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STRUCTURAL STABLE VECTOR FIELDS IN RIGID BODY DYNAMICS

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Abstract: This activity is devoted to the study of the relative structural stability (relative roughness) of dynamical systems considered not on the whole space of dynamical systems, but only on its certain subspace. Moreover, the space of deformations of systems also does not coincide with the whole space of admissible deformations. In particular, we consider systems of differential equations arising in the dynamics of a rigid body interacting with a resisting medium. We show that they are relatively rough under certain conditions, they can also have nonroughness property of different degrees.

1. Introduction

Rough (structurally stable) systems can be considered as simplest ones; they are most spread in the corresponding space of dynamical systems. Indeed, rough systems are isolated by inequality-type conditions; therefore, it is natural to consider them as the most general case. One can draw a far-reaching analogy between rough systems and functions of one variable that have only simple roots, as well as the curves having no singularities and considered in a finite part of a plane. This analogy is rather fruitful, in particular, for working out efficient methods of qualitative study.

Sometimes it is of interest to consider the relative roughness, namely, the roughness with respect to a certain class of dynamical systems, i.e., with respect to a certain subset of the space of systems. This concept of relative roughness can be used in isolating nonrough systems, i.e., systems of first degree on nonroughness. Note that from the point of view of such a classification of nonrough systems, conservative systems are systems of infinite degree of nonroughness, in other words, its degree of nonroughness is higher than any finite degree of nonroughness. Thus, conservative systems are rather «rare» from the standpoint of such a classification.

However, considering the class of conservative (or Hamiltonian) systems, we can introduce the concept of roughness of a system with respect to this class. Namely this concept was used in fact by Poincare (without the term «roughness»).

Systems of the first degree of nonroughness can be defined as systems which are relatively rough in the set of (relatively) nonrough systems (the precise definition will be given in what follows).

A nonrough (relatively nonrough) vector field can be topologically equivalent to the rough (relatively rough) vector field. For example, on a two-dimensional sphere, a situation is possible in which the vector field is (absolutely) nonrough, although it is topologically equivalent to a rough vector field (on the topological equivalence, see below). The main cause of nonroughness in the last case is the degeneracy of the derivative in a neighborhood of a limit set. In the case of analytic systems, requiring that the right-hand sides of a system have no less than five derivatives, one can define the systems of the second degree of roughness as systems relatively rough in the set of systems that are nonrough and are not systems of the first degree of nonroughness.

In an absolutely similar way one can define systems of 3rd, 4th, …, *n*-th degree of roughness. The definition is introduced by induction. In the case of systems with analytical right-hand sides, which we consider, the definition of system proximity is introduced. Thus, the dynamical system is called a system of the *n*-th degree of roughness in a closed region if it is a nonrough system that is not a nonrough system of degree less or equal to *n*-1 and if it is relatively rough in the set of nonrough systems that are not nonrough systems of degree less or equal to *n*-1.

2. Definition of the Relative Structural Stability (Relative Roughness)

The classical definition of roughness [1-4], and also the definition given in [5] operate with two objects, namely, with classes of systems and with their own space of deformations of systems with the topology. For the first time, the definition of roughness of a dynamical system on the plane was given under certain additional assumptions with respect to the set of systems being considered. Namely, it was additionally assumed that the boundary of the domain where the system was considered is a noncontractible cycle for the trajectory of this system, i.e., a simple smooth closed curve without contacts (not tangent to the trajectory of the system). Obviously, in this case, this curve is also a noncontractible cycle for the trajectory of any system sufficiently close to that under consideration. Although this assumption severely restricts the class of systems being considered, the meaning of the concept of system's roughness is retained, while the definition of roughness is much simple than that given under general assumptions on the boundary of the domain.

One can introduce the definition of roughness in such a way that the presence of nonrough trajectories lying on the boundary of the domain would not be forbidden. But this does not correspond to the meaning of the concept of roughness.

In our opinion, it is both natural and necessary from various points of view to introduce the concept of roughness without special assumptions about the boundary of a domain [6].

As for the concept of roughness (as well as that of various degrees of nonroughness), it is based on the concept of the topological equivalence of dynamical systems.

