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THE CASES OF INTEGRABILITY IN TERMS OF TRANSCENDENTAL FUNCTIONS IN DYNAMICS OF A RIGID BODY INTERACTING WITH A MEDIUM

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Abstract: In this activity some hypotheses on the properties of a medium are embodied in the three-dimensional dynamic model of interaction of a body with a medium. All of the interaction of a medium with a body is concentrated on that part of the surface of the body which has the form of a convex plane region. Since the interaction is governed by the laws of jet flow. New integrable cases and families of phase portraits in the plane dynamics of rigid bodies are discovered. Certain model cases of the motion of rigid bodies in a resisting medium are qualitatively studied and integrated. The first integrals of corresponding systems are found; these integrals are transcendental functions and functions that can be expressed in terms of elementary functions. New families of multidimensional phase portraits in the spatial dynamics are found. The problem of the dynamically symmetrically fixed rigid body placed in the flow of an incoming medium is integrated.

1. Introduction

Dynamic model of interaction of a rigid body with resisting medium provided jet flow [1,2], considered in activity, not only allows successfully to transfer outcomes appropriate problems from plane dynamics of a rigid body interacting with the medium [3,4] and to receive their spatial analogs, but also to detect new cases of integrability till Jacob. Thus in some cases the integrals express through elementary functions. In activity the integrability of classical is shown in the problems about a spherical pendulum, located in a flow by filling of a medium, about spatial motion of a body at availability constraint, and also the mechanical and topological analogies are shown in the last two problems.

The hypothesizes adduced in [3] and concerning of properties of a medium, have found the reflection in construction of spatial (3D) dynamic model of interaction of a rigid body with resisting medium. In this connection there is a capability of formalizing of the model suppositions and obtaining of a full system of ordinary differential equations.

2. On Interaction

All interaction of medium with a body is concentrated on that part of a surface of a body which has the shape of convex plane area *P*.

As the interaction happens under the laws of jet flow the force ***S*** of this interaction is directed on a normal line to area and the point *N* of the acting of this force is determined only in one parameter — by an angle of attack α which is measured between vector of velocity ***v*** of a point *D* of a plate and external normal line in this point (straight line *CD*). The point *D* is the interception of the straight line *CD* (*C* — center of mass) that is perpendicular to plane *P*. Thus, $DN = R(\alpha)$.

Size of force of resistance we shall accept as $S = sv^2$, where *v* is the module of speed of a point *D*, and coefficient of resistance *s* is the function only of angle of attack: $s = s(\alpha)$.

There is the additional force ***T***, which acts on a body on the straight line *CD*. Let's name it as "force of a thrust". The introduction of this force is used, as for maintenance of some specific classes of motions (thus *T* is the reaction of the possible (or probable) imposed constraint and in the methodical purposes, which pursue learning of interesting non-linear systems (having character of pendulum) arising at the reduction of the order. In case of absence external force *T* the body makes free braking (deceleration) in a resisting medium [3,4].

Systems of coordinates connected to a body shall designate through *Dxyz*. The last coordinate system connected to a point *D* is selected such that the tensor of inertia in the given system has diagonal type: diag{*A*,*B*,*C*}. Mass distribution we shall accept by such that longitudinal principal axis of inertia coincides an axis *CD* (it is an axis *Dx*), and the axes *Dy* and *Dz* lie in a plane *P* and will derivate with the right of coordinate system. Moreover, we shall consider case dynamically symmetrical rigid body, i.e. the equality *B* = *C* is executed.

3. Dynamical equations

In this case for the description of a position of a body in 3D space it is possible to select the Cartesian coordinates (x_0, y_0, z_0) of a point D and three angles (θ, ψ, φ), which are determined similarly to classical navigational angles.

