity of body point motion but the form the body itself. If the velocity is small, then with a sufficient accuracy, one can assert that the resistance is proportional to the first power of the velocity. For large velocities, the resistance is proportional to the velocity square.

Also, from the professor N. E. Zhukovskii studies, it is known an attempt to model the motion basing on experiments in self-rotations of plates falling in the air [1, 2] (the so-called "Hamburg cardboard"). Here, one needs to take into account such properties of interaction of the medium to the body as the resistance force and the body force. Precisely, the aerodynamic characteristics of a plate are also used for modelling the bird flight [2].

Professor N. E. Zhukovskii assumed the existence of a "bird body" dynamical equilibrium with respect to the center of masses such that the angle between the center mass velocity and the plate wing plane (angle of attack) serves as a control parameter, i.e., it can be given arbitrarily. This assumption is equivalent to the assumption on such a body motion separation under which the characteristic time of motion with respect to the center of masses is considerably smaller than the characteristic times of motion of the center itself.

The study of the body motion in a medium under the condition that its plane parallel motion a-is connected with the rotational motion is of interest. The problems mentioned above are far from to be exhaustive for all the possibilities of such a type.

Among the studies of professor S. A. Chaplygin, we also mention the statement of the problem of heavy body motion in an incompressible fluid [3, 4].

In the framework of this work, our *fundamental* problem is the study of the infinite length plate motion under the streamline flow conditions. First of all, this problem is important for further studying the motion of a body interacting with a medium through frontal plane part.

### 2.3 Various aspects of problem consideration

As is seen, in the historical past, only one aspect of the problem of the body motion in a resisting medium was considered. Precisely, the interests of investigators were aimed at the obtaining concrete trajectories, although in an approximate but explicit form. Moreover, in parallel, the problem of more precise modelling the interaction of a body with a resisting medium was considered.

Let us briefly illustrate the above problem by examining bodies of a simple form.

A plane plate is the simplest body that allows one to study various futures of the motion in a medium. The effects related to the influence of adhered masses (classical Kirchhoff problem) are demonstrated in [5] by examining the motion of a body–late in a fluid (as is known, the study was initiated by Thomson, Tate, and Kirchhoff).

The Kirchhoff problem posed in the second half of XIX century opened the second aspect of the problem consideration. It is related to the integrability problems for the system of differential equations, which describes this motion (the problems of existence of analytic (smooth, meromorphic) first integrals).

Up to the present, because of complexity, various variants of the Kirchhoff problem were almost always considered from the viewpoint of the integrability problem, and only in certain cases, the qualitative analysis of a number of trajectories was carried out. In the works of Kirchhoff, Clebsch, Steklov, Chaplygin, Lyapunov, Kharlamov, etc., some conditions for existence of an additional first integral were found.

Also, let us mention the third aspect of the above problem consideration, precisely, the qualitative analysis of the systems of differential equations describing this motion (phase space fibrations, the qualitative location of phase trajectories, symmetries, etc.). Although the listed problems are closely related to the integrability, its solution is of an independent nature. Moreover, this aspect stimulates the development of qualitative means.

### 2.4 Sequence of steps in modelling

Generally speaking, the general problem of studying the body motion in the resistance force field "is prevented" by the absence of any complete description of this force field. As is known, in principle, we can measure the positional component of the resistance force in a stationary experiment. But the component of the force field, which corresponds to the quasi-velocities of the system considered arises only under the non-stationary body motion.

Therefore, the process of describing the force field is a sequence of steps. We first study a preparatory model of the force field and construct a family of mechanical systems whose motion has different characteristics that essentially depend on model parameters such that the information about them is incomplete or does not exist at all. As a result of studying such a model, there arise questions such that the answers to them cannot be found in the framework of the accepted model. Then the elaborated objects become the subject of a detailed experimental study at the second step. Such an experiment presupposes the answers to the formulated questions and either introduces necessary corrections to the preparatory constructed model or reveals new questions, which lead to the necessity of the first step repetition but in a new level of the problem understanding.

Such an approach is related to the description of stationary motion regimes, their branching, bifurcation, stability and instability analysis, revealing surgery conditions, and appearance of regular or irregular (i.e., *chaotic*) oscillations.

Sometime, we can succeed in obtaining the answers to questions of qualitative character when discussing the traditional problem of analytic mechanics, the problem of existence of the full tuple of first integrals for the constructed dynamical system. At the same time, the study of the behavior of a dynamical system "as a whole" often forces us to use the numerical experiment. In this case, there arises the necessity of elaborating new computational algorithms or improving the known, as well as new qualitative methods.

In this work, we study the problem on the body motion under the condition that the line of the force applied to the body does not change its orientation with respect to the body and can only displace parallel to itself depending on the angle of attack and, possibly, on other phase variables. Such conditions arise under the plate motion with the so-called "large" angles of attack in a medium under a streamline flow (in this case, the fluid is assumed to be ideal in general, although all this are also true for fluids of a small viscosity, first of all, for the water) or under a separation flow (which is justified by an experiment completely satisfactory). Therefore, the *main objects of studying* is a family of bodies such that a part of the surface of each of which has a plane part that is flowed by a medium according to the streamline flow laws.

## 2.5 Model assumptions, quasi-stationarity hypothesis and phase variables

Assume that a rigid body of mass $m$ executes a plane-parallel motion in a medium with quadratic resistance law and that a certain part of the exterior body surface is a plane plate being under the medium streamline flow conditions. This means that the action of the medium on the plate reduces to the force **S** (applied at the point $N$) whose line of action is orthogonal to the plate. Let the remained par of the body surface be situated in a volume bounded by the flow surface that goes away from the plate boundary and is not subjected by the medium action. For example, similar conditions can arise after the body entrance into the water.

Assume that among the body motions, there exists a rectilinear plane-parallel drag regime. This in possible when the following two conditions hold: 1) the body velocity is orthogonal to the plate $AB$; 2) the perpendicular dropped from the body center of gravity $C$ on the plate plane belongs to the line of the action of the force **S** (Fig. 1).

Let us relate to the body the right coordinate system $Dxyz$ whose axis $z$ moves parallel to itself, and for simplicity, assume that the plane $Dzx$ is the geometric symmetry plane of the body. This ensures the fulfillment of property 2) under the motion satisfying condition 1) (Fig. (1)).

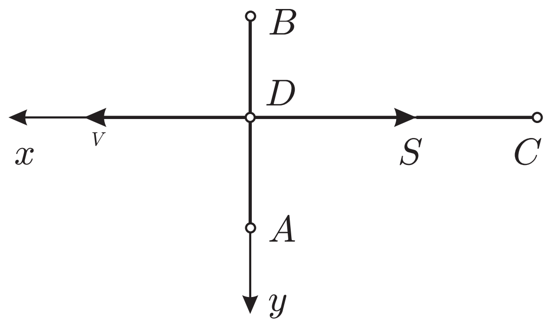


Figure 1: Unperturbed the plane-parallel interaction of the body with the medium.

