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STABILITY OF A RIGID BODY TRANSLATING IN A RESISTING MEDIUM*

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The paper discusses a nonlinear model that describes the interaction of a rigid body with a medium and takes into account (based on experimental data on the motion of circular cylinders in water) the dependence of the arm of the force on the normalized angular velocity of the body and the dependence of the moment of the force on the angle of attack. An analysis of plane and spatial models (in the presence or absence of an additional follower force) leads to sufficient stability conditions for translational motion, as one of the key types of motions. Either stable or unstable self-oscillation can be observed under certain conditions

Keywords: rigid body in a resisting medium, nonlinear model, sufficient stability conditions, stable and unstable self-oscillation

1. Introduction. Experimental data on the motion of homogeneous circular cylinders in water [7, 12] show that the dependence of the moment of force of the medium on the angular velocity of the body should also be taken into account. Then the equations of motion will include additional terms describing dissipation in the system.

The nonlinear analysis of the motion of a body with finite angles of attack concentrates on establishing the conditions under which there exist finite-amplitude oscillations near the unperturbed motion, which confirms the necessity of a comprehensive nonlinear analysis. Studies of the plane and spatial models describing the interaction of a rigid body and a medium (in the presence or absence of an additional follower force) have established sufficient stability conditions for translational motion, which is a key type of motion. It was shown that there are conditions under which stable or unstable self-oscillatory motion is possible.

Since the nonlinear analysis is complex, the initial stage of such a study disregards the dependence of the moment of force of the medium on the angular velocity of the body and considers the dependence on the angle of attack alone [9, 11, 16].

Of practical importance is the stability analysis of the so-called unperturbed (translational) motion such that the velocities of points of the body are perpendicular to the plate (cavitator).

All the results obtained under this elementary assumption allows concluding that there no conditions under which the systems would have solutions describing angular finite-amplitude oscillations of the body.

The present paper is the next stage in the study of a moving rigid body interacting with the medium only by the flat front area (plate). The force exerted by the medium is found using the properties of a quasistationary jet flow [9]. The motion of the medium is not studied, and the characteristic time of motion of the rigid body relative to the center of mass is commensurable with the characteristic time of motion of this center.

2. Plane-Parallel Motion of a Symmetric Rigid Body in a Resisting Medium: Problem Statement. Let a homogeneous rigid body of mass m undergo plane-parallel motion in a homogeneous flow of a medium. A portion of the outside

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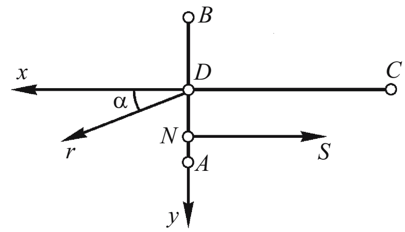

Fig. 1

surface of the body is a flat plate AB subject to a jet flow of the medium. This means that if tangent forces are absent, the effect of the medium on the plate is the force $\boldsymbol{S}$ (applied at the point N) orthogonal to it (Fig. 1). The other portion of the body surface can be placed in a volume bounded by the surface of the jet stalling at the edge of the plate and is not affected by the medium. Conditions are similar when the body enters water [4, 5, 7, 12, 14, 16]. The gravity is assumed negligible compared with the resistance of the medium (see also [20–24]).

To describe the plate, we choose a right-hand coordinate system $Dxyz$ with the z-axis perpendicular to the plane of the body. For simplicity, we assume that Dzx is the plane of geometrical symmetry of the body. Then the possible types of motions include translational deceleration (unperturbed motion) perpendicular to the plate AB, the median perpendicular Dx dropped from the body's center of gravity C to the plate being aligned with the line of action of the force $\boldsymbol{S}$. In this motion, the velocity vector $\boldsymbol{v}$ of the point D deviates, in the general case, from the axis DC of geometrical symmetry by an angle (of attack) α.

To construct a dynamic model, we introduce the first three phase coordinates: the velocity v of the point D relative to the flow of the medium (Fig. 1), the angle α, and the algebraic value Ω of the projection of the absolute angular velocity of the body onto the z-axis; $AB = \Delta$.

Let the force $\boldsymbol{S}$ be quadratically dependent ($S = s_1 v^2$) on v with a coefficient s_1 (Newtonian drag), which is usually expressed as $s_1 = \rho P c_x / 2$, where c_x is the (already dimensionless) drag coefficient (ρ is the density of the medium, P is the plate area). This coefficient depends on the angle of attack, the Strouhal number, and other quantities usually used as parameters in static models. We introduce a dimensionless Strouhal-type phase variable $\omega \cong \Omega\Delta / v$ and an auxiliary function $s = s_1 \operatorname{sgn} \cos \alpha$, the effect of the medium on the body being described by a pair of functions (y_N, s).

We will restrict ourselves to the dependence of the coefficient c_x on the angle of attack, i.e., we assume that s is a function of α, and $y_N = DN$ is a function of the pair of dimensionless variables (α, ω).

As already mentioned, only the dependence of the pair (y_N, s) on the angle of attack alone is taken into account in studying two-dimensional interaction in [9, 11]. In what follows, we will analyze the plane-parallel and spatial motions of the body with s depending on the angle of attack and the function y_N additionally depending on the reduced angular velocity ω.

The free-deceleration problem (the body is only subject to drag (see case (i) below)) for small angles of attack gives further insight into nonlinear dynamic systems that describe the interaction between a medium and a body and account for the so-called rotary derivatives of the moment of force of the medium with respect to the angular velocity of the body. The term "rotary derivative" is often used in hydrodynamics when dynamic functions are differentiated in a non-inertial coordinate frame. If the moment of force depends on the angular velocity, then it appears (linearly in this velocity) in the equations of motion.

The unperturbed motion is described by the equalities $\alpha(t) \equiv 0, \omega(t) \equiv 0$. Therefore, with small (α, ω), the function $y_N(\alpha, \omega)$ is represented as

$$y_N = \Delta(k\alpha - h\omega), \tag{2.1}$$

where k and h are some constants. We neglect the dependence of s on α because the body is geometrically symmetric and, hence, the function s is even.

The linearized model describing the mechanical effect of the medium includes three parameters $s = s_1, k$, and h determined by the planform of the plate. The coefficient s is dimensional, while the parameters k and h are dimensionless by definition. The values of s and k can be found experimentally by weight measurements in, for example, water or wind tunnels (see [23] for the theoretical determination of these values for some plates with $k > 0$). As far as the parameter h, which accounts for the dependence of the moment of force on the angular velocity, is concerned, it is not even obvious a priori that it should be included in the model.

