SOME CLASSICAL PROBLEMS IN A THREE-DIMENSIONAL DYNAMICS OF A RIGID BODY INTERACTING WITH A MEDIUM

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ABSTRACT - A generalization to the 3D case of plane-parallel motion of a solid interacting with resistant medium, for which the conditions of jet or escape flow hold, is considered. Because of using experimental information on the properties of jet flow, we are forced to consider a whole class of dynamic systems that possess the property of relative structural stability. An example of a dynamic system that has a complete set of first integrals expressible via elementary functions is given. The present paper is devoted to the study of the possible extension to the 3D case of results of plane-parallel dynamics of motion of a solid in resistant medium for which the conditions of jet or escape flow hold. Here the procedures of construction of three-dimensional phase portraits for systems with variable dissipation are applied. We consider an example of using these procedures in the study of a class of 3D motions of a solid in a resistant medium for which the system is subjected to a nonintegrable servoconstraint that makes it possible to consider the system of dynamical motion equations to be of smaller dimension.

NOTATION

P	Plane area
S	Resistant force
N	Point of an acting of the force
α	Angle of attack
v	*V*ector of velocity
D	Point of the plane area
C	Center of mass
R, s	Aerodynamic functions
T	Servoconstraint force
{A,B,C}	Main components of tensor of inertia
(x_0, y_0, z_0)	Cartesian coordinates
(θ, ψ, φ)	Angles for determining of the position of a body
(x, y, z)	Moving axes
(e_x, e_y, e_z)	System of coordinates
σ	Distance CD
(v, α, β)	Spatial polar coordinates
(p, q, r)	Projection of angular velocity
(ξ, η, r, q)	Coordinates of a pendulum

INTRODUCTION

Dynamic model of interaction of a rigid body with resisting medium provided jet flow (Lokshin *et al.,* 1986), considered in activity, not only allows successfully to transfer outcomes appropriate problems from plane dynamics of a rigid body interacting with the medium (Gurevich, 1979) and to receive their spatial analogs, but also to detect new cases of integrability till the Jacobi (Samsonov *et al.,* 1989, 1990). Thus in some cases the integrals express through elementary functions. In activity the integrability of classical is shown in the problems about a spherical pendulum, located in a flow by filling of a medium, about spatial motion of a body at availability constraint, and also the mechanical and topological analogies are shown in the last two problems.

The hypothesizes adduced in (Chaplygin, 1976) and concerning of properties of a medium, have found the reflection in construction of spatial (3D) dynamic model of interaction of a rigid body with resisting medium. In this connection there is a capability of formalizing of the model suppositions and obtaining of a full system of ordinary differential equations.

DYNAMIC SYSTEM

Area of interaction

All interaction of medium with a body is concentrated on that part of a surface of a body which has the shape of convex plane area P.

Force of interaction

As the interaction happens under the laws of jet flow the force S of this interaction is directed on a normal line to area and the point N of the acting of this force is determined only in one parameter - by an angle of attack α which is measured between vector of velocity v of a point D of a plate and external normal line in this point (straight line CD). The point D is the interception of the straight line CD (C - center of mass) that is perpendicular to plane P. Thus, $DN = R(\alpha)$.

Some of hyposesys

Size of force of resistance we shall accept as $S = sv^2$, where v is the module of speed of a point D, and coefficient of resistance s is the function only of angle of attack: $s = s(\alpha)$.

Description of servoconstraint

There is the additional force T, which acts on a body on the straight line CD. Let's name it as “force of a thrust”. The introduction of this force is used, as for maintenance of some specific classes of motions (thus T is the reaction of the possible (or probable) imposed constraint and in the methodical purposes, which pursue learning of interesting non-linear systems (having character of pendulum) arising at the reduction of the order. In case of absence external force T the body makes free braking (deceleration) in a resisting medium.

Systems of coordinates

Systems of coordinates connected to a body shall designate through $Dxyz$. The last coordinate system connected to a point D is selected such that the tensor of inertia in the given system has diagonal type: $diag\{A, B, C\}$. Mass distribution we shall accept by such that longitudinal principal axis of inertia coincides an axis CD (it is an axis Dx), and the axes Dy and Dz lie in a plane P and will derivate with the right of coordinate system. Moreover, we shall consider case dynamically symmetrical rigid body, i.e. the equality

$$B = C$$

is executed.