To construct the dynamical model, let us introduce the following phase coordinates: the value $v = |\mathbf{v}|$ of the velocity $\mathbf{v}$ of the point $D$ (see Fig. Fig. (2)), the angle $\alpha$ between the vector $\mathbf{v}$ and the axis $x$, and the algebraic value $\Omega$ of the projection of the body absolute angular velocity on the axis $z$.

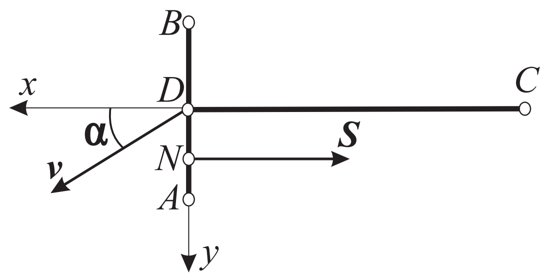


Figure 2: Perturbed the plane-parallel interaction of the body with the medium.

Assume that the value of the force **S** quadratically depend on $v$ with nonnegative coefficient $s_1$ ($S = s_1 v^2$). As usual, one represents $s_1$ in the form $s_1 = \rho P c_x/2$, where $c_x$ is now the dimension-free coefficient of the frontal resistance ($\rho$ is the medium density and $P$ is the plate area). This coefficient depends on the angle of attack, the *Struchal number*, and other quantities which are usually considered as parameters. In what follows, we also introduce the following additional phase variable of the "Struchal type": $\omega = \Omega\Delta/v$, where $\Delta$ is the characteristic plate transversal size. We restrict ourselves to the dependence of $c_x$ on $\alpha$, i.e., we assume that $s_1$ is the function of $\alpha$, and $y_N$ is a function of the pair $(\alpha, \omega)$ of dimension-free variables.

Let us define (purely formally for now) the dependence of $s_1$ and the ordinate $y_N$ of the point $N$ on the phase coordinates $(\alpha, \omega)$. The system of dynamical equations must admit a particular solution of the form $\alpha(t) \equiv 0,\ \omega(t) \equiv 0$. Therefore, we have the condition $y_N(0,0) = 0$ for the function $y_N(\alpha, \omega)$, and in the linear case, we need to assume that $y_N = \Delta(k\alpha - h\omega)$, where $k$ and $h$ are certain constants. Because the approximation is linear, we can ignore the dependence of $s_1$ on $\alpha$ and $\omega$.

In what follows, to take into account the action direction of the force $\mathbf{S}$, we introduce the following sigh-alternating auxiliary function $s(\alpha) = s_1(\alpha)\mathrm{sign}\cos\alpha.$

Therefore, the linearized model of the force medium action contains three parameters $s$, $k$, and $h$, which are determined by the plate form in the plan. As was already mentioned, the first of these parameters, the coefficient $s$, is *dimensional*. The parameters $k > 0$ and $h$ are *dimension-free* because of the method of their introduction. Note that the quantities $s$ and $k$ can be found experimentally by using weight measurements in devices of the hydro- or aerodynamic tubes type.

To describe the results and concrete properties of the body motion, in Institute of Mechanics of M. V. Lomonosov Moscow State University, V. A. Eroshin and V. M. Makarshin carried out experiments in registration of the motion of homogeneous circular cylinders in the water. Owing to the experiment, it becomes possible to find the dimension-free parameters $k$ and $h$ of the medium action on a rigid body.

The experiment allows one to make several important conclusions.

The first of them is as follows: the *rectilinear stationary free body drag* (*in the water*) *is unstable* at least with respect to the angle of attack and the angular velocity.

The second conclusion obtained from the carried out natural experiment is as follows: *in modelling the medium action on the body, it is necessary to take into account the additional parameter characterizing the rotational derivative of the moment with respect to the body angular velocity*. This parameter introduces the dissipation into the system. In our linear approximation, the accounting damping moment linearly depends on the body velocity.

For certain cases, the value of the damping moment coefficient under the body motion in the water was already estimated. This estimate confirms the instability of the body rectilinear motion in the water. Purely formally, increasing the value of the damping coefficient, we can attain the stability of a motion, but it is difficult to ensure this stability in realty. The rigid bode rectilinear motion is stable in certain media (for example in the clay), as the experiment shows. Possibly, this stability is attained due account for the existence of a considerable damping from the medium in the system or for the existence of forces tangent to the plate.

If the additional medium damping action on the body is of a purely dissipative character, then we can restrict ourselves to the region of positive damping moment parameter values, since a priori, its sign is not obvious.

### 2.6 Beginning of nonlinear analysis

The first conclusion made from the experiment forces us to consider the class of possible body motions for small angles of attack as a supporting class for studying the class of free body drags with finite angle of attack. For different bodies, under motion in a medium and under certain conditions, the angles of attack can practically assume any value from the interval $(0,\pi/2)$, and only for the angles close to $\pi/2$, the so-called washing out of the lateral surface is inevitable. Therefore, there arises the necessity of extending the functions $y_N$ and $s$ to finite angles of attack, i.e., the expansion of the domain of the pair of dynamical functions up to the interval $(0,\pi/2)$. But, in fact, it is necessary to extend the dynamical functions to the whole numerical line; this is clear from the following arguments.

Imagine an aircraft executing a plane-parallel motion over the water. In this case, the aircraft itself interact with the water through a certain construction (for example, a keel) containing a plane surface part (plate), which is flowed around by the water when the aircraft moves over it. We can assume that the plane plate interacts with the water according to the streamline flow laws. Such an aircraft is similar to the well-known screencraft.

The plane-parallel motion of an aircraft over the water is ensured by stabilizing its transversal oscillations, in particular, for a screencraft, by the existence of the screen itself.

Under the motion of such an aircraft (or a screencraft) over the water, the mention plane plate can move in the water with practically arbitrary (real) angles of attack,including the case where it "cuts" the water, i.e., it moves with angles of attack close to $\pi/2$, as well as it assumes this value. Therefore, the dynamical functions of such a plate can be extended to any angles of attack.

In order to pass to a more complete description of the free body motion, let us represent the dynamics equations as follows:

$$\dot{v}\cos\alpha - \dot{\alpha}v\sin\alpha - \Omega v\sin\alpha + \sigma\Omega^2 = F_x/m, \tag{1}$$

$$\dot{v}\sin\alpha + \dot{\alpha}v\cos\alpha + \Omega v\cos\alpha - \sigma\dot{\Omega} = 0, \tag{2}$$

$$I\dot{\Omega} = y_N(\alpha,\omega)F_x, \quad \omega = \Delta\Omega/v. \tag{3}$$

As a rule, for various variants of the body motion considered below, the generalized force $F_x$ is quadratic in velocities $(v,\Omega)$ and explicitly depends on the auxiliary sign-alternating function $s(\alpha)$ (for example, $F_x(\alpha,v,\Omega) = -s(\alpha)v^2$ in the case of the body free drag). Therefore, the class of conceptual bodies and their conceptual motions defines a certain pair of dynamical functions $(s(\alpha),\ y_N(\alpha,\omega))$ belonging to definite function classes.