The motion of the bodies under consideration was studied at the Research Institute of Mechanics of the M. V. Lomonosov Moscow State University [7, 12]. The studies began with experiments on homogeneous circular cylinders moving in water. The experiments lead us to the following conclusions: (i) the unperturbed motion of the body (in water) is unstable, at least with respect to the angle of orientation of the body; it is also possible to determine the dimensionless parameters k and h; (ii) in modeling the effect of the medium on the body, it is indeed necessary to account for an additional parameter that is equivalent to the rotary derivative of the moment of hydroaerodynamic forces with respect to the angular velocity of the body and responsible for additional dissipation in the system.

The damping moment coefficient was evaluated in [5] for some cases of motion of bodies in water. It was established that the unperturbed motion of a rigid body in water is unstable with respect to the angle of attack and angular velocity. By formally increasing this coefficient, we can make this motion stable, which is stable in the above sense in some media (such as clay) according to the experiment [1]. However, this stability may be due to intensive damping by the medium or forces tangential to the plate.

The assumptions on the body–medium interaction being the same, we will consider the class of problems where the body is subjected not only to drag, but also to a follower force (thrust) **T** along the straight line CD (Fig. 1). One of such problems was addressed in [6, 8] in the case of constant thrust, and it was shown that the unperturbed motion is unstable.

Noteworthy are the following two cases of motion analyzed in detail: (i) (free) deceleration, i.e., the body is subject only to drag (the follower force is absent) and (ii) the velocity of the center of the plate is constant all the time (a nonintegrable constraint is present), i.e.,

$$v \equiv \text{const}. \tag{2.2}$$

Note that in case (i), the unperturbed motion may also be called translational deceleration.

The position of the body on the plane is specified by the coordinates (x_0, y_0) of the point D and by an angle φ. The polar coordinates (v, α) of the tip of the velocity vector of the point D and the algebraic value of the projection of the angular velocity Ω are related to the variables $(x_0^{\bullet}, y_0^{\bullet}, \varphi^{\bullet}, \varphi)$ as follows:

$$\varphi^{\bullet} = \Omega, \quad x_0^{\bullet} = v\cos(\alpha + \varphi), \quad y_0^{\bullet} = v\sin(\alpha + \varphi), \tag{2.3}$$

which are (nonintegrable) kinematic relations.

Thus, the phase state of the system is determined by the functions $(v, \alpha, \Omega, x_0, y_0, \varphi)$, the first three quantities being considered quasivelocities.

Since the kinetic energy of the body and the generalized forces do not depend on its position on the plane, the coordinates (x_0, y_0, φ) are cyclic, which reduces the order of the system of equations of motion.

The equations of motion of the center of mass (for the projections onto the body axes Dxy) and the angular momentum equations in the König frame form a closed system of differential equations in the three-dimensional phase space of quasivelocities ($\sigma = DC$, I is the central moment of inertia; differentiation is with respect to time)

$$v^{\bullet} \cos\alpha - \alpha^{\bullet} v \sin\alpha - \Omega v \sin\alpha + \sigma\Omega^2 = -s(\alpha)v^2 / m, \tag{2.4}$$

$$v^{\bullet} \sin\alpha + \alpha^{\bullet} v \cos\alpha + \Omega v \cos\alpha - \sigma\Omega^{\bullet} = 0, \tag{2.5}$$

$$I\Omega^{\bullet} = y_N(\alpha, \omega) s(\alpha) v^2, \quad \omega \cong \Omega\Delta / v. \tag{2.6}$$

Systems (2.3), (2.4)–(2.6) form a complete system of equations to describe the quasistationarity plane-parallel motion of a rigid body in a resisting medium. In the case of problem (ii), the right-hand side of Eq. (2.4) includes $(T - s(\alpha)v^2)/m$.

In particular, for condition (2.2) to be satisfied, it is sufficient that

$$T = T(v, \alpha, \Omega) = m\sigma\Omega^2 + v^2[s(\alpha) - (m\sigma y_N(\alpha, \omega)s(\alpha)\sin\alpha)/I\cos\alpha],$$

the first equation in (2.4) holding identically. Note that case (ii) is of methodical importance only since it allows reducing the order of the system of equations of motion and leads to an important mechanical analogy [17, 19].

3. Influence Functions Dependent on the Angular Velocity of the Body. The dynamic system (2.4)–(2.6) includes the functions $y_N(\alpha,\omega)$ and $s(\alpha)$, which describe the effect of the medium on the body. The function y_N (compare with (2.1)) depend on both angle of attack α and reduced angular velocity ω. If the dependence on the angular velocity is neglected (which is the so-called elementary restriction on the influence functions, as in some previous studies), then y_N is a function of the angle of attack alone: $y_N = y(\alpha)$, and how it depends on the unique argument is determined from experimentally found properties of jet flow [4, 13, 14]. In this case, the problem can be "immersed" in a more general class of problems.

The objective of this paper is to account for the rotary derivatives of the moment of force of the medium with respect to the components of the angular velocity of the body by introducing additional arguments into the influence functions, which is a nontrivial modeling problem. As already mentioned, we will introduce the angular velocity as an argument into the function y_N and will not into the coefficient s.

By analogy with (2.1), y_N is represented as

$$y_N(\alpha,\omega) \cong y_N(\alpha,\Omega/v) = y(\alpha) - H\Omega/v, \tag{3.1}$$

where $H > 0$ due to the experimental results [1, 7, 12, 13].

Then Eq. (2.6) becomes

$$I\Omega^{\bullet} = F(\alpha)v^2 - Hs(\alpha)\Omega v, \qquad F(\alpha) = y(\alpha)s(\alpha). \tag{3.2}$$

System (2.4), (2.5), (3.2) contains the functions $F(\alpha)$ and $s(\alpha)$ that are rather difficult to describe analytically even for plates of simple shape. For this reason, we will "immerse" the problem in a wider class of problems that accounts only for the qualitative properties of the functions $F(\alpha)$ and $s(\alpha)$.