System of dynamical equations

In this case for the description of a position of a body in 3D space it is possible to select the Cartesian coordinates (x_0, y_0, z_0) of a point D and three angles (θ, ψ, φ), which are determined similarly to classical navigational angles.

By virtue of the theorem of motion of center of mass in space in projections on moving axes (x, y, z) and theorem of change of kinetic moment of rather these axes, we receive a full system of differential equations considered in dynamic space of quasivelocities

$$
\begin{gathered}
v'\cos\alpha - \alpha' v\sin\alpha + qv\sin\alpha\sin\beta - rv\sin\alpha\cos\beta + \sigma(q^2 + r^2) = \frac{T}{m} - \frac{s(\alpha)}{m}v^2 \\
v'\sin\alpha\cos\beta + \alpha' v\cos\alpha\cos\beta - \beta' v\sin\alpha\sin\beta + rv\cos\alpha - pv\sin\alpha\sin\beta - \sigma pq - \sigma r' = 0 \\
v'\sin\alpha\sin\beta + \alpha' v\cos\alpha\sin\beta + \beta' v\sin\alpha\cos\beta - qv\cos\alpha + pv\sin\alpha\cos\beta - \sigma pr + \sigma q' = 0 \\
Ap' + (C - B)qr = 0 \\
Bq' + (A - C)pr = -z_N s(\alpha)v^2 \\
Cr' + (B - A)pq = y_N s(\alpha)v^2
\end{gathered}
$$

Here coordinates of a point N in a system (e_x, e_y, e_z) will accept as: $(0, y_N(\alpha,\beta), z_N(\alpha,\beta))$ where $y_N(\alpha,\beta) = R(\alpha)\cos\beta,$ $z_N(\alpha,\beta) = R(\alpha)\sin\beta,$ σ is the distance CD.

In a general dynamic system of the twelfth order by virtue of cyclic character of positional coordinates the splitting of independent subsystem of sixth order happens in a phase space of quasivelocities $\mathrm{T}^2\{\alpha,\beta\} \times \Re^1\{v\} \times \Re^3\{p,q,r\}$. Here (v,α,β) are the spatial polar coordinates of the velocity of point D, (p,q,r) is the projection of angular velocity to coordinate system connected with a body.

DYNAMICALLY SYMMETRICAL RIGID BODY WITH CONSTRAINT

Dynamic equations of motion of a free rigid body

Dynamic equations of motion of a free rigid body at availability of servoconstraint of a type

$$v = const$$

(plane version of the given problem see in (Shamolin, 1994, 1996)) accept the first integral

$$p = p_0$$

and look like

$$
\begin{gathered}
\alpha' = -z_2 + \sigma\frac{v}{B}\frac{F(\alpha)}{\cos\alpha} + \frac{\sigma}{v}\frac{A}{B}p_0\frac{z_1}{\cos\alpha} \\
z_2' = \frac{F(\alpha)}{B}v^2 - z_1\left[z_1\frac{\cos\alpha}{\sin\alpha} - \frac{A}{B}p_0 + \frac{\sigma}{v}\frac{A}{B}p_0\frac{z_2}{\cos\alpha}\right]
\end{gathered} \qquad (1)
$$

$$z_1' = z_2\left[z_1\frac{\cos\alpha}{\sin\alpha} - \frac{A}{B}p_0 + \frac{\sigma}{v}\frac{A}{B}p_0\frac{z_2}{\cos\alpha}\right]$$

$$\beta' = -p_0 + \left[z_1\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma}{v}\frac{A}{B}p_0\frac{z_2}{\cos\alpha}\right] \qquad (2)$$

Here $z_1 = q\cos\beta + r\sin\beta, z_2 = r\cos\beta - q\sin\beta.$

The function in a dynamic system (1),(2) has the following properties: for qualitative description of its we use being available the experimental information on properties of jet flow.

The function F is smooth, odd, π - periodic, satisfying to a property: $F(\alpha) > 0$ at $\alpha \in (0, \frac{\pi}{2})$.