### 2.7 Classes of dynamical functions

The fist stage of the complete nonlinear study of the body motion in a medium under the quasi-stationarity conditions is the study of the corresponding dynamical systems in which the damping is not taken into account (in particular, $h = 0$ in the linear case). The account for the damping is the next labor-consuming stage of studying the problem, which is presented in this work in a sufficient detail.

To begin with, we consider the case where the pair of dynamical functions $(y_N, s)$ depends only on the angle of attack. In this case, to qualitatively describe this pair of functions, we use the experimental information about the streamline flow properties.

The classes of dynamical functions to be introduced are sufficiently wide. They consists of smooth, $2\pi$-periodic ($y_N(\alpha)$ is odd and $s(\alpha)$ is even) functions satisfying the following conditions: $y_N(\alpha) > 0$ for $\alpha \in (0,\pi)$, and, moreover, $y'_N(0) > 0$ and $y'_N(\pi) < 0$ (the function class $\{y_N\} = Y$); $s(\alpha) > 0$ for $\alpha \in (0,\pi/2)$, $s(\alpha) < 0$ for $\alpha \in (\pi/2,\pi)$, and, moreover, $s(0) > 0$ and $s'(\pi/2) < 0$ (the function class $\{s\} = \Sigma$). Both $y_N$ and $s$ change the sign under the replacement of $\alpha$ on $\alpha+\pi$. Therefore, $y_N \in Y$, $s \in \Sigma$.

In particular, the analytic functions

$$y_N(\alpha) = y_0(\alpha) = A\sin\alpha \in Y,\ s(\alpha) = s_0(\alpha) = B\cos\alpha \in \Sigma,\ A, B > 0, \tag{4}$$

serve as typical representatives of the described classes and correspond to the medium interaction functions obtained by professor S. A. Chaplygin in studying the plane-parallel flow around of a plane infinite length by a homogeneous medium flow.

In what follows, there rises the product $F(\alpha) = y_N(\alpha)s(\alpha)$ in the dynamical systems considered. It follows from the conditions listed above that $F$ is a sufficiently smooth odd $\pi$-periodic functions satisfying the following conditions: $F(\alpha) > 0$ for $\alpha \in (0, \pi/2)$, $F'(0) > 0$, and $F'(\pi/2) < 0$ (the function class $\{F\} = \Phi$). Therefore, $F \in \Phi$.

In particular, the analytic function

$$F = F_0(\alpha) = AB\sin\alpha\cos\alpha \in \Phi \tag{5}$$

is also a typical representative of the a function class $\Phi$ arisen (and also corresponds to the professor S. A. Chaplygin case mentioned above).

Let us explain the necessity of a wide choice of the function classes $Y$ and $\Sigma$. A plane plate is a geometric section of the part of the body surface that interacts with the medium and is plane. The geometric form of such a plane domain can be arbitrary. Moreover, the chord lying in the plane of the domain can differently determine the plane of the body motion itself (in the case of the plane-parallel motion). The latter circumstances allow us to refer the dynamical functions arisen to definite classes. As was noted above, sufficiently weak conditions are imposed on these function classes, and, therefore, these classes are sufficiently wide. In advance, they include admissible concrete functions take for each conceptual body and each conceptual motion.

Therefore, to study the medium flow around a plate, we use the *classes of dynamical systems* defined by a pair of dynamical functions, which considerably complicates the global analysis performance.

But certainly, it is not possible to associate a conceptual rigid body with its motion with each concrete pair of dynamical functions. Therefore, the study of this problem for sufficiently wide classes of dynamical functions allows us to speak about a relatively complete consideration of the problem of the body motion in a medium in the framework of these model assumptions under the quasi-stationarity conditions.

### 2.8 Directions developed in the work

Let us indicate the following directions developed in the work. The first two directions are traditional for analytical mechanics.

1. *Elaboration of a qualitative methodology for studying nonlinear systems of dissipative character.*

Such a methodology allows us to answer a number of questions of nonlinear analysis, in particular, to the following question, which is principal for us: *is it possible to find a pair of functions $y_n$ and $s$ from certain classes such that in a neighborhood of the origin of the phase plane, there exist stable limit cycles of the split system*?

2. *Search for possible integrable cases.*

It seems to be not possible to construct the general theory of studying systems of ordinary differential equation, even for systems of not very general form. Therefore, this work is not a next attempt in this direction. It only generalizes some qualitative studies in the dynamics of a rigid body interacting with a medium, which were initiated long ago. Therefore, we meet dynamical systems having very interesting properties.

The third direction is characteristic for applied aerodynamics and is specific in the framework of this work.

3. *Search for possible analogs between the clamped and free bodies dynamics*.

### 2.9 Principal applied question of nonlinear analysis

The instability of the rectilinear translational drag allose us to pose the principal question of nonlinear analysis in studying a finite neighborhood of such a motion. Precisely, *is it possible to find a pair of dynamical functions $y_N$ and $s$ for describing the conceptual body motion such that in a finite neighborhood of this stationary motion, there exist stable limit cycles?*

One of the main results of this work is a *partly negative answer* to this question, precisely, *for a quasi-static description of the interaction between the medium and the body, when the dynamical quantities $y_N$ and $s$ depend only on the angle of attack, for any admissible pair of obtained dynamical functions $y_N(\alpha)$ and $s(\alpha)$, in the whole range of finite angles of attack $\alpha \in (0, \pi/2)$, there are no any auto-oscillations in the system considered*.

To attain a possible *affirmative answer to the principal question of nonlinear analysis*, in modelling the interaction of a body with a medium, we use an additional damping medium action, which gives a dissipation to the system. Therefore, in principle, under certain conditions, the appearance of stable auto-oscillations is possible in the framework of the model considered, but, however, the search for a body exhibiting the necessary properties requires an additional experiment. This result is not only one of the main results of the present work, but it opens a *new direction* in analytical studying the interaction of a body with a medium with account for the medium damping actions.

### 2.10 On variable dissipation in system

After certain simplifications, the general system (1)–(3) reduces to second-order pendulum systems in which there is a linear dissipative force with variable coefficient alternating the sign for different angles of attack.

In this case, we therefore speak about the system with the so-called *variable dissipation*, where the term "variable" mainly refers not to the value of the dissipation coefficient but to its *sign*.

In the mean, during the period with respect to angle of attack, the dissipation can be positive, negative, or zero. In the latter case we speak about the *system with variable dissipation in the mean*.