A reference result is due to Chaplygin who have derived analytic expressions of the functions $y(\alpha)$ and $s(\alpha)$ for a plane-parallel jet flow past an infinite plate [13]:

$$y(\alpha) = y_0(\alpha) = A\sin\alpha \in \{y\}, \qquad A > 0, \tag{3.3}$$

$$s(\alpha) = s_0(\alpha) = B\cos\alpha \in \{s\}, \qquad B > 0. \tag{3.4}$$

This result helps constructing the functional classes $\{y\}$ and $\{s\}$. Combining (3.3) and (3.4) with the experimental data on the properties of jet flow [1, 7, 12, 13], we will formally describe these classes, which consist of sufficiently smooth, 2π-periodic functions ($y(\alpha)$ is odd, while $s(\alpha)$ is even) such that $y(\alpha) > 0$ for $\alpha \in (0,\pi)$, with $y'(0) > 0$, $y'(\pi) < 0$ (class of functions $\{y\} = \mathrm{Y}$); $s(\alpha) > 0$ for $\alpha \in (0,\pi/2)$, $s(\alpha) < 0$ for $\alpha \in (\pi/2,\pi)$, with $s(0) > 0, s'(\pi/2) < 0$ (class of functions $\{s\} = \Sigma$). As α is replaced with $\alpha + \pi$, both y and s reverse sign. Thus, $y \in \mathrm{Y}$ and $s \in \Sigma$ and $y \in \mathrm{Y}$.

It follows from the above conditions that F in (3.2) is a sufficiently smooth, odd, π-periodic function such that $F(\alpha) > 0$ for $\alpha \in (0,\pi/2), F'(0) > 0, F'(\pi/2) < 0$ (class of functions $\{F\} = \Phi$).

The analytic function [13] $F = F_0(\alpha) = AB\sin\alpha\cos\alpha \in \Phi$ is a typical representative of the class of functions Φ.

Since unperturbed motion is unstable (as noted in [7, 12]), the question arises of whether there exist finite-amplitude angular oscillations of the body's axis of symmetry. The more general question is whether there exists a pair of functions y and s such that $0 < \alpha(t) < \alpha^* < \pi/2$ for some solution of the dynamic part of the equations of motion beginning with $t = t_1$.

As shown in [17, 19], with the elementary restriction on the functions y_N and s and in the quasistationary case of interaction between a medium and a symmetric body (y_N and s are only dependent on the angle of attack), the system has no oscillatory solutions of finite amplitude for an arbitrary pair of functions (y,s) in the entire range $(0 < \alpha < \pi/2)$ of finite angles of attack.

Thus, to find whether it is possible to answer in the affirmative the above question, we will take into account the dependence of the moment of force of the medium on the reduced angular velocity, using formula (3.1) with $H \neq 0$. It turns out that a yes answer may be expected under some assumptions.

Certainly, only the analysis of the equations of motion in the neighborhood of unperturbed motion is of practical importance since the lateral surface is wetted at some critical angles of attack, which makes our model invalid. For bodies with different lateral surfaces, however, the critical angles are generally different and unknown. Therefore, the entire range of angles has to be examined.

Thus, to study a plane-parallel flow past a plate, use is made of classes of dynamic systems defined in terms of a pair of influence functions, which considerably complicates the qualitative analysis.

4. Motion of a Body in a Resisting Medium under Follower Force. Consider a body moving through a medium under the action of a follower force so that condition (2.2) is satisfied all the time, i.e., this force is such that the first equation in (2.4) holds identically. Then the parameters introduced above are supplemented with the positive parameter v, and the dynamic part of the equations of motion in the case of (2.2) is reduced to the second-order system

$$\alpha^{\bullet} v\cos\alpha + \Omega v\cos\alpha - \sigma\Omega^{\bullet} = 0, \quad I\Omega^{\bullet} = F(\alpha)v^2 - Hs(\alpha)\Omega v, \quad H > 0,$$

which is equivalent to the following normal-form system:

$$\alpha^{\bullet} = -\Omega + \frac{\sigma v}{I}\frac{F(\alpha)}{\cos\alpha} - \frac{\sigma}{I}H\frac{s(\alpha)}{\cos\alpha}\Omega, \quad \Omega^{\bullet} = \frac{v^2}{I}F(\alpha) - H\frac{v}{I}\Omega s(\alpha) \tag{4.1}$$

outside and only outside the union of straight lines

$$O = \{(\alpha,\Omega) \in R^2 : \alpha = \pi/2 + \pi k, k \in Z\}.$$

Let us analyze its trivial solution representing unperturbed motion for stability. To this end, we will write the characteristic equation near the origin of coordinates (here, $A = y'_N(0)$, $B = s(0)$, $n_0^2 = F'(0)/I = y'_N(0)s(0)/I = AB/I$) $\lambda^2 + \lambda[BH/I - \sigma n_0^2]v + n_0^2 v^2 = 0$).

Let us introduce the following three positive dimensionless parameters:

$$\mu_1 = 2\frac{B}{mn_0} > 0, \quad \mu_2 = \sigma n_0 > 0, \ \text{S} \ \mu_3 = \frac{BH}{In_0} > 0.$$

The following proposition is obvious.

Proposition 1. For $\mu_3 > \mu_2$ ($\mu_3 < \mu_2$), the trivial solution of (4.1) is asymptotically stable (is repulsing).

To ascertain whether a limit cycle can be born near the origin of coordinates, let us analyze the trivial solution of system (4.1) for stability at the critical ratio of the parameters: $\mu_3 = \mu_2$. To this end, we change phase variables as $(\alpha,\Omega) \to (a,w)$ in (4.1):

$$\alpha = a, \quad \Omega = (n_0 v\mu_2 a)/(1+\mu_2^2) - (n_0 vw)/(1+\mu_2^2),$$

which leads to the following system:

$$a^{\bullet} = \omega_0 w + \frac{\sigma v}{I}\left(\frac{f_3 + 3AB}{6}\right)a^3 - \frac{\sigma H}{I}\left(\frac{s_2 + B}{2}\right)a^2\left(\frac{n_0 v\mu_2}{1+\mu_2^2}a - \frac{n_0 v}{1+\mu_2^2}w\right) + \bar{\bar{o}}_1((a^2 + w^2)^{3/2}),$$

$$w^{\bullet} = -\omega_0 a - \frac{v^2}{I}\frac{f_3}{6}\left(\frac{n_0 v}{1+\mu_2^2}\right)^{-1}a^3 + \frac{Hv}{2I}s_2 a^2(\mu_2 a - w) + \bar{\bar{o}}_2((a^2 + w^2)^{3/2}). \tag{4.2}$$

We now introduce the following auxiliary index [17, 19]:

$$\text{In} = \frac{\sigma v}{I}(f_3 + 3AB) - 3\frac{\sigma H}{I}(B + s_2)\frac{n_0 v\mu_2^2}{1+\mu_2^2} - \frac{Hv}{I}s_2, \quad s_2 = s''(0), \quad f_3 = F'''(0).$$

Proposition 2. If In < 0 (In > 0) and the inequality

$$|\mu_3 - \mu_2| < 2, \tag{4.3}$$

holds, then the origin of coordinates of the phase plane $R^2\{a, w\}$ of system (4.2) ((4.1)) is a weak stable (unstable) focus when $\mu_3 = \mu_2$.