Let $X^r(M)$ be the space of C^r-vector field on a compact manifold M with C^r-topology, $r \geq 1$. Two vector fields X, $Y \in X^r(M)$ are called *topologically equivalent* if there exists a homeomorphism $h: M \to M$ that takes trajectories of the field X to trajectories of the field Y preserving their orientations; this last condition means that if $p \in M$ and $\delta > 0$, then there exists $\varepsilon > 0$ such that if $0 < t < \delta$, then $hX_t(p) = Y_{t'}(h(p))$ for a certain $t' \in (0, \varepsilon)$. We call h a *topological equivalence* between X and Y. Thus, we have defined an equivalence relation on $X^r(M)$. Another, more strong relation is the conjugacy of vector field flows. Two vector fields X and Y are called *conjugate*, if there exists a topological equivalence h that preserves the parameter t; this means that $hX_t(p) = Y_t(h(p))$ for all $p \in M$ and $t \in R$.

The definition given by Andronov and Pontryagin, along with the closeness in a certain topology of the system under consideration and its deformation requires that a homeomorphism that realizes the topological equivalence of the last two systems should be close to the identity operator. On the other hand, the definition given by Peixoto does not require such a closeness property.

If the system is rough in the Andronov-Pontryagin sense, then it is rough in the Peixoto sense as well. Moreover, the necessary and sufficient conditions of roughness in the Andronov-Pontryagin sense coincide with the necessary and sufficient conditions of roughness in the Peixoto sense. The latter definition has the following advantage: it directly implies that rough systems fill in regions in the space of dynamical systems. On the other hand, if the first definition is used, this property is to be proved relying on the necessary and sufficient conditions of roughness.

Let Ξ be a sufficiently small neighborhood of the vector field X under consideration in $X^r(M)$. As was already noted in brief, in the original definition of roughness given by Andronov and Pontryagin, it is also required that for sufficiently small Ξ, the homeomorphism realizing the topological equivalence between X and Y can be made arbitrarily close to the identity operator in the C^0-topology (i.e., it can shift points of M arbitrarily small). Since the version of this requirement was proposed by Peixoto, in the cases where it is necessary to clear out exactly which version of roughness is meant, one speaks about the roughness in the Andronov-Pontryagin sense and about the roughness in the Peixoto sense. However, at present, it is still not clear whether these versions differ from one another and whether one of them has any tangible advantages over the other.

The presented definition depends of *r*. If it is necessary to explicitly point this dependence, one can speak about the roughness in the class C^r.

Up to now, we spoke about the global properties of vector fields on manifolds. One can analyze the local topological behavior of trajectories of vector fields. For vector fields from a certain open dense subset in the space $X^r(M)$, one can describe the behavior of trajectories in a neighborhood of each point of the manifold. In addition, the local structure of trajectories is not changed under small perturbations of the field (the so-called local roughness). Thus, the topological conjugacy yields a complete classification.

In higher dimensions, the set of rough fields is still vast, but it ceases to be everywhere dense. Here, there exist versatile and more complex phenomena that are preserved under small perturbations of the original field. Even for rough fields the structure of trajectories is not completely known, and its description is still a field of active research.

By virtue of the classical definitions of the structural stability criteria for the latter were discussed both for linear nonatonomous systems and for the classes of nonlinear systems. The attributes of the structural stability for lower-dimensional systems are formulated as hypotheses due to Smale. They are extended to higher dimensions.

Several other modified definitions of roughness appeared recently. All of them have one feature in common, namely, deformations of dynamical systems considered on a certain manifold M^n are taken in the whole space of smooth vector fields $\chi(C^r)$ in the C^r-topology (most often, $r = 1$).

We consider vector fields (dynamical systems) that are deformed not over the whole class $X^r(M^n)$ of fields, but only over a certain subclass $X(Q)$ defined via a class of functions $Q \in C^r$.

Definition 1. *A vector field v on a manifold M^n is said to be relatively structurally stable (relatively rough or rough with respect to the class of fields $X(Q)$ defined via the class of functions Q) if for any neighborhood Θ of the homeomorphism 1_{M^n}, in the space of all homeomorphisms with C^0-topology, there exists a neighborhood $U \in X(Q)$ of the vector field v under consideration such that the latter is equivalent to any vector field from $U \in X(Q)$ via a certain homeomorphism from Θ.*

Note that the closeness of vector fields is understood in the C^1-topology, and the closeness of homeomorphisms is understood in the C^0-topology. Moreover, here, we deal not with the conjugacy but with the equivalence.