### 2.11 General character of symmetries in system for plane and spatial dynamics

In what follows, it is necessary to note at once an important mechanical analogy arising on the basis of qualitative properties of the free body stationary motion in the medium flow and the pendulum equilibrium. Such an analogy allows us to extend the properties of pendulum nonlinear dynamical systems to the dynamical systems of a free body and obtain some topological analogies. For example, under condition (5), the angle of turn of the pendulum is completely equivalent to the angle of attack in the free body motion. In condition (5) violates (the group of conditions (4)), then the angle of attack of the free body and the angle of turn of the pendulum are trajectorially topologically equivalent.

Moreover, under additional condition such an equivalence also extends to the spatial case, which allows us to speak about the *general character of symmetries which are in the system under the plane-parallel, as well as spatial motions* (for the plane and spatial variants of the pendulum.

We now arrive at one more problem: *the extension of the model for describing a free drag to the spatial case*. This problem is the next labor-consuming stage of the study, and, moreover, the complexity extent of the natural experiment considerably increases in this case. Also, it should be noted that in studying the spatial free drag, we can obtain an analogous linear dynamical system, which allows us to use the methodology for finding the parameters of the medium action on the rigid body under the spatial motion (for such a methodology for the plane motion. In this case, the trajectory beams from the plane dynamics are the projections of analogous spatial beams.

Note that in realty, in performing every natural experiment, the information about only one point is obtained. If the secondary repeated experiment is possible, then one can obtain the information, which is different from, although close to the above information. Thus, the condensation of a large number of experimental points creates not only a complete picture of the real trajectory but certain difficulties in finding the real initial conditions , as well as dimension-free parameters of the medium action on the body. To overcome such difficulties, it seems to be appropriate to obtain the information about at least two point for each experimental trajectory that correspond to the same initial conditions of motion.

Since the principal question of nonlinear analysis was formulated above, after discussing the linearized system, we form a number of nonlinear dynamical system in the quasi-velocity space depending on two dynamical functions and describing various classes of the body motion in the medium under the quasi-stationarity conditions. A complete nonlinear analysis of such systems is performed in the next sections by the methods of qualitative theory known early, as well as by new methods obtained exclusively for the arisen systems with variable dissipations.

### 2.12 General methodology of study

The assertions obtained in the work for variable dissipation system are a continuation of the Poincarè–Bendixon theory for systems on closed two-dimensional manifolds and the topological classification of such systems.

The problems considered in the work stimulate the development of qualitative tools of studying, and, therefore, in a natural way, there arises a qualitative variable dissipation system theory.

In this case, we mainly focus on the topological classification of trajectory types and domains of trajectory location in the phase space. The analysis is performed analogously to the classical works, as well to recent works.

## 3 COMPLETE INTEGRABILITY OF CERTAIN CLASSES OF NONCONSERVATIVE SYSTEMS

Also, we show that for homogeneous circular cylinders moving in the water, the rectilinear translational drag is not stable for any dynamical and geometric parameters of such cylinders. Probably, this is related to the motion of the cylinders in the water, when the water damping is inessential, which does not allow us to speak about the stability of the rectilinear translational damping. However, for cylinders having a hole in their interior, the attainment of the above stability is possible under certain conditions.

Therefore, under certain conditions, the account for the medium damping action on a rigid body leads to an affirmative answer to the principal question of nonlinear analysis: *under the* *body motion in a medium with finite angles of attack, in principle, the appearance of stable auto-oscillations is possible. Moreover, for circular cylinder, the appearance of stable and unstable auto-oscillations is possible!*

All what said above, allows us to estimate the results of the work as a whole as a *new direction in analytical mechanics of a rigid body interacting with a medium.*

The results of the presented work are appeared owing to the study of the applied problem on the rigid body motion in a resisting medium where complete lists of transcendental first integrals expressed through a finite combination of elementary functions were obtained. This circumstance allows one to perform a complete analysis of phase trajectories and show those properties of them which exhibit the *roughness* and preserve for systems of a more general form. The complete integrability of that systems is related to symmetries of latent type. Therefore, the study of sufficiently wide classes of systems having analogous latent symmetries is of interest.

As is known, the concept of integrability is sufficiently broad and indefinite in general. In its constructing, it is necessary to take into account what it means (we mean a certain criterion according to which one makes a conclusion about that the structure of trajectories of the system considered is especially "attractive and simple"), in which function classes the first integrals are sought for, etc.

In this work, we follow the approach, which takes the transcendental and, moreover, elementary functions as the function class for first integrals. Here, the *transcendence is understood not in the sense of elementary function theory* (fir example, trigonometrical functions), but in the sense that these function have essentially singular points (by the classification accepted in theory of functions of one complex variable, when a function has essentially singular points). Moreover, we need to formally continue them to the complex domain. As a rule, such systems are strongly nonconservative.

### 3.1 Preliminaries

Of course, in the general case, it is too difficult to construct a certain theory of integrating nonconservative systems (at least of lower dimension). But in a number of cases where the systems under study exhibit additional symmetries, we can succeed in finding their first integrals through finite combinations of elementary functions.

In our more early works, we consider the class of problems from the rigid body dynamics in which the characteristic time of the body motion with respect to its center of masses is comparable with the characteristic time of motion of the center of masses itself. As was noted early, the complexity of solving such problems depends on many factors, including the character of the exterior force field. For example, in the case of conservative (gravity) force field, the motion of a body around its center of masses can be strongly chaotic (the classical problem of the heavy body motion around a fixed point). In this case, it is not possible to construct a somewhat general theory of integration; a natural possibility to proceed further is to impose some restriction on the rigid body geometry and the necessity of existence of some groups of even latent symmetries for the force field considered.

The presented work arises from the problem of the heavy body motion in a resisting medium, where the contact surface of the body with the medium is only a plane part of its exterior surface. In this case, the force field is constructed from the reasons for the medium action on the body in the streamline (or interrupted) flow around under the quasi-stationarity conditions. it turns out that the study of such a class of bodies reduces either to systems with energy scattering (*dissipative systems*) or to systems with energy supplying (the so-called *antidissipative systems*). Note that similar problems were already appeared in applied aerodynamics in the studies of

Central Aero- and Hydrodynamics Institute named after professor N. E. Zhukovskii.

Also, the classes of plane-parallel and spatial motions of rigid bodies interacting with a medium were considered early; among them (depending on the number of degrees of freedom), we can highlight the following: the motion of bodies free in a medium that is immovable at infinity and bodies partially clamped and situated in the running-up medium flow. One of such problems having the greatest applied significance was studied in detail; this is the problem of the body free drag in a resisting medium. Moreover, the problem of free body motion in the presence of a tracking force and also the problem of oscillation of a clamped pendulum in the running-up medium flow were considered.