Condition (4.3) is necessary since the origin of coordinates on the plane $R^2\{a, w\}$ will be a (stable or unstable, strong or weak) focus only if this condition is satisfied.

Theorem 1. Let inequality (4.3) hold for system (4.1). Then:

(i) If In < 0, then for any fixed μ_2 there exist $\delta_1, \delta_2 > 0$ such that the origin of coordinates is a strong stable focus for $\mu_3 \in (\mu_2, \mu_2 + \delta_1)$. When $\mu_3 \in (\mu_2 - \delta_2, \mu_2)$, the origin is a strong unstable focus surrounded by a stable limit cycle that expands as μ_3 decreases from μ_2 to $\mu_2 - \delta_2$ as $\sqrt{|\mu_2 - \mu_3|}$.

(ii) If In > 0, for any fixed μ_2 there exist $\delta_1, \delta_2 > 0$ such that the origin is a strong unstable focus for $\mu_3 \in (\mu_2 - \delta_2, \mu_2)$. When $\mu_3 \in (\mu_2, \mu_2 + \delta_1)$, the origin is a strong stable focus surrounded by an unstable limit cycle that expands as μ_3 increases from μ_2 to $\mu_2 + \delta_2$ as $\sqrt{|\mu_2 - \mu_3|}$.

The condition $\mu_3 > \mu_2$ ($\mu_3 < \mu_2$) can easily be tested since in each specific case, the parameters depend either on the first derivatives of the influence functions (y_N, s) or on their values. The condition In < 0 (In > 0), however, is rather difficult to test in each specific case since not only the explicit expressions but also highest derivatives, even at separate points, of the influence functions (y_N, s) are unknown for each specific body.

5. Free Deceleration of a Rigid Body in a Resisting Medium. Consider a body moving freely (with vectoring thrust inactive) through a resisting medium.

Using the formula $dq = vdt$ to introduce a new independent variable (the length q traveled by the point D), i.e., introducing new differentiation, we rearrange system (2.4)–(2.6) as follows (differentiation is with respect to the length traveled):

$$v' = \Psi(\alpha, \omega)v, \tag{5.1}$$

$$\alpha' = -\omega + \sigma\omega^2 \sin\alpha + \frac{\sigma}{I}F(\alpha)\cos\alpha + \frac{s(\alpha)}{m}\sin\alpha - \frac{\sigma}{I}H\omega s(\alpha)\cos\alpha,$$

$$\omega' = \frac{1}{I}F(\alpha) + \sigma\omega^3\cos\alpha - \frac{\sigma}{I}\omega F(\alpha)\sin\alpha + \omega\frac{s(\alpha)}{m}\cos\alpha - \frac{B}{I}H\omega\cos\alpha + \frac{\sigma}{I}H\omega^2 s(\alpha)\sin\alpha, \tag{5.2}$$

where the prime denotes differentiation with respect to q, $\omega = d\varphi/dq$,

$$\Psi(\alpha, \omega) = -\sigma\omega^2\cos\alpha + \frac{\sigma}{I}F(\alpha)\sin\alpha - \frac{s(\alpha)}{m}\cos\alpha - \frac{\sigma}{I}H\omega s(\alpha)\sin\alpha.$$

Equations (5.2) form an independent subsystem of the second-order on the phase cylinder $S^1\{\alpha \bmod 2\pi\} \times R^1\{\omega\}$.

As above, we will analyze the trivial solution of system (5.2) for stability. Obviously, this solution describes translational deceleration (unperturbed motion).

Let us write the characteristic equation near the origin of coordinates:

$$\lambda^2 - \lambda[2B/m + \sigma n_0^2 - BH/I] + (B/m + \sigma n_0^2)(B/m - BH/I) + n_0^2(1 + \sigma BH/I) = 0.$$

Proposition 3. Let inequality (4.3) hold. Then the trivial solution of system (5.2) is asymptotically stable (is repulsing) for $\mu_3 > \mu_1 + \mu_2$ ($\mu_3 < \mu_1 + \mu_2$).

Figure 2 shows the general pattern of trajectories of the vector field of system (5.2) near the origin (*1* is the attracting point; *2* is the saddle point; and *3* is the repulsing point).

To ascertain whether a limit cycle can be born around the origin of coordinates, we will analyze the trivial solution of (5.2) for stability at the critical combination of parameters: $\mu_3 = \mu_1 + \mu_2$. To this end, we will change phase variables as $(\alpha, \omega) \to (a, w)$ in (5.2):

$$\alpha = a, \quad \omega = [n_0(\mu_2 + \mu_1/2)a]/[1 + \mu_2(\mu_1 + \mu_2)] - [n_0\sqrt{1 + \mu_1^2/4}w]/[1 + \mu_2(\mu_1 + \mu_2)],$$

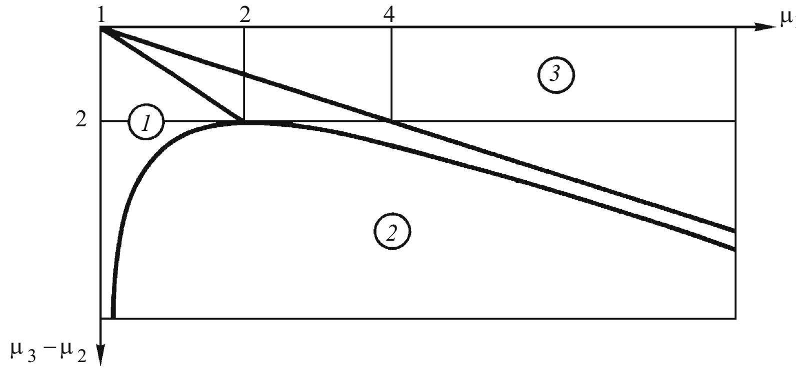

Fig. 2

which leads to the following system:

$$a' = \omega_0 w + \left(-\frac{\sigma n_0^2}{2} + \frac{\sigma f_3}{6I} - \frac{B}{6m} + \frac{s_2}{2m} \right) a^3$$