Also, we note that up to now, in the definition given above, the following aspects are important:

1. The homeomorphism realizing the equivalence is sufficiently small;

2. The C^1-topology in the space of vector fields under consideration.

3. Relative Structural Instability (Relative Nonroughness) of Various Degrees

Similar to the definition of a vector field of the first degree of nonroughness, we can define the fields of the first degree of relative nonroughness by considering vector field deformations in the subspace *X(Q)* of the space of all vector fields.

Definition 2. *A vector field v on the manifold M^n is called a vector field of the first degree of nonroughness if it is not a relatively rough vector field and if, for any neighborhood Θ of the homeomorphism 1_{M^n} in the space of all homeomorphisms with the C^0-topology, there exists a neighborhood $U \in X(Q)$ of the vector field such that the field v is topologically equivalent to any field from $U \in X(Q)$ that is not relatively rough. This equivalence relation is realized by a certain homeomorphism from Θ.*

Note that the closeness of vector fields in this case is understood in the C^3-topology.

In a similar way, one can define vector fields which are fields of the *n* degree of relative nonroughness. Here, the C^{2n+1}-topology in the space of vector fields is used.

Definition 3. *A vector field v on the manifold M^n is called a vector field of the n-th degree of relative nonroughness if it is not a relatively nonrough vector field of degree less than or equal to n-1 and if for any neighborhood Θ of the homeomorphism 1_{M^n} in the space of all homeomorphisms with C^0-topology, there exists a neighborhood $U \in X(Q)$ of the vector field v such that the field v is topologically equivalent to any field from $U \in X(Q)$ that is not relatively rough or relatively nonrough vector field of degree less than or equal to n-1 via a certain homeomorphism from Θ.*

4. Examples from the Dynamics of a Rigid Body Interacting with a Medium

Consider systems arising in the plane dynamics of a rigid body interacting with a resisting medium. Since certain phase variables are cyclic, the general sixth-order system admits a separation of the independent subsystem of the third order. In turn, in this subsystem, by using the well-known technique, the system of the second order is isolated. Such systems have one property in common. Since, as a rule, variable dissipation systems with zero mean have additional symmetries; these systems have separatrices connecting hyperbolic saddle equilibrium states. That is why (absolutely) such systems cannot be structurally stable. Since deformations of such systems are only considered over a certain subset of all systems defined through a certain subclass of functions (right-hand sides) that makes it possible to preserve all the symmetries in the system, the systems under consideration remain relatively rough in certain domains of parameters.

Example 1. Consider systems of the following form on the two-dimensional cylinder

$$\alpha = -\Omega + A_1 F(\alpha)/\cos\alpha,\ \Omega^{\bullet} = A_2 F(\alpha),\ A_1 > 0,\ A_2 > 0, \quad (1)$$

under the following condition: F – smooth odd π-periodic function such that $F'(0) > 0$, $F'(\pi/2) < 0$, $F(\alpha) > 0$ if $\alpha \in (0, \pi/2)$ and $F(\alpha) < 0$ if $\alpha \in (\pi/2, \pi)$. Thus $Q = \{F\}$.

Lemma 1. *System (1) is relatively structurally stable. Moreover, any two systems of the form (1) are topologically equivalent.*

Sketch of the Proof. We define the set of vector fields *X(Q)* corresponding to system (1); moreover, the function F runs over the whole class Q. The space of system parameters is infinite-dimensional. Lemma 1 follows from the following observations.

(a) For any $F \in Q$, the phase portrait of system (1) is of one and the same topological type:

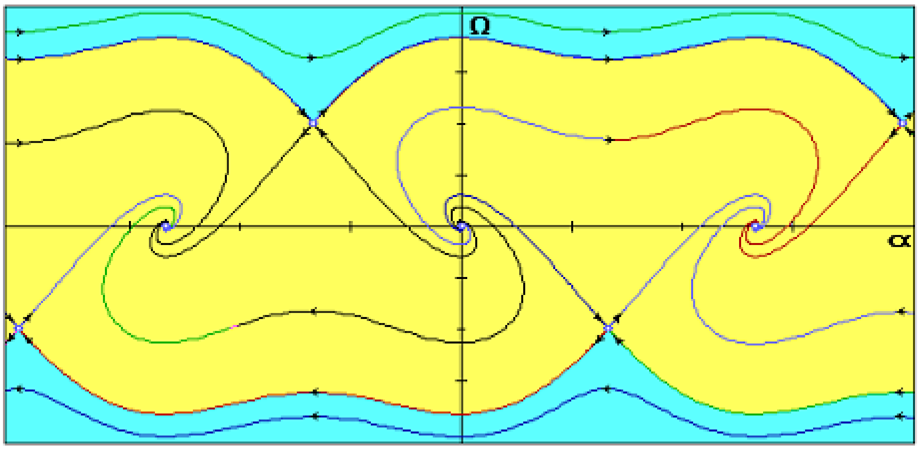

Fig. 1

(b) in every region of the phase cylinder (oscillatory and rotational) (see Fig. 1) the equivalence of its own is constructed; on the «key» separatrices, these equivalencies are «sewed».

