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Spatial motion of a pendulum in a jet flow: qualitative aspects and integrability

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The presented work is devoted to the problem of motion in a resisting medium of a rigid body whose surface of contact with the medium is only a plane part of its exterior surface. In this case, the force field is constructed from the reasons for the medium action on the body under the streamline (or separation) flow around and quasi-stationarity conditions. It turns out that studying the motion of such a class of bodies reduces to systems with energy scattering (dissipative systems) or with its pumping (the so-called anti-dissipative systems) [1, 2]. Note that similar problems were already appeared in applied aerodynamics in the studies of Central Aero-Hydrodynamic Institute named after professor N. E. Zhukovskii.

1 Introduction

This work is devoted to the development of qualitative methods in the theory of nonconservative systems that arise, e.g., in such fields of science as the dynamics of a rigid body interacting with a resisting medium, oscillation theory, etc. This material can call the interest of specialists in the qualitative theory of ordinary differential equations, in rigid body dynamics, as well as in fluid and gas dynamics since the work uses the properties of motion of a rigid body in a medium under the streamline flow around conditions.

The author obtains a full spectrum of complete integrability cases for nonconservative dynamical systems having nontrivial symmetries. Moreover, in almost all cases of integrability, each of the first integrals is expressed through a finite combination of elementary functions and is a transcendental function of its variables, simultaneously. In this case, the transcendence is meant in the complex analysis sense, i.e., after the continuation of the functions considered to the complex domain, they have essentially singular points. The latter fact is stipulated by the existence of attracting and repelling limit sets in the system considered (for example, attracting and repelling foci) [3, 4].

The author obtains new families of phase portraits of systems with variable dissipation on lower- and higher-dimensional manifolds. He discusses the problems of their absolute or relative roughness, He discovers new integrable cases of the rigid body motion, including those in the classical problem of motion of a spherical pendulum placed in the over-running medium flow.

2 Spatial Motion of an Axially Symmetric Rigid Body in a Resisting Medium

Consider a moving homogeneous body of mass m. A portion of its surface is a flat disk. A jet flow is past the body [1, 2]. The other portion of the body's surface is inside the volume bounded by the jet stalling at the disk edge and is not affected by the medium. Conditions are similar when homogeneous circular cylinders enter water. Assume that there are no tangential forces. Then the force $\mathbf{S}$ applied by the medium to the body at the point N does not change the orientation relative to the body (is normal to the disk) and is quadratic with respect to the velocity of its center D.

If the above conditions are satisfied, the motions of the body include translational deceleration similar to the case of plane-parallel (unperturbed) motion: the body can undergo translational motion along its axis of symmetry, i.e., perpendicularly to the disk plane. We choose the right-hand coordinate system $Dxyz$ with the Dx-axis aligned with the axis of geometrical symmetry of the body and the $Dy-$ and $Dz-$axes fixed to the disk. The components of the angular velocity vector Ω in the system $Dxyz$ are denoted by $\{\Omega_x, \Omega_y, \Omega_z\}$. The inertia tensor of the dynamically symmetric body is diagonalized in the body axes $Dxyz$: diag$\{I_1, I_2, I_2\}$.

We will also use the quasi-stationarity hypothesis and assume for simplicity that $R = DN$ is defined at least by the attack angle α between the velocity vector $\mathbf{v}$ of the center D of the disk and the straight line Dx. Thus, $DN = R(\alpha, \ldots)$. Moreover, we assume that $S = |\mathbf{S}| = s_1(\alpha)v^2,\ v = |\mathbf{v}|$. For convenience, we introduce (as in the case of plane-parallel motion) an auxiliary alternating function $s(\alpha)$: $s_1 = s_1(\alpha) = s(\alpha)\mathrm{sgn}\cos\alpha > 0$ instead of the coefficient $s_1(\alpha)$. Thus, the pair of functions $R(\alpha, \ldots)$ and $s(\alpha)$ defines the forces and moments exerted by the medium on the disk under such assumptions.

2.1 Dynamic Part of the Equations of Spatial Motion

Let us use the spherical coordinates (v, α, β_1) of the tip of the velocity vector $\mathbf{v} = \mathbf{v}_D$ of the point D relative to the flow to measure the angle β_1 in the plane of the disk. Expressing the quantities (v, α, β_1), using nonintegrable relations, in terms

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of the cyclic kinematic variables and velocities and supplementing them with the projections $(\Omega_x, \Omega_y, \Omega_z)$ of the angular velocity onto the body axes, we consider them as quasivelocities. Using the theorems on the motion of the center of mass (in the body-fixed frame of reference $Dxyz$) and variation in the angular momentum in the same frame, we obtain the dynamic part of the differential equations of motion in the six-dimensional phase space of quasivelocities ($\sigma = DC$). The first group of equations describes the motion of the center of mass, while the second group the motion around the center of mass [1].