$$+ \frac{\sigma H}{2I} (B - s_2) a^2 \left(\frac{n_0 (\mu_2 + \mu_1 / 2)}{1 + \mu_2 (\mu_1 + \mu_2)} a - \frac{n_0 \sqrt{1 + \mu_1^2 / 4}}{1 + \mu_2 (\mu_1 + \mu_2)} w \right)$$

$$+ \sigma a \left(\frac{n_0 (\mu_2 + \mu_1 / 2)}{1 + \mu_2 (\mu_1 + \mu_2)} a - \frac{n_0 \sqrt{1 + \mu_1^2 / 4}}{1 + \mu_2 (\mu_1 + \mu_2)} w \right)^2 + \bar{\bar{o}}_1 ((a^2 + w^2)^{3/2}), \tag{5.3}$$

$$w' = -\omega_0 a - \frac{f_3}{6I} \left(\frac{n_0 \sqrt{1 + \mu_1^2 / 4}}{1 + \mu_2 (\mu_1 + \mu_2)} \right)^{-1} a^3 - \left(-\sigma n_0^2 - \frac{H s_2}{2I} - \frac{B}{2m} + \frac{s_2}{2m} \right) a^2 \left(\frac{(\mu_2 + \mu_1 / 2)}{\sqrt{1 + \mu_1^2 / 4}} a - w \right)$$

$$- \frac{\sigma H B}{I} a \left(\frac{n_0 (\mu_2 + \mu_1 / 2)}{1 + \mu_2 (\mu_1 + \mu_2)} a - \frac{n_0 \sqrt{1 + \mu_1^2 / 4}}{1 + \mu_2 (\mu_1 + \mu_2)} w \right)^2 \left(\frac{n_0 \sqrt{1 + \mu_1^2 / 4}}{1 + \mu_2 (\mu_1 + \mu_2)} \right)^{-1}$$

$$- \sigma \left(\frac{n_0 (\mu_2 + \mu_1 / 2)}{1 + \mu_2 (\mu_1 + \mu_2)} a - \frac{n_0 \sqrt{1 + \mu_1^2 / 4}}{1 + \mu_2 (\mu_1 + \mu_2)} w \right)^3 \left(\frac{n_0 \sqrt{1 + \mu_1^2 / 4}}{1 + \mu_2 (\mu_1 + \mu_2)} \right)^{-1} + \bar{\bar{o}}_2 ((a^2 + w^2)^{3/2}). \tag{5.4}$$

We now introduce the following auxiliary index [17, 19]:

$$\mathrm{In} = -5\mu_2 - \mu_1 + \frac{\sigma f_3}{\mathrm{In}_0} + 2 \frac{s_2}{m n_0} - \frac{H s_2}{I n_0} + 8\mu_2 \frac{1 + \mu_1^2 / 4}{(1 + \mu_2 (\mu_1 + \mu_2))^2}$$

$$+ \mu_2 \frac{H}{\mathrm{In}_0} (5B - 3 s_2) \frac{\mu_2 + \mu_1 / 2}{1 + \mu_2 (\mu_1 + \mu_2)} + 12\mu_2 \left(\frac{\mu_2 + \mu_1 / 2}{1 + \mu_2 (\mu_1 + \mu_2)} \right)^2.$$

Proposition 4. If In < 0 (In > 0) and inequality (4.3) holds, then the origin of coordinates of the phase plane $R^2 \{a, w\}$ of system (5.3), (5.4) ((5.2)) is a weak stable (unstable) focus when $\mu_3 = \mu_1 + \mu_2$.

Theorem 2. Let inequality (4.3) hold for (5.2). Then:

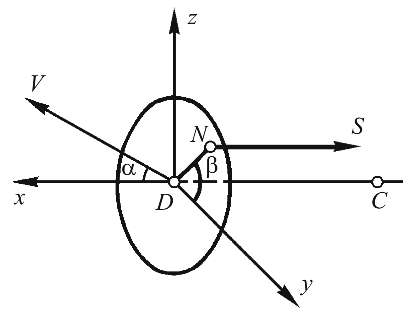

Fig. 3

(i) If In < 0, then for any fixed μ_2 there exist $\delta_1, \delta_2 > 0$ such that the origin of coordinates is a strong stable focus for $\mu_3 \in (\mu_2, \mu_2 + \delta_1)$. When $\mu_3 \in (\mu_2 - \delta_2, \mu_2)$, the origin is a strong unstable focus surrounded by a stable limit cycle that expands as μ_3 decreases from μ_2 to $\mu_2 - \delta_2$ as $\sqrt{|\mu_2 - \mu_3|}$.

(ii) If In > 0, then for any fixed μ_2 there exist $\delta_1, \delta_2 > 0$ such that the origin of coordinates is a strong unstable focus for $\mu_3 \in (\mu_2 - \delta_2, \mu_2)$. When $\mu_3 \in (\mu_2, \mu_2 + \delta_1)$, the origin is a strong stable focus surrounded by an unstable limit cycle that expands as μ_3 increases from μ_2 to $\mu_2 + \delta_2$ as $\sqrt{|\mu_2 - \mu_3|}$.

6. Spatial Motion of an Axisymmetric Rigid Body in a Resisting Medium: Problem Statement. Consider a moving homogeneous body of mass m. A portion of its surface is a flat disk. A jet flow is past the body [4, 13, 14, 17, 19]. The other portion of the body's surface is inside the volume bounded by the jet stalling at the disk edge and is not affected by the medium. Conditions are similar when homogeneous circular cylinders enter water [7, 12].

Assume that there are no tangential forces. Then the force $\boldsymbol{S}$ applied by the medium to the body at the point N does not change the orientation relative to the body (is normal to the disk) and is quadratic with respect to the velocity of its center D (Newtonian drag, Fig. 3). The gravity is assumed negligible compared with the resistance (effect) of the medium.

If the above conditions are satisfied, the motions of the body include translational deceleration similar to the case of plane-parallel (unperturbed) motion: the body can undergo translational motion along its axis of symmetry, i.e., perpendicularly to the disk plane.

We choose the right-hand coordinate system $Dxyz$ (Fig. 3) with the Dx-axis aligned with the axis of geometrical symmetry of the body and the Dy- and Dz-axes fixed to the disk. The components of the angular velocity vector Ω in the system $Dxyz$ are denoted by $\{\Omega_x, \Omega_y, \Omega_z\}$. The inertia tensor of the dynamically symmetric body is diagonalized in the body axes $Dxyz$: $\mathrm{diag}\{I_1, I_2, I_2\}$.