By virtue of the theorem of motion of center of mass in space in projections on moving axes (x, y, z) and theorem of change of kinetic moment of rather these axes, we receive a full system of differential equations considered in dynamic space of quasivelocities

$$v'\cos\alpha - \alpha' v\sin\alpha + qv\sin\alpha\sin\beta - rv\sin\alpha\cos\beta + \sigma(q^2 + r^2) = T/m - s(\alpha)v^2/m\,,$$

$$v'\sin\alpha\cos\beta + \alpha' v\cos\alpha\cos\beta - \beta' v\sin\alpha\sin\beta + rv\cos\alpha - pv\sin\alpha\sin\beta - \sigma pq - \sigma r' = 0\,,$$

$$v'\sin\alpha\sin\beta + \alpha' v\cos\alpha\sin\beta + \beta' v\sin\alpha\cos\beta - qv\cos\alpha + pv\sin\alpha\cos\beta - \sigma pr + \sigma q' = 0\,,$$

$Ap'+(C-B)qr=0$, $Bq'+(A-C)pr=-z_N s(\alpha)v^2$, $Cr'+(B-A)pq=y_N s(\alpha)v^2$.

Here coordinates of a point N in a system (e_x, e_y, e_z) will accept as $(0, y_N(\alpha,\beta), z_N(\alpha,\beta))$, where $y_N(\alpha,\beta)=R(\alpha)\cos\beta$, $z_N(\alpha,\beta)=R(\alpha)\sin\beta$, σ is the distance CD.

In a general dynamic system of the twelfth order by virtue of cyclic character of positional coordinates the splitting of independent subsystem of sixth order happens in a phase space of quasivelocities $\mathrm{T}^2\{\alpha,\beta\}\times\Re^1\{v\}\times\Re^3\{p,q,r\}$. Here (v,α,β) are the spatial polar coordinates of the velocity of point D , (p,q,r) is the projection of angular velocity to coordinate system connected with a body.

4. Dynamically Symmetric Rigid Body with Constraint

Dynamic equations of motion of a free rigid body at availability of servo constraint of a type $v=const$ (plane version of the given problem also see in [3,4]) accept the first integral $p=p_0$ and look like

$$\alpha'=-z_2+\sigma\frac{v}{B}\frac{F(\alpha)}{\cos\alpha}+\frac{\sigma}{v}\frac{A}{B}p_0\frac{z_1}{\cos\alpha},$$

$$z_2'=\frac{F(\alpha)}{B}v^2-z_1\left[z_1\frac{\cos\alpha}{\sin\alpha}-\frac{A}{B}p_0+\frac{\sigma}{v}\frac{A}{B}p_0\frac{z_2}{\cos\alpha}\right], \tag{1}$$

$$z_1'=z_2\left[z_1\frac{\cos\alpha}{\sin\alpha}-\frac{A}{B}p_0+\frac{\sigma}{v}\frac{A}{B}p_0\frac{z_2}{\cos\alpha}\right],$$

$$\beta'=-p_0+\left[z_1\frac{\cos\alpha}{\sin\alpha}+\frac{\sigma}{v}\frac{A}{B}p_0\frac{z_2}{\cos\alpha}\right]. \tag{2}$$

Here $z_1=q\cos\beta+r\sin\beta, z_2=r\cos\beta-q\sin\beta.$

The function in a dynamic system (1), (2) has the following properties: for qualitative description of its we use being available the experimental information on properties of jet flow [1,2].

5. Main theorems

Theorem 1. *The dynamic system (1), (2) is equivalent (in trajectory sense) topologically to a system (1),(2) under such condition:*

$$F=F_0(\alpha)=A'B'\sin\alpha\cos\alpha, A', B'>0. \tag{3}$$

The system (1), (2) under condition of (3) will accept a type of analytical ($n_0^2=\frac{A'B'}{B}$):

$$\alpha'=-z_2+\sigma n_0^2\sin\alpha+\frac{\sigma}{v}\frac{A}{B}p_0\frac{z_1}{\cos\alpha},$$