Main theorems

Proposition 1. *The dynamic system (1),(2) is equivalent (in trajectory sense) topologically to a system (1),(2) under such condition:*

$$F = F_0(\alpha) = A'B'\sin\alpha\cos\alpha, A', B' > 0 \qquad (3)$$

The system (1),(2) under condition of (3) will accept a type of analytical:

$$\alpha' = -z_2 + \sigma n_0^2 \sin\alpha + \frac{\sigma}{v}\frac{A}{B}p_0\frac{z_1}{\cos\alpha}$$

$$z_2' = n_0^2 v^2 \sin\alpha\cos\alpha - z_1\left[z_1\frac{\cos\alpha}{\sin\alpha} - \frac{A}{B}p_0 + \frac{\sigma}{v}\frac{A}{B}p_0\frac{z_2}{\cos\alpha}\right]$$

$$z_1' = z_2\left[z_1\frac{\cos\alpha}{\sin\alpha} - \frac{A}{B}p_0 + \frac{\sigma}{v}\frac{A}{B}p_0\frac{z_2}{\cos\alpha}\right]$$

$$\beta' = -p_0 + \left[z_1\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma}{v}\frac{A}{B}p_0\frac{z_2}{\cos\alpha}\right]$$

Here $n_0^2 = \frac{A'B'}{B}$.

Let's consider the capabilities of an integration of a system (1),(2) at a level $p_0 = 0$. At this field of vectors of a system (1) has three kinds of symmetry:

1) *A central symmetry*. Such symmetry near the points $(\pi k, 0, 0), k \in Z$ in space $\Re^3\{\alpha, z_2, z_1\}$ arise for the reason that the vector field in coordinates $\{\alpha, z_2, z_1\}$ changes the sign at replacement

$$\begin{pmatrix} \pi k - \alpha \\ -z_2 \\ -z_1 \end{pmatrix} \Rightarrow \begin{pmatrix} \pi k + \alpha \\ z_2 \\ z_1 \end{pmatrix}$$

2) *Some mirror symmetry* (SMS). Such symmetry is related to the planes $\Lambda_i, i \in Z$ where $\Lambda_i = \{(\alpha, z_2, z_1) \in \Re^3 : \alpha = \frac{\pi}{2} + \pi i\}$ arises for the reason that α - making component of field of vectors of our system in coordinates $\{\alpha, z_2, z_1\}$ is saved at replacement

$$\begin{pmatrix} \frac{\pi}{2} + \pi k - \alpha \\ z_2 \\ z_1 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\pi}{2} + \pi k + \alpha \\ z_2 \\ z_1 \end{pmatrix}$$

and z_2 - and z_1 - making components change the sign;

3) *by a symmetry is related to the planes* $\{(\alpha, z_2, z_1) \in \Re^3 : z_1 = 0\}$, namely, z_2 - and α - making of components of vector field of a system are saved at replacement

$$\begin{pmatrix} \alpha \\ z_2 \\ z_1 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha \\ z_2 \\ -z_1 \end{pmatrix}$$

and z_1 - making component changes the sign.

In activities (Shamolin, 1994) the first integral of a system from plane dynamics expressed through elementary functions.

Theorem 1. *The system (1) at $p_0 = 0$ has a full set of the first integrals, one from which is meromorphic function, and second is transcendental. The system (1),(2) at $p_0 = 0$ also is quite integrated till Jacobi, two from which first integral are integrals of systems (1) at $p_0 = 0$ and third is analytical function.*

The meromorphic integral of a system (1) at $p_0 = 0$ will look like

$$\frac{z_1^2 + z_2^2 - \sigma n_0^2 v z_2 \sin\alpha + n_0^2 v^2 \sin^2\alpha}{z_1 \sin\alpha} = C_1 \qquad (4)$$

As the system (1),(2) at $p_0 = 0$ has a variable dissipation and also is analytical, for its it is possible in an obvious kind to find two other additional integrals. The following identity is executed

$$u_1 = \frac{1}{2}\{C_1 \pm G\}$$

Here $G = \sqrt{C_1^2 - 4[u_2^2 - \sigma n_0^2 v u_2 + n_0^2 v^2]}, u_1 = z_1\tau, u_2 = z_2\tau, \tau = \sin\alpha$ (for search of additional integrals it is used the meromorphic first integral (4)). A quadrature for search of a unknown quantity of an integral linking the sizes u_2 and τ is received by a kind

$$\int \frac{d\tau}{\tau} = \int \frac{(\sigma n_0^2 v - u_2)du_2}{2[u_2^2 - \sigma n_0^2 v u_2 + n_0^2 v^2] - \frac{C_1}{2}(C_1 \pm G)} \tag{5}$$

If $w_1 = u_2 - \frac{\sigma n_0^2 v}{2}$ the right member (5) accept a kind

$$\int \frac{(\frac{\sigma n_0^2}{2} - w_1)dw_1}{2[w_1^2 - \frac{n_0^2 v^2(\sigma^2 n_0^2 - 4)}{4}] - \frac{C_1}{2}(C_1 \pm G)} \tag{6}$$