The problems considered early stimulate the development of the qualitative tools for studying, which essentially complement the qualitative theory of nonconservative systems with dissipation and antidissipation (i.e., *systems with dissipative and dispersive forces*).

Therefore, we have studied the dynamical equations of motion arising in studying the plane and spatial dynamics of a rigid body interacting with a medium, which leads to a possible generalization of the method for studying obtained early to the general systems arising in qualitative theory of ordinary differential equations, in dynamical system theory, as well as in oscillation theory. Also, we have studied qualitatively nonlinear effects in the plane and spatial dynamics of a rigid body interacting with a medium; on the qualitative level, we have justified the necessity of introducing the concepts of relative roughness and relative non-roughness of various degrees.

Many results of this work were regularly reported at a number of workshops, including the workshop ‘Actual Problems of Geometry and Mechanics’ named after professor V. V. Trofimov, which is headed by professor D. V. Georgievskii and professor M. V. Shamolin.

Therefore, the work deals with certain aspects of mathematical modelling the medium action on a rigid body under the quasi-stationarity conditions, These aspect composes an initial idea about the problems of qualitative nature arising further.

### 3.2 Variable dissipation dynamical systems and their general properties

Generally speaking, the dynamics of a rigid body interacting with a medium is just a field, where there arise either dissipative systems or systems with the so-called *antidissipation* (energy supporting inside the system itself). Therefore, it becomes urgent to construct a methodology precisely for those classes of systems which arise in modelling body motion the contact surface of which is a plane part, the simplest part of their exterior surface.

Since in such a modelling, one uses the experimental information about the streamline flow around properties, there arises a necessity of studying the class of dynamical systems that exhibit the (relative) *structural stability* property. Therefore, it is quite natural to introduce the definitions of relative roughness for such systems. In this case, many of the system considered are turned out to be (absolutely) rough in the Andronov–Pontryagin sense.

After certain simplifications, we can reduce the system of equations for the plane-parallel motion to the second-order pendulum systems in which there is a linear dissipative force with variable coefficient whose sign alternates for different values of the periodic phase variable in the system.

As was already noted early, there are important mechanical analogies arising when comparing qualitative properties of the free body stationary motion and the pendulum equilibrium in the medium flow. Such analogies have a deep supporting meaning, since they allows us to extend the properties of the pendulum nonlinear dynamical systems to the free body dynamical systems. Both classes os systems belong to the class of the so-called *pendulum dynamical* *systems with zero mean variable dissipation*.

Under additional conditions, we also extend the above equivalence to the case of spatial motion, which allows us to speak about the *general character of symmetries* in a system with zero mean variable dissipation under the plane-parallel, as well as spatial motions.

### 3.3 Examples from dynamics

Below, we highlight the classes of essentially nonlinear systems of the second and third orders integrable in transcendental (in the sense of theory of functions of one complex variable) elementary functions. For example such systems are five-parametric dynamical systems including the majority of systems that are studied early in the dynamics of a rigid body interacting with a medium [6, 7]:

$$\dot{\alpha} = a\sin\alpha + b\omega + \gamma_1\sin^5\alpha + \gamma_2\omega\sin^4\alpha + \gamma_3\omega^2\sin^3\alpha + \gamma_4\omega^3\sin^2\alpha + \gamma_5\omega^4\sin\alpha,$$

$$\dot{\omega} = c\sin\alpha\cos\alpha + d\omega\cos\alpha + \gamma_1\omega\sin^4\alpha\cos\alpha + \gamma_2\omega^2\sin^3\alpha\cos\alpha +$$
$$+\gamma_3\omega^3\sin^2\alpha\cos\alpha + \gamma_4\omega^4\sin\alpha\cos\alpha + \gamma_5\omega^5\cos\alpha.$$

In this connection, it is of reason to introduce the definitions of relative structural stability (relative roughness) and relative structural instability (relative non-roughness) of various degrees. The latter two properties are proved for the systems arising in the dynamics of a rigid body interacting with a medium, e.g.

As is known, the (purely) dissipative dynamical systems (as well as (purely) antidissipative systems), which in our case can belong to the class of systems with zero mean variable dissipation are as a rule structurally stable ((absolutely) rough), and, on the contrary, the systems with zero mean variable dissipation (which, as a rule, have additional symmetries) are either structurally unstable (non-rough) or only relatively structurally stable (relatively rough). It is difficult to prove the latter assertion in the general case. However, the introduction of the concept relative roughness (and also relative non-roughness of various degrees) allows us to present the classes of concrete systems from the rigid body dynamics that exhibit the above properties.

For example a dynamical system of the form

$$\dot{\alpha} = \Omega + \beta\sin\alpha, \quad \dot{\Omega} = -\beta\sin\alpha\cos\alpha$$

is relatively structurally stable (relatively rough) and topologically equivalent to the system describing a clamped pendulum placed in the running-out medium flow [6, 7] (its phase portrait is depicted in Fig. (3), $\alpha \to -\alpha$).

We can find its first integral being a transcendental (in the sense of theory of functions of one complex variable such that it has essentially singular points after continuing it into the complex domain) functions of phase variables that is expressed through a finite combination of elementary functions.

As is seen, the phase cylinder $\mathbf{R}^2\{\alpha, \Omega\}$ of quasi-velocities of the system considered exhibits an interesting topological structure of partition into trajectories.

On the cylinder, there are two domains (whose closure is just the phase cylinder) filled in by trajectories of perfectly different character.

The first domain called oscillatory or finitary (it is simply connected (see Fig. (3))) is entirely filled in by trajectories of the following type. Almost every such trajectory starts at the repelling point $(2\pi k, 0)$ and ended at the attracting point $((2k+1)\pi, 0)$, $k \in \mathbf{Z}$. An exception is the fixed points $(\pi k, 0)$ and separatrices that either emanate from the repelling point $(2\pi k, 0)$ and enter

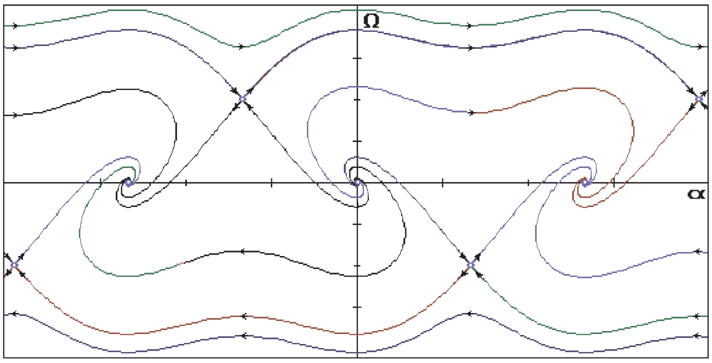


Figure 3: Relatively rough dynamical system.

the saddles $S_{2k}$ and $S_{2k+1}$ or emanate from the saddles $S_{2k+1}$ and $S_{2k+2}$ and enter the attracting points $((2k+1)\pi, 0)$. Here,

$$S_k = \left(-\frac{\pi}{2} + \pi k, (-1)^k \beta\right).$$

The second domain called rotational (it is two-connected (see Fig. (3)) is entirely filled in by rotational motions similar to rotations on the mathematical pendulum plane. These phase trajectories envelope the phase cylinder and are periodic on it.