$$z_2{}'=n_0^2v^2\sin\alpha\cos\alpha - z_1\left[z_1\frac{\cos\alpha}{\sin\alpha}-\frac{A}{B}p_0+\frac{\sigma}{v}\frac{A}{B}p_0\frac{z_2}{\cos\alpha}\right],$$

$$z_1{}'=z_2\left[z_1\frac{\cos\alpha}{\sin\alpha}-\frac{A}{B}p_0+\frac{\sigma}{v}\frac{A}{B}p_0\frac{z_2}{\cos\alpha}\right],\ \beta'=-p_0+\left[z_1\frac{\cos\alpha}{\sin\alpha}+\frac{\sigma}{v}\frac{A}{B}p_0\frac{z_2}{\cos\alpha}\right].$$

At the beginning let's consider the capabilities of an integration of a system (1), (2) at a level $p_0=0$. At this field of vectors of a system (1) has three kinds of symmetry:

1) *A central symmetry*. Such symmetry near the points $(\pi k,0,0), k\in Z$, in space $\Re^3\{\alpha,z_2,z_1\}$ arise for the reason that the vector field in coordinates $\{\alpha,z_2,z_1\}$ changes the sign at replacement

$$\begin{pmatrix}\pi k-\alpha\\ -z_2\\ -z_1\end{pmatrix}\Rightarrow\begin{pmatrix}\pi k+\alpha\\ z_2\\ z_1\end{pmatrix}.$$

2) *Some mirror symmetry* (SMS). Such symmetry is related to the planes $\Lambda_i, i\in Z$ where $\Lambda_i=\{(\alpha,z_2,z_1)\in\Re^3:\alpha=\frac{\pi}{2}+\pi i\}$ arises for the reason that α -making component of field of vectors of our system in coordinates $\{\alpha,z_2,z_1\}$ is saved at replacement $\begin{pmatrix}\frac{\pi}{2}+\pi k-\alpha\\ z_2\\ z_1\end{pmatrix}\Rightarrow\begin{pmatrix}\frac{\pi}{2}+\pi k+\alpha\\ z_2\\ z_1\end{pmatrix}$ and z_2 - and z_1 -making components change the sign;

3) *by a symmetry is related to the planes* $\{(\alpha,z_2,z_1)\in\Re^3:z_1=0\}$, namely, z_2 - and α -making of components of vector field of a system are saved at replacement $\begin{pmatrix}\alpha\\ z_2\\ z_1\end{pmatrix}\Rightarrow\begin{pmatrix}\alpha\\ z_2\\ -z_1\end{pmatrix}$ and z_1 -making component changes the sign.

In activities [3,4] the first integral of a system from plane dynamics expressed through elementary functions.

Theorem 2. *The system (1) at* $p_0=0$ *has a full set of the first transcendental integrals. The system (1), (2) at* $p_0=0$ *also is quite integrated till Jacob, two from which first integral are integrals of systems (1) at* $p_0=0$ *and third is analytical function.*

The meromorphic integral of a system (1), (2) at $p_0=0$ will look like

$$\frac{z_1^2+z_2^2-\sigma n_0^2vz_2\sin\alpha+n_0^2v^2\sin^2\alpha}{z_1\sin\alpha}=C_1. \tag{4}$$

As the system (1), (2) at $p_0 = 0$ has a *variable dissipation* and also is analytical, for its it is possible in an obvious kind to find two other additional integrals. The following identity is executed $u_1 = \{C_1 \pm G\}/2$, where $G = \sqrt{C_1^2 - 4[u_2^2 - \sigma n_0^2 v u_2 + n_0^2 v^2]}$, $u_1 = z_1 \tau,\ u_2 = z_2 \tau,\ \tau = \sin\alpha$ (for search of additional integrals it is used the first integral (4)). A quadrature for search of a unknown quantity of an integral linking the sizes u_2 and τ is received by a kind

$$\int \frac{d\tau}{\tau} = \int \frac{(\sigma n_0^2 v - u_2) du_2}{2[u_2^2 - \sigma n_0^2 v u_2 + n_0^2 v^2] - \frac{C_1}{2}(C_1 \pm G)}\ . \tag{5}$$