Let us consider the class of problems where a rigid body moves through a medium under a follower force acting along the axis of geometrical symmetry of the body and producing classes of motions (imposed constraints) of interest, this force being the reaction of the constraints imposed. Here, the follower force is such that condition $v = \text{const}$ is satisfied all the time. The Routh-cyclic invariant relation $\Omega_x \equiv \Omega_{x0} = \text{const}$ holds at all instants of time. In what follows, we will examine the case where the rigid body does not rotate about its longitudinal axis, i.e., $\Omega_{x0} = 0$. Then the independent dynamic part of the equations of motion in the four-dimensional phase space is given by

$$\begin{aligned} &\dot{\alpha}\cos\alpha\cos\beta_1 - \dot{\beta}_1 v\sin\alpha\sin\beta_1 + \Omega_z v\cos\alpha - \sigma\dot{\Omega}_z = 0, \\ &\dot{\alpha}\cos\alpha\sin\beta_1 + \dot{\beta}_1 v\sin\alpha\cos\beta_1 - \Omega_y v\cos\alpha + \sigma\dot{\Omega}_y = 0,\ I_2\dot{\Omega}_y = -z_N s(\alpha)v^2,\ I_2\dot{\Omega}_z = y_N s(\alpha)v^2, \end{aligned} \tag{1}$$

where y_N and z_N are Cartesian coordinates, in the plane of the disk, of the point N of application of the resisting force. System (1) includes the influence functions y_N, z_N, and s. To determine them qualitatively (by analogy with the case of plane-parallel motion), we will use experimental data on the properties of jet flow.

We will analyze system (1) for the following influence functions; such an analysis can be performed for an arbitrary pair of functions y_N, z_N, and s: $y_N = A\sin\alpha\cos\beta_1 - h\Omega_z/v,\ z_N = A\sin\alpha\sin\beta_1 + h\Omega_y/v,\ s(\alpha) = B\cos\alpha,\ A, B, h > 0.$

The resultant system will be called a reference one. The coefficient h appears in the terms proportional to the rotary derivatives of the moment of hydroaerodynamic forces with respect to the components of the angular velocity of the body.

System (1) is a dynamic system with variable dissipation and with zero mean (with respect to the angle of attack) [3]. This means that the integral of the divergence of its right-hand side over the period of the angle of attack, which describes the variation in the phase volume (after the appropriate reduction of the system), is equal to zero. The system is semiconservative.

Projecting the angular velocities onto the moving axes not fixed to the body so that $z_1 = \Omega_y\cos\beta_1 + \Omega_z\sin\beta_1,\ z_2 = -\Omega_y\sin\beta_1 + \Omega_z\cos\beta_1$ and introducing dimensionless variables $w_k,\ k = 1, 2$, and parameters by the formulas $h_1 = hB,\ \sigma h_1/I_2 = H_1,\ \beta = \sigma^2 AB/I_2,\ \sigma z_k = v w_k$ ($\sigma < \cdot >= v <'>$), we obtain the forth order system:

$$\begin{aligned} &\alpha' = -(1 + H_1)w_2 + \beta\sin\alpha,\ w_2' = \beta\sin\alpha\cos\alpha - (1 + H_1)w_1^2\cot\alpha - H_1 w_2\cos\alpha, \\ &w_1' = (1 + H_1)w_1 w_2\cot\alpha - H_1 w_1\cos\alpha, \end{aligned} \tag{2}$$

$$\beta_1' = (1 + H_1)w_1\cot\alpha, \tag{3}$$

which includes the independent third-order subsystem (2).

If $\beta = H_1$ then after the change of variables $w^* = \ln|w_1|$, the divergence of the right-hand side of (2) ((2), (3)) will become identically equal to zero, which allows considering the system(s) to be conservative.

2.2 Complete List of Transcendental First Integrals

Theorem. *The system (2), (2) possesses three invariant relations (the complete tuple) which are the transcendental functions from the complex analysis view of point. Herewith, all the relations express in terms of the finite combination of the elementary functions.*

For example, one of the first integrals has the following form:

$$\frac{(1 + H_1)(w_2^2 + w_1^2) - (\beta + H_1)w_2\sin\alpha + \sin^2\alpha}{w_1\sin\alpha} = C_1 = \text{const}.$$

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