Although the dynamical system considered is not conservative, in the rotational domain of its phase plane $\mathbf{R}^2\{\alpha, \Omega\}$ it admits the preservation of an invariant measure with variable density. This property characterizes the system considered as a system with zero mean variable dissipation.

Key separatrices (for example, the separatrix emanating from the point $(-\pi/2, \beta)$ and entering the point $(3\pi/2, \beta)$) are boundaries of domains in which the motion is of different character. So, in the oscillatory domain containing the repelling and attracting equilibrium points, almost all trajectories have attractors and repellers as limits sets. Hence there are no even absolutely continuous function being the density of an invariant measure in this domain.

The matter is different for the domain entirely filled in by rotational motions. As was shown early, there exists a smooth function being the density of the invariant measure in the domain entirely filled in by periodic trajectories not contracting to a point along the phase cylinder.

### 3.4 First results

Some results of the dynamics if plane-parallel motion also extend to the spatial case; in this connection, the author has posed the spatial problem early. In particular, we have found a complete list of integrals in the problem of the spatial motion of a dynamically symmetric clamped rigid body placed in the running-out medium flow. This system with zero mean variable dissipation is topologically equivalent to the spatial motion of a rigid body in a resisting medium in which a non-integrable constraint is imposed on the body (it is realized by a certain additional tracking force).

The spatial motion of a rigid body in a resisting medium under which the center of masses executes the rectilinear uniform motion is also a dynamical system with zero mean variable dissipation. its qualitative study allows us to present a convenient comparison system for many systems with zero mean variable dissipation.

We have obtained a family of phase portraits in the problem of the spatial body free drag in a resisting medium. In these works, we develop the technique for studying a neighborhood of a singular equilibrium state, i.e., an equilibrium state such that the right-hand sides of dynamical systems are extended to it only by continuity. For example, for small phase variables $\alpha$ and $\Omega$, the right-hand side of the system has a singularity of the form $1/\alpha$. This difficulty is overcame by a specific constructing a Lyapunov function.

### 3.5 One of the definitions of systems with zero mean variable dissipation

We study the system of ordinary differential equations having a periodic phase coordinate. The systems under study have those symmetries for which in the mean during the period of the periodic coordinate, their phase volume preserves. So, for example, the following pendulum system with smooth right-hand side $\mathbf{V}(\alpha, \omega)$ periodic in $\alpha$ of period $T$:

$$\dot{\alpha} = -\omega + f(\alpha),\ \dot{\omega} = g(\alpha),\ f(\alpha + T) = f(\alpha),\ g(\alpha + T) = g(\alpha),$$

preserves its phase area on the phase cylinder during the period $T$:

$$\int_0^T \mathrm{div}\mathbf{V}(\alpha, \omega)\, d\alpha = \int_0^T \left( \frac{\partial}{\partial \alpha}(-\omega + f(\alpha)) + \frac{\partial}{\partial \omega} g(\alpha) \right) d\alpha = \int_0^T f'(\alpha)\, d\alpha = 0.$$

The system considered is equivalent to the pendulum equation

$$\ddot{\alpha} - f'(\alpha)\dot{\alpha} + g(\alpha) = 0,$$

in which the integral if the coefficient $f'(\alpha)$ standing by the dissipative term $\dot{\alpha}$ vanishes in the mean during the period.

It is seen that the system considered has those symmetries for which it becomes a system *with zero mean variable dissipation* in the sense of the following definition.

*Definition.* Let us consider a smooth autonomous system of the $(n+1)$th order and normal form defined on the cylinder $\mathbf{R}^n\{x\} \times \mathbf{S}^1\{\alpha \bmod 2\pi\}$, where $\alpha$ is a periodic coordinate of period $T > 0$. Denote by div$(x, \alpha)$ the divergence of the right-hand side (which is a function of all phase variables in general and is not identically equal to zero) of this system. Such a system is called a system with zero (nonzero) mean variable dissipation if the function

$$\int_0^T \mathrm{div}(x, \alpha)\, d\alpha$$

is (is not) identically equal to zero. Moreover, in some cases (for example, when at certain points of the circle $\mathbf{S}^1\{\alpha \bmod 2\pi\}$, there arise singularities), this integral is understood in the principal value sense .

It should be noted that it is sufficiently difficult to give the general definition of a system with zero (nonzero) mean variable dissipation. The definition just presented uses the concept of divergence (as is known, the divergence of the right-hand side of a normal form system characterizes the variation of the phase volume in the phase space of the system).

### 3.6 Systems with symmetries and zero mean variable dissipation

Let us consider systems of the form (the dot denotes the derivative in time)

$$\dot{\alpha} = f_\alpha(\omega, \sin\alpha, \cos\alpha),\ \dot{\omega}_k = f_k(\omega, \sin\alpha, \cos\alpha),\ k = 1, \ldots, n, \tag{6}$$

defined on the set $\mathbf{S}^1\{\alpha \bmod 2\pi\} \setminus K \times \mathbf{R}^n\{\omega\}$, $\omega = (\omega_1, \ldots, \omega_n)$, where the functions $f_\lambda(u_1, u_2, u_3)$, $\lambda = \alpha, 1, \ldots, n$, of three variables $u_1$, $u_2$, and $u_3$ are as follows:

$$f_\lambda(-u_1, -u_2, u_3) = -f_\lambda(u_1, u_2, u_3),\ f_\alpha(u_1, u_2, -u_3) = f_\alpha(u_1, u_2, u_3),$$

$$f_k(u_1, u_2, -u_3) = -f_k(u_1, u_2, u_3).$$

The set $K$ is either empty or consists of finitely many points of the circle $\mathbf{S}^1\{\alpha \bmod 2\pi\}$.

The latter two variables $u_2$ and $u_3$ in the functions $f_\lambda(u_1, u_2, u_3)$ depend on one parameter $\alpha$, but they are distinguished in separate groups for the following reasons. First, not in the whole their domain, they are uniquely expressed from one another, and, second, the first of them is odd, whereas the second is an even function of $\alpha$, which influences on the symmetry of system (6) differently.

To this system, we put in correspondence the non-autonomous system

$$\frac{d\omega_k}{d\alpha} = \frac{f_k(\omega, \sin\alpha, \cos\alpha)}{f_\alpha(\omega, \sin\alpha, \cos\alpha)};$$

by the substitution $\tau = \sin\alpha$, it reduces to the form ($k = 1, \ldots, n$)

$$\frac{d\omega_k}{d\tau} = \frac{f_k(\omega, \tau, \varphi_k(\tau))}{f_\alpha(\omega, \tau, \varphi_\alpha(\tau))},\ \varphi_\lambda(-\tau) = \varphi_\lambda(\tau), \lambda = \alpha, 1, \ldots, n.$$

In particular, the right-hand side of latter system can be algebraic (i.e., it can be a ration between two polynomials); sometimes, this helps to search for its first integrals in explicit form.