If $w_1 = u_2 - \frac{\sigma n_0^2 v}{2}$ the right member (5) accept a kind

$$\int \frac{(\frac{\sigma n_0^2}{2} - w_1) dw_1}{2[w_1^2 - \frac{n_0^2 v^2 (\sigma^2 n_0^2 - 4)}{4}] - \frac{C_1}{2}(C_1 \pm G)}\ . \tag{6}$$

The size (6) is broken into a part where $\frac{\sigma n_0^2 v}{2}\int\limits_{(1)} - \int\limits_{(2)}$; here $\int\limits_{(1)} = \int \frac{dw_1}{G_1}$, $\int\limits_{(2)} = \int \frac{dw_1}{2G_1}$; where $G_1 = 2[w_1^2 - n_0^2 v^2 \frac{(\sigma^2 n_0^2 - 4)}{4}] - \frac{C_1}{2}(C_1 \pm G)$. If $a = \frac{n_0^2 v^2 (\sigma^2 n_0^2 - 4)}{4}, \bar{x} = w_1^2, \bar{y}^2 = C_1^2 - 4(\bar{x} - a)$ that $\int\limits_{(2)} = \frac{1}{2}\ln\left|\bar{y} + C_1\right| + const$. Furthermore, $\int\limits_{(1)} = \pm \int \frac{dy}{(\bar{y} + C_1)\sqrt{C_1^2 - \bar{y}^2 + 4a}}$.

Let us for a determinacy $C_1^2 + 4a \geq 0$. Then

$$\int\limits_{(1)} = \pm \frac{1}{n_0^2 v^2 \sqrt{4 - \sigma^2 n_0^2}} \arcsin \frac{C_1 \bar{y} + C_1^2 + n_0^2 v^2 (\sigma^2 n_0^2 - 4)}{(\bar{y} + C_1)\sqrt{C_1^2 + n_0^2 v^2 (\sigma^2 n_0^2 - 4)}} + const,\ if\ \sigma n_0 < 2\ ,$$

$$\int\limits_{(1)} = \mp \frac{1}{C_1(\bar{y} + C_1)} \sqrt{C_1^2 - \bar{y}^2} + const,\ if\ \sigma n_0 = 2\ ,$$

$$\mp \int\limits_{(1)} = -\frac{1}{2n_0^2 v^2 \sqrt{\sigma^2 n_0^2 - 4}} \ln\left|\frac{n_0 v \sqrt{\sigma^2 n_0^2 - 4} + G_1}{\bar{y} + C_1} + \frac{C_1}{n_0 v \sqrt{\sigma^2 n_0^2 - 4}}\right| +$$

$$+ \frac{1}{2n_0^2 v^2 \sqrt{\sigma^2 n_0^2 - 4}} \ln\left|\frac{n_0 v \sqrt{\sigma^2 n_0^2 - 4} - G_1}{\bar{y} + C_1} + \frac{C_1}{n_0 v \sqrt{\sigma^2 n_0^2 - 4}}\right| + const,\ if\ \sigma n_0 > 2.$$

Additional the first integral of a systems found above being by transcendental function of state variables makes together with (4) a full set of the first integrals of a system (1) at $p_0 = 0$. For the system (1), (2) at $p_0 = 0$ the one more first integral is necessary.

Remark. Everywhere is higher instead of it is necessary to insert left-hand part of equality (4).