We will use the quasitationarity hypothesis and assume for simplicity that $R = DN$ is defined at least by the attack angle α between the velocity vector $\boldsymbol{v}$ of the center D of the disk and the straight line Dx. Thus, $DN = R(\alpha, \dots)$.

Moreover, we assume that $S = s_1(\alpha)v^2$, $v = |\boldsymbol{v}|$. For convenience, we introduce (as in the case of plane-parallel motion) an auxiliary alternating function $s(\alpha)$: $s_1 = s_1(\alpha) = s(\alpha)\,\mathrm{sgn}\cos\alpha \ge 0$ instead of the coefficient $s_1(\alpha)$. Thus, the pair of functions $R(\alpha, \dots)$ and $s(\alpha)$ defines the forces and moments exerted by the medium on the disk under such assumptions.

6.1. Dynamic Part of the Equations of Spatial Motion. Let us use the spherical coordinates (v, α, β_1) of the tip of the velocity vector $\boldsymbol{v} = \boldsymbol{v}_D$ of the point D relative to the flow to measure the angle β_1 in the plane of the disk (Fig. 3). Expressing the quantities (v, α, β_1), using nonintegrable relations, in terms of the cyclic kinematic variables and velocities and supplementing them with the projections $(\Omega_x, \Omega_y, \Omega_z)$ of the angular velocity onto the body axes, we consider them as quasivelocities. Obviously, $\boldsymbol{v}_D = \{v\cos\alpha, v\sin\alpha\cos\beta_1, v\sin\alpha\sin\beta_1\}$.

Using the theorems on the motion of the center of mass (in the body-fixed frame of reference $Dxyz$) and variation in the angular momentum in the same frame, we obtain the dynamic part of the differential equations of motion in the six-dimensional phase space of quasivelocities (σ is the distance DC). The first group of equations describes the motion of the center of mass, while the second group the motion around the center of mass

$$v^{\bullet}\cos\alpha - \alpha^{\bullet} v\sin\alpha + \Omega_y v\sin\alpha\sin\beta_1 - \Omega_z v\sin\alpha\cos\beta_1 + \sigma(\Omega_y^2 + \Omega_z^2) = -s(\alpha)v^2 / m,$$

$$v^{\bullet}\sin\alpha\cos\beta_1 + \alpha^{\bullet} v\cos\alpha\cos\beta_1 - \beta_1^{\bullet} v\sin\alpha\sin\beta_1 + \Omega_z v\cos\alpha - \Omega_x v\sin\alpha\sin\beta_1 - \sigma\Omega_x\Omega_y - \sigma\Omega_z^{\bullet} = 0,$$

$$v^{\bullet}\sin\alpha\sin\beta_1 + \alpha^{\bullet} v\cos\alpha\sin\beta_1 + \beta_1^{\bullet} v\sin\alpha\cos\beta_1 + \Omega_x v\sin\alpha\cos\beta_1 - \Omega_y v\cos\alpha - \sigma\Omega_x\Omega_z + \sigma\Omega_y^{\bullet} = 0,$$

$$I_1\Omega_x^{\bullet} = 0, \quad I_2\Omega_y^{\bullet} + (I_1 - I_2)\Omega_x\Omega_z = -F(\alpha,\dots)\sin\beta_1 \cdot v^2,$$

$$I_2\Omega_z^{\bullet} + (I_2 - I_1)\Omega_x\Omega_y = F(\alpha,\dots)\cos\beta_1 \cdot v^2,$$

$$F(\alpha,\dots) = R(\alpha,\dots)s(\alpha). \tag{6.1}$$

6.2. Equations of Motion of a Symmetric Body Subject to Newtonian Drag and Follower Forces. Let us consider the class of problems where a rigid body moves through a medium under a follower force acting along the axis of geometrical symmetry of the body (compare with the case of plane-parallel motion) and producing (under some conditions) classes of motions (imposed constraints) of interest, this force being the reaction of the constraints imposed. If the follower force is absent, the body undergoes spatial free deceleration due to drag (see also [10, 15, 18]). Here, the follower force is such that condition (2.2) is satisfied all the time.

In view of Eqs. (6.1), the Routh-cyclic invariant relation $\Omega_x = \Omega_{x_0} = \text{const}$ holds at all instants of time.

6.3. Equations of Motion of a Rigid Body without Rotation about the Longitudinal Axis. In what follows, we will examine the case where the rigid body does not rotate about its longitudinal axis, i.e., $\Omega_{x_0} = 0$.

Then the independent dynamic part of the equations of motion in the four-dimensional phase space is given by

$$\alpha^{\bullet} v\cos\alpha\cos\beta_1 - \beta_1^{\bullet} v\sin\alpha\sin\beta_1 + \Omega_z v\cos\alpha - \sigma\Omega_z^{\bullet} = 0, \tag{6.2}$$

$$\alpha^{\bullet} v\cos\alpha\sin\beta_1 + \beta_1^{\bullet} v\sin\alpha\cos\beta_1 - \Omega_y v\cos\alpha + \sigma\Omega_y^{\bullet} = 0, \tag{6.3}$$

$$I_2\Omega_y^{\bullet} = -z_N s(\alpha)v^2, \quad I_2\Omega_z^{\bullet} = y_N s(\alpha)v^2, \tag{6.4}$$

where y_N and z_N are Cartesian coordinates, in the plane of the disk, of the point N of application of the resisting force. System (6.2)–(6.4) includes the influence functions y_N, z_N, and s. To determine them qualitatively (by analogy with the case of plane-parallel motion), we will use experimental data on the properties of jet flow.

For simplicity, we will analyze system (6.2)–(6.4) for the following influence functions (compare with (3.6); such an analysis can be performed for an arbitrary pair of functions y_N, z_N, and s):

$$y_N = A\sin\alpha\cos\beta_1 + h\Omega_z / v, \qquad z_N = A\sin\alpha\sin\beta_1 - h\Omega_y / v,$$

$$s(\alpha) = B\cos\alpha, \qquad A, B, h > 0. \tag{6.5}$$

The resultant system will be called a reference one. The coefficient h in (6.5) appears in the terms proportional to the rotary derivatives of the moment of hydroaerodynamic forces (drag) with respect to the components of the angular velocity of the body (see also [2, 3]).