(c) For instance, in the oscillatory region (see Fig. 1), the equivalence is constructed as follows. We construct not only an equivalence, i.e., a homeomorphism h of the phase cylinder, but, what is more, the conjugacy. In the oscillatory domain, there exist only two singular points, (0,0) and (π,0) (the first of them is repelling, and the second is an attracting one). Thus, we consider two systems (1) for the function $F_1(\alpha)$ and $F_2(\alpha)$. The corresponding phase flows of the phase cylinder are denoted by g_1^t and g_2^t. We require that the homeomorphism h take the origin to the origin. Consider a small circle S^1 around the origin. It can be chosen transversal to both fields of systems (1) for $F = F_1(\alpha)$ and $F = F_2(\alpha)$, simultaneously. We define $h(p) = p$ (accurate up to a linear contraction or dilation) for all $p \in S^1$ in such a way that $h(p_1^1) = h(p_2^1)$ and $h(p_1^2) = h(p_2^2)$. Here, $h(p_1^1) = h(p_2^1)$, $k = 1, 2$, are two points on the circle S^1; for $F = F_k$, the separatrices of the vector field of system (1) which emanate from the origin and enter the saddles (in the central strip) pass through them. If q is not the origin, then there exists a unique $t \in R$ such that $g_1^t(q) = p \in S^1$. We set $h(q) = g_2^{-t}(p) = g_2^{-t} g_1^t(q)$. It is immediately seen that h is continuous and has a continuous inverse.

(d) By virtue of the constructed mapping h, the point $(\pi,0)$ passes to the point $(\pi,0)$ by continuity.

Example 2. Consider systems of the following form on the two-dimensional cylinder

$$\alpha' = -\omega + \sigma F(\alpha)\cos\alpha / I + \sigma\omega^2 \sin\alpha, \ \omega' = F(\alpha)/I - \omega\Psi(\alpha,\omega), \ \sigma, I > 0 \quad (2)$$

where $\Psi(\alpha,\omega) = \sigma F(\alpha)\sin\alpha / I - \sigma\omega^2 \cos\alpha$, under the previous conditions on the function F. It is likewise a variable dissipation system with zero mean.

Lemma 2. *The infinite-dimensional space of vector fields X(Q) corresponding to the system (2) is partitioned into the disjoint union* $X(Q) = X(Q_1) \coprod X(Q_2) \coprod X(Q_3)$ *having the following properties:*

(a) the system (2) defined via the spaces $X(Q_1)$, $X(Q_3)$ *, is relatively rough in the space X(Q);*

(b) the system (2) defined via the space $X(Q_2)$ *is a system of the first degree of nonroughness in the space X(Q);*

(c) the set $X(Q_2)$ *has zero measure in the space X(Q);*

(d) the sets $X(Q_1)$, $X(Q_3)$ *have a finite measure in the space X(Q).*

The topological equivalence in this case is constructed depending on the region of the phase cylinder and also depending on the classes $X(Q_k)$, $k = 1, 2, 3$.

6. Conclusions

In general, the dynamics of a body interacting with a medium is exactly a domain where there arise either dissipative systems or systems with the so-called antidissipation. It is very difficult to construct any method that would be general enough for studying such systems; therefore, it becomes urgent to elaborate such a technique namely for those classes of systems which appear in simulating the motions of such bodies which contact with a medium on a plane part of their outer surface.

Since in such a simulation, one uses the experimental data on the properties of jet flow, it becomes necessary to study the class of dynamical systems that have the (relative) structural stability property. Therefore, it is quite natural to introduce the definitions of relative roughness for such systems.

Thus, we speak about systems with the so-called *variable dissipation*; here, the term «variable» refers rather to a possible change of sign of the dissipation factor than to its magnitude. On average over the period in the attack angle, the dissipation can be both positive and negative; it can also be zero. In the last case, we *speak about variable dissipation systems with zero mean*.

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