The following assertion immerses the class of systems (6) in the class of dynamical systems with zero mean variable dissipation. The inverse embedding does not hold in general.

**Theorem.** *Systems of the form (6) are dynamical system with zero mean variable dissipation.*

This proposition is proved by using the symmetries of system (6) listed above.

The converse assertion is not true in general, since we can present a set of dynamical systems on a two-dimensional cylinder that are system with zero mean variable dissipation but do not satisfies the properties listed above.

In this work, we mainly consider the case where the functions $f_\lambda(\omega, \tau, \varphi_k(\tau))$ are polynomials in $\omega$ and $\tau$.

To begin with, let us consider a certain class of autonomous systems on the two-dimensional circular cylinder $\mathbf{S}^1\{\alpha \bmod 2\pi\} \times \mathbf{R}^1\{\omega\}$. For example, to the following pendulum systems (arising in the dynamics of a rigid body interacting with a medium [6, 7]) with parameter $\beta > 0$ and the same phase variables (see also the above section 2):

$$\dot\alpha = -\omega + \beta\sin\alpha, \quad \dot\omega = \sin\alpha\cos\alpha, \tag{7}$$

$$\dot\alpha = -\omega + \beta\sin\alpha\cos^2\alpha + \beta\omega^2\sin\alpha,\ \dot\omega = \sin\alpha\cos\alpha - \beta\omega\sin^2\alpha\cos\alpha + \beta\omega^3\cos\alpha \tag{8}$$

in the variables $(\omega, \tau)$, we put in correspondence the following equations with algebraic right-hand side, respectively:

$$\frac{d\omega}{d\tau} = \frac{\tau}{-\omega + \beta\tau}, \frac{d\omega}{d\tau} = \frac{\tau + \beta\omega[\omega^2 - \tau^2]}{-\omega + \beta\tau + \beta\tau[\omega^2 - \tau^2]}.$$

In this case, systems (7) and (8) are dynamical systems with zero mean variable dissipation, which is easy to verify directly.

Moreover, each of them have a first integral being a transcendental (in the sense of theory of functions of one complex variable) function that expresses through a finite combination of elementary functions.

For example, system (7) has a first integral of the following form (depending on the value of $\beta$, three cases are possible that corresponds to the existence of foci, nodes, or degenerate nodes in the phase portrait of the system):

$$\beta^2 - 4 < 0:$$
$$\{\Omega^2 + \beta\Omega\sin\alpha + \sin^2\alpha\} \times \exp\left\{\frac{2\beta}{\sqrt{-\beta^2+4}} \arctan\frac{2\Omega+\beta\sin\alpha}{\sqrt{-\beta^2+4}\sin\alpha}\right\} = \text{const};$$
$$\beta^2 - 4 > 0:$$
$$|2\Omega + (\beta + \sqrt{\beta^2-4}\sin\alpha)|^{\sqrt{\beta^2-4}-\beta} \times |2\Omega + (\beta - \sqrt{\beta^2-4}\sin\alpha)|^{\sqrt{\beta^2-4}+\beta} = \text{const};$$
$$\beta^2 - 4 = 0:$$
$$|2\Omega + \beta\sin\alpha| \times \exp\left\{-\frac{\beta\sin\alpha}{2\Omega+\beta\sin\alpha}\right\} = \text{const}.$$

The phase portrait of system (8) can be of three different types depending on the values of the parameter $\beta$.

In the expression of its first integral, also three cases are possible depending on the value of the constant $\beta$ and corresponding to the existence of foci, nodes, and degenerate nodes in the phase portrait of the system.

Let us represent the parameter $\beta$ as the product: $\beta = \sigma^2 n_0^2$; after that, to system (8), we put in correspondence a differential equation of the form

$$\frac{d\omega}{d\tau} = \frac{-n_0^2\tau + \sigma\omega[\omega^2 - n_0^2\tau^2]}{\omega + \sigma n_0^2\tau + \sigma\tau[\omega^2 - n_0^2\tau^2]},\ \tau = -\sin\alpha.$$

Introduce the following notation: $C_1 = 2 - \sigma n_0,\ C_2 = \sigma n_0,\ C_3 = -2 - \sigma n_0$. Performing a number of changes by the formulas $\omega - n_0\tau = u_1,\ \omega + n_0\tau = v_1,\ u_1 = v_1 t_1,\ v_1^2 = p_1$, where $v_1 \neq 0$, we obtain the Bernoulli-type equation

$$2p_1\left[C_1 t_1 + C_2 + \frac{2\sigma}{n_0}t_1 p_1\right] = \frac{dp_1}{dt_1}[C_3 - C_1 t_1^2].$$

By the known change $p^{-1} = q_1$ for $p_1 \neq 0$, we reduce the latter equation to the form

$$\dot{q_1} = a_1(t_1)q_1 + a_2(t_1),$$

where

$$a_1(t_1) = \frac{2(C_1 t_1 + C_2)}{C_1 t_1^2 - C_3},\ a_2(t_1) = \frac{4\sigma t_1}{n_0(C_1 t_1^2 - C_3)}.$$

(Here, the dot denotes the derivative in $t_1$.)

The general solution of the linear homogeneous equation is represented in the form

$$q_{1\,HOM}(t_1) = k(C_1 t_1^2 - C_3)Q(t_1),\ k = \text{const},$$

where the function $Q$ has the following form depending on the value of the constant $C_1$:

$$Q(t_1) =:$$
$$e^{t_1}, C_1 = 0;$$
$$e^{2\frac{C_2}{\sqrt{-C_1C_3}}\arctan\sqrt{-\frac{C_1}{C_3}}t_1}, C_1 > 0;$$
$$\left(\frac{\sqrt{-C_1}t_1 + \sqrt{-C_3}}{\sqrt{-C_1}t_1 - \sqrt{-C_3}}\right)^{\frac{C_2}{\sqrt{C_1C_3}}}, C_1 < 0.$$

To obtain the solution of the inhomogeneous equation, we assume that the quantity $k$ is a function of $t_1$; we find it by the quadrature

$$k(t_1) = \frac{4\sigma}{n_0} \int Q^{-1}(t_1) \frac{t_1}{(C_1 t_1^2 - C_3)^2}\, dt_1.$$

Therefore, the transcendental first integral of system (8) becomes

$$Q^{-1}(t_1) q_1 (C_1 t_1^2 - C_3)^{-1} - \tfrac{4\sigma}{n_0} \int_{t_0}^{t_1} Q^{-1}(\tau_1) \tfrac{\tau_1}{(C_1\tau_1^2 - C_3)^2}\, d\tau_1 = C^0,$$

where $C^0 = \text{const}$.