The size (6) is broken into a part where $\frac{\sigma n_0^2 v}{2}\int\limits_{(1)} - \int\limits_{(2)}$; here $\int\limits_{(1)} = \int \frac{dw_1}{G_1}$, $\int\limits_{(2)} = \int \frac{dw_1}{2G_1}$; here $G_1 = 2[w_1^2 - n_0^2 v^2 \frac{(\sigma^2 n_0^2 - 4)}{4}] - \frac{C_1}{2}(C_1 \pm G)$.

If $a = \frac{n_0^2 v^2(\sigma^2 n_0^2 - 4)}{4}, \bar{x} = w_1^2, \bar{y}^2 = C_1^2 - 4(\bar{x} - a)$ that $\int\limits_{(2)} = \frac{1}{2}\ln\left|\bar{y} + C_1\right| + const.$ Furthermore,

$$\int\limits_{(1)} = \pm\int \frac{dy}{(\bar{y} + C_1)\sqrt{C_1^2 - \bar{y}^2 + 4a}}$$

Let us for a determinancy $C_1^2 + 4a \geq 0$. Then

$$\int\limits_{(1)} = \pm\frac{1}{n_0^2 v^2\sqrt{4 - \sigma^2 n_0^2}} \arcsin\frac{C_1\bar{y} + C_1^2 + n_0^2 v^2(\sigma^2 n_0^2 - 4)}{(\bar{y} + C_1)\sqrt{C_1^2 + n_0^2 v^2(\sigma^2 n_0^2 - 4)}} + const, if\ \sigma n_0 < 2$$

$$\int\limits_{(1)} = \mp\frac{1}{C_1(\bar{y} + C_1)}\sqrt{C_1^2 - \bar{y}^2} + const, if\ \sigma n_0 = 2$$

$$\mp\int\limits_{(1)} = -\frac{1}{2n_0^2 v^2\sqrt{\sigma^2 n_0^2 - 4}}\ln\left|\frac{n_0 v\sqrt{\sigma^2 n_0^2 - 4} + G_1}{\bar{y} + C_1} + \frac{C_1}{n_0 v\sqrt{\sigma^2 n_0^2 - 4}}\right| +$$

$$+\frac{1}{2n_0^2 v^2\sqrt{\sigma^2 n_0^2 - 4}}\ln\left|\frac{n_0 v\sqrt{\sigma^2 n_0^2 - 4} - G_1}{\bar{y} + C_1} + \frac{C_1}{n_0 v\sqrt{\sigma^2 n_0^2 - 4}}\right| + const, if\ \sigma n_0 > 2$$

Additional the first integral of a systems found above being by transcendental function of state variables makes together with (4) a full set of the first integrals of a system (1) at $p_0 = 0$. For the system (1),(2) at $p_0 = 0$ the one more first integral is necessary.

Remark. Everywhere is higher instead of it is necessary to insert left-hand part of equality (4).

For search of the last integral of a system (1),(2) at $p_0 = 0$ we shall remark, that as $\frac{dz_1}{d\beta} = z_2$ that $\frac{du_1}{d\beta} + [-u_2 + \sigma n_0^2 v] = u_2$. Therefore

$$\frac{du_1}{d\beta} = \pm\sqrt{\sigma^2 n_0^4 v^2 - 4[u_1^2 - C_1 u_1 + n_0^2 v^2]}$$

and, therefore, the required quadrature receives a kind

$$\mp\int \frac{du_1}{\sqrt{\sigma^2 n_0^4 v^2 - 4[u_1^2 - C_1 u_1 + n_0^2 v^2]}} = \beta + C_3,\, C_3 = const$$

The left-hand part of the last equality (without the sign) has a kind

$$\frac{1}{2}\arcsin\frac{(u_2 - \frac{\sigma n_0^2 v}{2})^2}{\sqrt{C_1^2 + n_0^2 v^2(\sigma^2 n_0^2 - 4)}}$$

After substitutions we have a unknown quantity an invariant ratio

$$\cos^2[2(\beta + C_3)] = \frac{(u_2 - \frac{\sigma n_0^2 v}{2})^2 u_1^2}{G_2} \qquad (7)$$

where $G_2 = [u_2^2 - \sigma n_0^2 v u_2]^2 + 2[u_2^2 - \sigma n_0^2 v u_2][u_1^2 + n_0^2 v^2] + [u_1^2 - n_0^2 v^2]^2 + \sigma^2 n_0^4 v^2 u_1^2$ which is analytical relation.