6.4. Analysis of the Reference System. System (6.2)–(6.4) is a dynamic system with variable dissipation and with zero mean (with respect to the angle of attack) [17, 19]. This means that the integral of the divergence of its right-hand side over the period of the angle of attack, which describes the variation in the phase volume (after the appropriate reduction of the system), is equal to zero. The system is semiconservative in a sense.

Projecting the angular velocities onto the moving axes not fixed to the body so that $z_1 = \Omega_y\cos\beta_1 + \Omega_z\sin\beta_1$, $z_2 = -\Omega_y\sin\beta_1 + \Omega_z\cos\beta_1$ and introducing dimensionless variables w_k, $k = 1, 2$, and parameters by the formulas $h_1 = hB$, $\sigma h_1 / I_2 = H_1$, $\beta = \sigma^2 AB / I_2$, $\sigma z_k = v w_k$ (with $\alpha' = v / \sigma\alpha^{\bullet}$, etc.), we obtain the following analytic dynamic system (reference system) of the fourth order:

$$\alpha' = -(1 + H_1)w_2 + \beta\sin\alpha, \tag{6.6}$$

$$w_2' = \beta \sin\alpha\cos\alpha - (1+H_1)w_1^2 \frac{\cos\alpha}{\sin\alpha} - H_1 w_2 \cos\alpha, \tag{6.7}$$

$$w_1' = (1+H_1)w_1 w_2 \frac{\cos\alpha}{\sin\alpha} - H_1 w_1 \cos\alpha, \tag{6.8}$$

$$\beta_1' = (1+H_1)w_1 \frac{\cos\alpha}{\sin\alpha}, \tag{6.9}$$

which includes the independent third-order subsystem (6.6)–(6.8).

If $\beta = H_1$, then after the change of variables $w_1^* = \ln|w_1|$, the divergence of the right-hand side of (6.6)–(6.8) ((6.6)–(6.9)) will become identically equal to zero, which allows considering the system(s) to be conservative.

6.5. Stability of Translational Motion. Consider the following positive definite function in the phase space of the third-order system (6.6)–(6.8):

$$V(\alpha, w_1, w_2) = w_2^2 + (1+\beta)w_1^2 + \beta[w_2 - \sin\alpha]^2. \tag{6.10}$$

Theorem 3. Function (6.10) is a Lyapunov (Chetaev) function for system (6.6)–(6.8), i.e., its derivative is negative definite for $\beta < H_1$ and positive definite for $\beta > H_1$.

Corollary. The origin of coordinates of system (6.6)–(6.8) (after the right-hand side is redefined at it) is an attracting singular point for $\beta < H_1$ and a repulsing singular point for $\beta > H_1$.

Proof. Indeed, by virtue of (6.6)–(6.8), the derivative of function (6.10) is $2(\beta - H_1)\cos\alpha[w_1^2 + w_2^2]$.

Note once again that a similar theorem is also valid for the general system with arbitrary influence functions y_N, z_N, and s. The asymptotic stability condition for the origin of coordinates of the system of reduced dynamic equations with respect to the variables (α, w_1, w_2) remains the same, $\beta < H_1$.

In the more general case (see (3.5)) where the influence functions are represented as

$$y_N = R(\alpha)\cos\beta_1 + h_1\Omega_z / v, \qquad z_N = R(\alpha)\sin\beta_1 - h_1\Omega_y / v, \qquad s = s(\alpha),$$

the equations of motion become

$$\alpha^\bullet = -z_2 + \frac{\sigma v}{I_2}\frac{F(\alpha)}{\cos\alpha} - \frac{\sigma h_1}{I_2} z_2 \frac{s(\alpha)}{\cos\alpha}, \qquad z_2^\bullet = \frac{F(\alpha)}{I_2} v^2 - z_1^2 \frac{\cos\alpha}{\sin\alpha} - \frac{\sigma h_1}{I_2} z_1^2 \frac{s(\alpha)}{\sin\alpha} - \frac{h_1 v}{I_2} z_2 s(\alpha),$$

$$z_1^\bullet = z_1 z_2 \frac{\cos\alpha}{\sin\alpha} + \frac{\sigma h_1}{I_2} z_1 z_2 \frac{s(\alpha)}{\sin\alpha} - \frac{h_1 v}{I_2} z_1 s(\alpha), \qquad \beta_1^\bullet = z_1 \frac{\cos\alpha}{\sin\alpha} + \frac{\sigma h_1}{I_2} z_1 \frac{s(\alpha)}{\cos\alpha}. \tag{6.11}$$

Consider the function

$$V_1(\alpha, z_1, z_2) = z_2^2 + (1+\beta)z_1^2 + \beta\left[z_2 - \frac{v}{\sigma}\sin\alpha\right]^2, \tag{6.12}$$

that is positive definite in the neighborhood of the origin.

Theorem 4. Function (6.12) is a Lyapunov (Chetaev) function for system (6.11), i.e., by virtue of (6.11), its derivative in the neighborhood of the origin is negative definite for $\sigma R'(0) < h_1$ and is positive definite for $\sigma R'(0) > h_1$.

Corollary. The origin of coordinates of system (6.11) (after the right-hand side is redefined at it) is an attracting singular point for $\sigma R'(0) < h_1$ and a repulsing singular point for $\sigma R'(0) > h_1$.

Proof. Indeed, the derivative of function (6.12) is given by

$$2v\left(\frac{\beta}{\sigma}\cos\alpha - \frac{h_1}{I_2}s(\alpha)\right)[z_1^2 + z_2^2] + 2v^2 z_2 \left\{\frac{F(\alpha)}{I_2} - \frac{\beta}{\sigma^2}\sin\alpha\cos\alpha\right\}$$

and can be represented as follows in the neighborhood of the origin:

$$2v\left(\frac{\beta}{\sigma}-\frac{h_1 B}{I_2}\right)[z_1^2+z_2^2]+\bar{\bar{o}}(\alpha^2+z_1^2+z_2^2).$$

The asymptotic-stability condition for moving homogeneous circular cylinders will be satisfied if $\sigma k < hD_1$, where D_1 is the diameter of the cylinder, k and h are dimensionless influence parameters, and σ is the distance DC.

7. Spatial Free Deceleration of a Rigid Body in a Resisting Medium. Let us analyze the spatial translational motion (namely, deceleration) of a body for stability using three quasivelocities: angle of attack and two components of the angular velocity.