For search of the last integral of a system (1), (2) at $p_0 = 0$ we shall remark, that as $\frac{dz_1}{d\beta} = z_2$ that $\frac{du_1}{d\beta} + [-u_2 + \sigma n_0^2 v] = u_2$. Therefore $\frac{du_1}{d\beta} = \pm\sqrt{\sigma^2 n_0^4 v^2 - 4[u_1^2 - C_1 u_1 + n_0^2 v^2]}$ and, therefore, the required quadrature receives a kind $\mp\int \frac{du_1}{\sqrt{\sigma^2 n_0^4 v^2 - 4[u_1^2 - C_1 u_1 + n_0^2 v^2]}} = \beta + C_3, C_3 = const.$

The left-hand part of the last equality (without the sign) has a kind $\frac{1}{2}\arcsin\frac{(u_2 - (\sigma n_0^2 v)/2)^2}{\sqrt{C_1^2 + n_0^2 v^2(\sigma^2 n_0^2 - 4)}}$. After substitutions we have a unknown quantity an invariant ratio

$$\cos^2[2(\beta + C_3)] = \frac{(u_2 - (\sigma n_0^2 v)/2)^2 u_1^2}{G_2} \tag{7}$$

where $G_2 = [u_2^2 - \sigma n_0^2 v u_2]^2 + 2[u_2^2 - \sigma n_0^2 v u_2][u_1^2 + n_0^2 v^2] + [u_1^2 - n_0^2 v^2]^2 + \sigma^2 n_0^4 v^2 u_1^2$ which is analytical relation.

Example. *If* $\sigma n_0 = 2$ *the equality (7) accept a following kind*

$$\cos^2[2(\beta + C_3)] = \frac{(z_2 - n_0 v \sin\alpha) z_1}{(z_2 - n_0 v \sin\alpha)^2 + z_1^2}.$$

In the last case, we *speak about variable dissipation systems with zero mean*. See above.

4. Another Examples from Plane Dynamics of a Rigid Body Interacting with a Medium

Consider systems also arising in the plane dynamics of a rigid body interacting with a resisting medium [3,4]. Since certain phase variables are cyclic, the general sixth-order system admits a separation of the independent subsystem of the third order. In turn, in this subsystem, by using the well-known technique, the system of the second order is isolated. Such systems have one property in common. Since, as a rule, variable dissipation systems with *zero mean* have additional symmetries; these systems have separatrices connecting hyperbolic saddle equilibrium states. That is why (absolutely) such systems cannot be structurally stable. Since deformations of such systems are only considered over a certain subset of all systems defined through a certain subclass of functions (right-hand sides) that makes it possible to preserve all the symmetries in the system, the systems under consideration remain relatively rough in certain domains of parameters.

Example 1. Consider systems of the following form on the two-dimensional cylinder

$$\alpha = -\Omega + A_1 F(\alpha)/\cos\alpha, \Omega^{\bullet} = A_2 F(\alpha), A_1 > 0, A_2 > 0, \tag{8}$$

under the following condition: F – smooth odd π-periodic function such that $F'(0) > 0$, $F'(\pi/2) < 0$, $F(\alpha) > 0$ if $\alpha \in (0, \pi/2)$ and $F(\alpha) < 0$ if $\alpha \in (\pi/2, \pi)$. Thus $Q = \{F\}$.

Lemma 1. *System (8) is relatively structurally stable. Moreover, any two systems of the form (8) are topologically equivalent.*

(a) For any F $\in$ Q, the phase portrait of system (8) is of one and the same topological type:

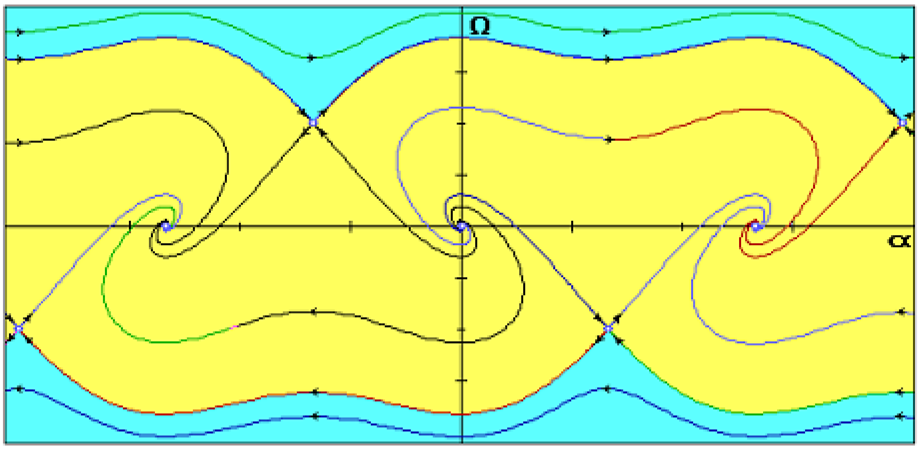

Fig. 1

(b) in every region of the phase cylinder (oscillatory and rotational) (see Fig. 1) the equivalence of its own is constructed; on the «key» separatrices, these equivalencies are «sewed».

(c) For instance, in the oscillatory region (see Fig. 1), the equivalence is constructed as follows. We construct not only an equivalence, i.e., a homeomorphism h of the phase cylinder, but, what is more, the conjugacy. In the oscillatory domain, there exist only two singular points, (0,0) and (π,0) (the first of them is repelling, and the second is an attracting one). Thus, we consider two systems (8) for the function $F_1(\alpha)$ and $F_2(\alpha)$. The corresponding phase flows of the phase cylinder are denoted by g_1^t and g_2^t. We require that the homeomorphism h take the origin to the origin. Consider a small circle S^1 around the origin. It can be chosen transversal to both fields of systems (1) for $F = F_1(\alpha)$ and $F = F_2(\alpha)$, simultaneously. We define $h(p) = p$ (accurate up to a linear contraction or dilation) for all $p \in S^1$ in such a way that $h(p_1^1) = h(p_2^1)$ and $h(p_1^2) = h(p_2^2)$. Here $h(p_1^1) = h(p_2^1)$, k = 1, 2, are two points on the circle S^1; for $F = F_k$, the separatrices of the vector field of system (8) which emanate from the origin and enter the saddles (in the central strip) pass through them. If q is not the origin, then there exists a unique $t \in R$ such that $g_1^t(q) = p \in S^1$. We set $h(q) = g_2^{-t}(p) = g_2^{-t} g_1^t(q)$. It is immediately seen that h is continuous and has a continuous inverse.

(d) By virtue of the constructed mapping h, the point (π,0) passes to the point (π,0) by continuity.

Example 2. Consider systems of the following form on the two-dimensional cylinder

$$\alpha' = -\omega + \sigma F(\alpha)\cos\alpha / I + \sigma\omega^2 \sin\alpha\,,\ \ \omega' = F(\alpha)/I - \omega\Psi(\alpha,\omega)\,,\ \ \sigma, I > 0 \qquad (9)$$

where $\Psi(\alpha,\omega) = \sigma F(\alpha)\sin\alpha / I - \sigma\omega^2 \cos\alpha$, under the previous conditions on the function *F*. It is likewise a variable dissipation system with *zero mean*.

Lemma 2. *The infinite-dimensional space of vector fields X(Q) corresponding to the system (2) is partitioned into the disjoint union* $X(Q) = X(Q_1) \coprod X(Q_2) \coprod X(Q_3)$ *having the following properties:*

(a) the system (2) defined via the spaces $X(Q_1)$, $X(Q_3)$ *, is relatively rough in the space X(Q);*

(b) the system (2) defined via the space $X(Q_2)$ *is a system of the first degree of nonroughness in the space X(Q);*

(c) the set $X(Q_2)$ *has zero measure in the space X(Q);*

(d) the sets $X(Q_1)$, $X(Q_3)$ *have a finite measure in the space X(Q).*

The topological equivalence in this case is constructed depending on the region of the phase cylinder and also depending on the classes $X(Q_k)$, k = 1, 2, 3.

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