Example. *If* $\sigma n_0 = 2$ *the equality (7) accept a following kind*

$$\cos^2[2(\beta + C_3)] = \frac{(z_2 - n_0 v \sin\alpha) z_1}{(z_2 - n_0 v \sin\alpha)^2 + z_1^2}$$

CLASSICAL PROBLEM ABOUT A SPARTIAL PENDULUM IN A FLOW

By analogy to plane case, we shall consider the problem about a dynamically symmetrical spatial pendulum, located in a flow of filling medium. At first we shall consider case of zero curliness along a centerline of a dynamic symmetry.

Let convex plane area is fixed perpendicularly of segment on the spherical hinge also is in a flow of filling medium which is gone from a constant by speed $v_\infty \neq 0$. Let's assume that the segment does not create a resistance.

The total force S of effect of a flow of medium on a body is directed in parallel to segment and the point N of the appendix of this force is determined only in one parameter -

angle of attack α, which is measured between the vector of speed v_A of a point A concerning of a flow and the segment. Thus the force S is directed to the normal line to that side from it which is opposite to a direction of vector of speed v_A and crosses through some point N of plane area biased from a point A forwards on to the relation to a direction of v_A. The similar conditions arise at use the models of jet flow of spatial bodies.

The vector e determines the orientation of the segment. Thus $S = s(\alpha)v_A^2 e$ where a resistant coefficient $s = s(\alpha)$.

Let $Ox_0y_0z_0$ is the fixed coordinate system. A direction of the filling flow the hours coincide a direction of an axis Ox_0. Let's connect to a body the coordinate system $Axyz$ where the axis Ax is directed along the segment and axes Ay and Az hardly are connected to plane area.

The coordinates of a point N in a system $Axyz$ look like $(0, y_N z_N)$. On the analogies to a problem about motion of a free body its are entered a function $R(\alpha)$ and also the angle β which is measuring in a plane Ayz. Thus let for a simplicity the property (3) is executed. For any allowed function $R(\alpha)$ the analysis is carried out similarly.

If the body is symmetric dynamically ($A, B = C$ are the main moments of inertia in a system $Axyz$), (p,q,r) are the projections of angular velocities in the system $Axyz$ that the equations of motion will accept as a kind

$$q' = -n_0^2 v_A^2 \sin\alpha\cos\alpha\sin\beta$$
$$r' = n_0^2 v_A^2 \sin\alpha\cos\alpha\cos\beta \tag{8}$$

The force of resistance accepts an availability of the first integral $p = p_0$ thus in equations (8) the condition $p_0 = 0$ is taken into account.

Let's consider the angles (ξ,η) which are determining an orientation of a pendulum. An angle ξ let's measure from an axis Ox_0 up to the segment and η is measured from a projection of the segment on a plane Ox_0z_0 up to an axis Oy_0. Then

$$\begin{gathered} \cos\xi = \cos\psi\cos\varphi \\ \sin\xi\cos\eta = \cos\psi\sin\varphi \\ \sin\xi\sin\eta = \sin\psi \end{gathered} \tag{9}$$

Full set of equations

Ratioes linking $(v_A\alpha,\beta)$ and (ξ,η,r,q) (l is the length of the segment) look like

$$\begin{gathered} v_A\cos\alpha = -v_\infty\cos\xi \\ v_A\sin\alpha\cos\beta = lr + v_\infty\sin\xi\cos\eta \\ v_A\sin\alpha\sin\beta = -lq - v_\infty\sin\xi\sin\eta \end{gathered} \tag{10}$$

By virtue of kinematic ratios, we have the following relations

$$\theta\acute{y} = -q \frac{\sin\varphi}{\cos\psi}$$

$$\varphi' = r + q \sin\varphi \frac{\sin\psi}{\cos\psi}$$

$$\psi' = q \cos\varphi$$

whence easily it is injected that

$$q = \frac{q'}{\cos\varphi}$$

$$r = \varphi' - \psi' \frac{\sin\varphi}{\cos\varphi} \frac{\sin\psi}{\cos\psi} \qquad (11)$$

Using properties (9) and (11) we have the following identities

$$q = \xi' \sin\eta + \eta' \frac{\sin\xi}{\cos\xi} \cos\eta$$

$$r = \xi' \cos\eta - \eta' \frac{\sin\xi}{\cos\xi} \sin\eta \qquad (12)$$

The equations from (8),(10) and (12) will derivate a full system for the determination of the motion of a pendulum at a level of an integral $p_0 = 0$.