As is seen, the final form of the first integral depends on the sign of the constant $C_1$, and, therefore, three variants are possible. Let us examine each of them.

FIRST VARIANT. $C_1 = 0$. After an elementary calculation, we obtain an additional integral in the form

$$e^{-\frac{u_1}{v_1}} \left( v_1^{-2} + \frac{\sigma^2}{2} \left(\frac{u_1}{v_1} + 1\right) \right) = \text{const}.$$

Therefore, for $C_1 = 0$, the transcendental first integral of system (8) is expressed through elementary functions.

SECOND VARIANT. $C_1 > 0$. The integration leads to the function

$$-\frac{\sigma}{4n_0} e^{-2\frac{C_2}{\sqrt{-C_1C_3}}\zeta} \left( \frac{C_2}{\sqrt{-C_1C_3}} \sin 2\zeta + \cos 2\zeta \right),$$

where

$$\zeta = \arctan \sqrt{-\frac{C_1}{C_3}} t_1.$$

As is seen, in the case $C_1 > 0$, the additional first integral is expressed through elementary functions.

THIRD VARIANT. $C_1 < 0$. By equivalent transformations, the integral transforms into

$$\frac{\sigma}{C_1 C_2 n_0} \left( 2\frac{\zeta^{1-\gamma}}{\gamma - 1} - 3\frac{\zeta^{-\gamma}}{\gamma} + \frac{\zeta^{-1-\gamma}}{\gamma + 1} \right),$$

where

$$\gamma = \frac{C_2}{\sqrt{C_1 C_3}} > 1,\ \zeta = \frac{\sqrt{-C_1} t_1 + \sqrt{-C_3}}{\sqrt{-C_1} t_1 - \sqrt{-C_3}}.$$

Therefore, in the case $C_1 < 0$, the additional first integral is also expressed through elementary functions.

And so, we study the connection between the following three properties, which are independent for the first glance, but they are sufficiently harmonically combined on systems from the rigid body dynamics:

1. The distinguished class of systems (6) with the above;
2. The fact that this class of systems consists of systems with zero mean variable dissipation (in the variable $\alpha$), which allows us to consider them as "almost" conservative systems;
3. In certain (although lower-dimensional) cases, these systems have first integrals, which are transcendental in general.

Let us present one more important example of a higher-order system that has the properties just listed [6, 7].

To the system

$$c\dot{\alpha} = -z_2 + \beta \sin\alpha,\ \dot{z}_2 = \sin\alpha\cos\alpha - z_1^2 \frac{\cos\alpha}{\sin\alpha},\ \dot{z}_1 = z_1 z_2 \frac{\cos\alpha}{\sin\alpha}, \tag{9}$$

which is distinguished considered in the three dimensional domain

$$\mathbf{S}^1\{\alpha \bmod 2\pi\} \setminus \{\alpha = 0,\ \alpha = \pi\} \times \mathbf{R}^2\{z_1, z_2\}$$

(such a system can reduce to an equivalent system on the tangent bundle of the two-dimensional sphere), and describes the spatial motion of a rigid body in a resisting medium, we put in correspondence the following system with algebraic right-hand side:

$$\frac{dz_2}{d\tau} = \frac{\tau - z_1^2/\tau}{-z_2 + \beta\tau}, \quad \frac{dz_1}{d\tau} = \frac{z_1 z_2/\tau}{-z_2 + \beta\tau}. \tag{10}$$

In this case, it is also seen that system (9) is a system with zero mean variable dissipation; in order to obtain a complete correspondence with the definition, it suffices to introduce the new phase variable $z_1^* = \ln|z_1|$.

Moreover, this system has two first integrals (i.e., the full list), which are transcendental functions and are expressed through a finite combination of elementary functions; as was mentioned above, this become possible after establishing its correspondence to the (non-autonomous in general) system of equations (10) with algebraic (polynomial) right-hand side.

Therefore, the systems from the rigid body dynamics presented above not only enter the class of systems (6) and have the mean zero variable dissipation, but they have a full list of transcendental first integrals expressed through a finite combination of elementary functions. In this case, the integration of systems (7) and (8) reduces to the integration of the corresponding equations with algebraic right-hand side.

As was noted, to seek for first integrals of the systems considered, it is better to reduce systems of the form (6) to systems with polynomial right-hand sides; the possibility of integrating the initial system in elementary functions depends on their form. Therefore, we proceed as follows: we seek for sufficient conditions for integrability in elementary functions of systems of equations with polynomial right-hand sides studying systems of the most general form in this case.

## 4 CONCLUSION

The results of the presented work were appeared owing to the study the applied problem of the rigid body motion in a resisting medium, where we have obtained complete lists of transcendental first integrals expressed through a finite combination of elementary functions. This circumstance allows the author to carry out the analysis of all phase trajectories and show those their properties which have the roughness and are preserved for systems of a more general form. The complete integrability of such system is related to their symmetries of latent type. Therefore, it is of interest to study a sufficiently wide class of dynamical systems having analogous latent symmetries.

So, for example, the instability of the simplest body motion, the rectilinear translational drag, is used for methodological purposes, precisely, for finding the unknown parameters of the medium action on a rigid body under the quasi-stationarity conditions.

The experiment on the motion of homogeneous circular cylinders in the water carried out in Institute of Mechanics of M. V. Lomonosov State University justified that in modelling the medium action on the rigid body, it is also necessary to take into account an additional parameter that brings a dissipation to the system.

In studying the class of body drags with finite angle of attack, the principal problem is finding those conditions under which there exist auto-oscillations in a finite neighborhood of the rectilinear translational drag. Therefore, there arises the necessity of a complete nonlinear study.

The initial stage of such a study is the neglecting of the medium damping action on the rigid body. Functionally, this means the assumption that the pair of dynamical functions determining the medium action depends only on one parameter, the angle of attack. The dynamical systems arising under such a nonlinear description are variable dissipation systems. Therefore, there arises the necessity to create the methodology for studying such systems.

Generally speaking, the dynamics of a rigid body interacting with a medium is just the field where there arise either nonzero mean variable dissipation systems or systems in which the energy loss in the mean during a period can vanish. In the work, we have obtained such a methodology owing to which it becomes possible to finally and analytically study a number of plane and spatial model problems.

In qualitative describing the body interaction with a medium, because of using the experimental information about the properties of the streamline flow around, there arises a definite dispersion in modelling the force-model characteristics. This makes it natural to introduce the definitions of relative roughness (relative structural stability) and to prove such a roughness for the system studied. Moreover, many systems considered are merely (absolutely) Andronov–Pontryagin rough.

## 5 ACKNOWLEDGEMENTS

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