7.1. Equations of Motion of a Nonrotating Symmetric Body Subject to Newtonian Drag (Spatial Free Deceleration). The dynamic part of the equations of motion is

$$v'=v\Psi_1(\alpha,Z_1,Z_2),\qquad \alpha'=-Z_2+\sigma(Z_1^2+Z_2^2)\sin\alpha+\frac{\sigma}{I_2}F(\alpha)\cos\alpha+\frac{s(\alpha)}{m}\sin\alpha,$$

$$Z_2'=\frac{1}{I_2}F(\alpha)-Z_2\Psi_1(\alpha,Z_1,Z_2)-Z_1^2\frac{\cos\alpha}{\sin\alpha},\qquad Z_1'=-Z_1\Psi_1(\alpha,Z_1,Z_2)+Z_1Z_2\frac{\cos\alpha}{\sin\alpha},$$

where $\Psi_1(\alpha,Z_1,Z_2)=-\sigma(Z_1^2+Z_2^2)\cos\alpha+\frac{\sigma}{I_2}F(\alpha)\sin\alpha-\frac{s(\alpha)}{m}\cos\alpha.$

In the case of Chaplygin influence functions, the analytic system of equations has the form

$$v'=v\Psi_1(\alpha,Z_1,Z_2),$$

$$\alpha'=-Z_2+\sigma(Z_1^2+Z_2^2)\sin\alpha+\frac{\sigma}{I_2}F(\alpha)\cos\alpha-\frac{\sigma h_1}{I_2}Z_2 s(\alpha)\cos\alpha+\frac{s(\alpha)}{m}\sin\alpha, \tag{7.1}$$

$$Z_2'=\frac{F(\alpha)}{I_2}-Z_2\Psi_1(\alpha,Z_1,Z_2)-Z_1^2\frac{\cos\alpha}{\sin\alpha}-\frac{\sigma h_1}{I_2}Z_1^2\frac{s(\alpha)}{\sin\alpha}-\frac{h_1}{I_2}Z_2 s(\alpha), \tag{7.2}$$

$$Z_1'=-Z_1\Psi_1(\alpha,Z_1,Z_2)+Z_1Z_2\frac{\cos\alpha}{\sin\alpha}+\frac{\sigma h_1}{I_2}Z_1Z_2\frac{s(\alpha)}{\sin\alpha}-\frac{h_1}{I_2}Z_1 s(\alpha), \tag{7.3}$$

$$\beta_1'=Z_1\frac{\cos\alpha}{\sin\alpha}+\frac{\sigma h_1}{I_2}Z_1\frac{s(\alpha)}{\sin\alpha},$$

where $\Psi_1(\alpha,Z_1,Z_2)=-\sigma(Z_1^2+Z_2^2)\cos\alpha+\frac{\sigma}{I_2}F(\alpha)\sin\alpha-\frac{s(\alpha)}{m}\cos\alpha-\frac{\sigma h_1}{I_2}Z_2 s(\alpha)\sin\alpha.$

Consider the function

$$V_2(\alpha,Z_1,Z_2)=\frac{\sigma}{2}(Z_1^2+Z_2^2)+Z_1\sin\alpha, \tag{7.4}$$

that is positive definite in the neighborhood of the origin of the phase space $\{(\alpha,Z_1,Z_2)\in R^3: Z_1\geq 0, \sin\alpha>0\}$.

Theorem 5. Function (7.4) is a Lyapunov function for system (7.1)–(7.3), i.e., its derivative in the neighborhood of the origin is negative definite if the conditions $\mu_3>\mu_1+\mu_2$ and (4.3) are satisfied (see also [15, 16]).

Indeed, the derivative of function (7.4) by virtue of (7.1)–(7.3) is given by

$$(-\mu_3+\mu_1+\mu_2)\alpha Z_1+\alpha Z_2+\sigma(Z_1^2+Z_2^2)\left(-\mu_3+\frac{\mu_1}{2}\right)+\bar{\bar{o}}(\alpha^2+Z_1^2+Z_2^2)$$

and it is a negative definite function in a sufficiently small neighborhood of the origin.

7.2. Stability of Spatial Translational Motion. Theorem 5 leads to the same conditions of asymptotic stability in the variables (α, z_1, z_2) as Proposition 3 does, which deals with plane-parallel motion.

In the case of spatial motion, the systems obtained have uncertainty at the origin, which is due to the degeneracy of the spherical coordinates of the extreme point of the velocity vector $\boldsymbol{v}$ of the center of the front disk (cavitator) and can be overcome by redefining the right-hand sides of the dynamic systems.

8. Conclusions. The instability of translational deceleration (elementary motion of a body) is used for methodical purposes [7, 12], namely, to determine the unknown influence parameters under quasistationarity conditions.

The experiment on the motion of homogeneous circular cylinders in water conducted at the Research Institute of Mechanics of the M. V. Lomonosov Moscow State University confirmed that in modeling the influence of a medium on a rigid body, it is necessary to introduce an additional parameter to account for dissipation in the system.

In studying the deceleration of a body with finite angles of attack, a key task is to establish the conditions under which self-oscillations occur in a finite neighborhood of translational deceleration. Thus, a comprehensive nonlinear analysis is of necessity.

The initial stage of such an analysis is neglecting the damping effect of the medium. This corresponds to the assumption that the pair of dynamic functions describing the influence of the medium depends on a unique parameter (angle of attack). The dynamic systems resulting from such a nonlinear description behave as systems with variable dissipation. Therefore, it is necessary to develop a procedure for analyzing such systems [17, 19].

In general, the dynamics of a rigid body interacting with a medium usually produces either systems with variable dissipation with nonzero mean (free deceleration) or systems with zero period-average loss of energy (motion of a rigid body in a resisting medium under a follower force). We have used an approach that makes it possible to conduct a complete analytical study of the plane-parallel and spatial motions of a rigid body.

Since experimental data on the properties of jet flow is used, there is some scatter in forces and moments characteristics in the qualitative description of the body–medium interaction. This makes it natural to define relative robustness (relative structural stability) and to prove such robustness for the systems under study [17, 19]. Many of the systems turn out to be simply (absolutely) Andronov–Pontryagin robust in the ordinary sense.

Analyzing all the results obtained under the elementary assumption that there is no damping, we conclude that it is impossible to establish the conditions under which there exist self-oscillations in a finite neighborhood of translational deceleration [9, 11].

Our study continues the analysis of the motion of a body in a medium that exerts a damping moment. This moment introduces additional dissipation into the system, which may make the translational deceleration of the body stable.

Thus, allowing for the damping effect of the medium on the rigid body, we can answer in the affirmative the key question of whether stable self-oscillations can occur when a body moves through a medium with finite angles of attack.

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