Proposition 2. *A full set of equations of the motion of a pendulum at condition (3) has a kind*

$$\xi'' + ln_0^2 v_\infty \xi' \cos\xi + n_0^2 v_\infty^2 \sin\xi \cos\xi - \eta'^2 \frac{\sin\xi}{\cos\xi} = 0$$

$$\eta'' + \xi' \eta' \frac{1 + \cos^2\xi}{\cos\xi \sin\xi} + ln_0^2 v_\infty \eta' \cos\xi = 0 \qquad (13)$$

As well as in case of a free body the system (13) has some symmetries. It also has a full set of the first integrals, expressed through elementary functions and the angle η is a cyclical coordinate.

Theorem 2. *The system (13) is topologically equivalent to (1) at* $p_0 = 0$. *Thus, as well as in plane case it is fair the mechanical analogy between a pendulum in a flow of a medium and the free body at availability of the servoconstraint.*

REMARKS ON A RIGID BODY OF THE LAGRANGE IN THE SPECIAL FIELD OF THE FORCES

Forces

One more analogy to a rigid body of the Lagrange in special field of forces is fair. Let on a rigid body of the Lagrange in case when longitudinal making angular rate is equal to zero the following force acts. It is perpendicular of an equatorial plane and its size is equal to $C_1 s(\theta), C_1 > 0$ (θ is the angle of nutation) and the distance from the point of the acting up to an axis of a dynamic symmetry is equal to $C_2 R(\theta), C_2 > 0$.

Then the dynamic equations of motion (for a simplicity in case (3)) will accept as a kind

$$\theta' = -z_2$$

$$z_2' = n_0^2 v^2 \sin\theta\cos\theta - z_1^2 \frac{\cos\theta}{\sin\theta} \tag{14}$$

$$z_1' = z_1 z_2 \frac{\cos\theta}{\sin\theta}$$

$$\varphi' = z_1 \frac{\cos\theta}{\sin\theta} \tag{15}$$

Corollary. *The system (14),(15) is equivalent to (1),(2) at* $p_0 = 0$ *and at* $\sigma = 0$.

Thus, we have three problems which are equivalent among themselves:

à) Free rigid body at availability of servoconstraint;

b) a pendulum in a flow of a medium;

c) a rigid body of the Lagrange in special a field of the forces.

TRAJECTORIES OF MOTION OF A SPHERICAL PENDULUM AND THE CASE OF NON-ZERO OF ITS CURLING ABOUT A CENTERLINE

Trajectories of a pendulum on a sphere

Pursuant to properties of the splitting on trajectories of phase spaces of a pendulum at a zero own curling the typical trajectories of a point D in the plane area are divided into classes.

à) Trajectories appropriate to oscillatory area. Such trajectories represent the curves on a sphere which beyond all bounds approaching to the poles of a sphere (on a flow) at $t \to \pm\infty$.

b) Trajectories appropriate to rotary area. Such trajectories represent the curves almost always are everywhere dense and filling ring-shaped areas on an sphere and are symmetrical relatively the equator.

Spherical pendulum at a non-zero own curling

Let's consider the equations of the motion of a pendulum under condition of when $p_0 = 0$:

$$\xi'' + l n_0^2 v_\infty \xi' \cos\xi + n_0^2 v_\infty^2 \sin\xi\cos\xi - \eta'^2 \frac{\sin\xi}{\cos\xi} - \frac{A}{B} p_0 \eta' \frac{\sin\xi}{\cos\xi} = 0$$

$$\eta'' + \xi'\eta' \frac{1+\cos^2\xi}{\cos\xi\sin\xi} + l n_0^2 v_\infty \eta' \cos\xi + \frac{A}{B} p_0 \xi' \frac{\cos\xi}{\sin\xi} = 0$$

Let's proceed to a classification of possible paths of a pendulum on a sphere.

à) Trajectories which are similar to trajectories à) for case $p_0 = 0$.

Asymptotics of behaviour of such curve are former.

b) Trajectories which are similar to trajectories b) for case $p_0 = 0$. Such trajectories almost always are everywhere dense on the whole sphere.

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