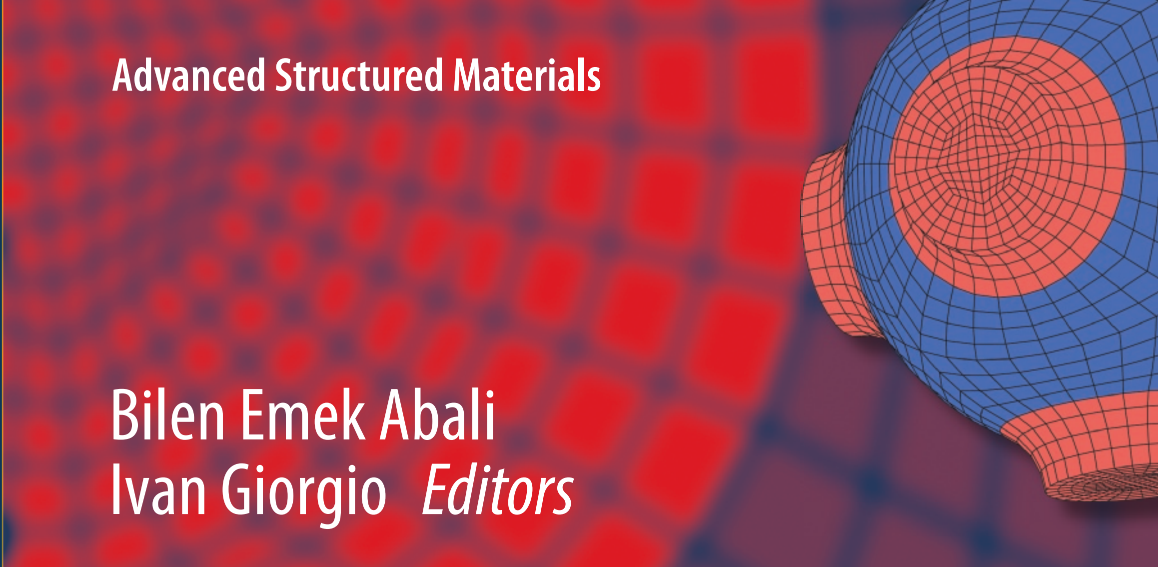
Advanced Structured Materials



Bilen Emek Abali
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Developments and Novel Approaches in Nonlinear Solid Body Mechanics

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Advanced Structured Materials

Volume 130

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Developments and Novel Approaches in Nonlinear Solid Body Mechanics

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Preface

The ICoNSOM 2019, International Conference on Nonlinear Solid Mechanics, took place at Palazzo Argiletum, Rome, Italy, from June 16 to June 19, 2019. Over 200 participation from the whole globe, the urge of this proceedings became clear. With the aid of the organizers, Marco Amabili, Francesco dell'Isola, Ivan Giorgio, Nicola Rizzi, and Luca Placidi, the scientific community did show a great interest allowing us to bring together this proceedings collected in two volumes:

- Developments and Novel Approaches in Nonlinear Solid Body Mechanics
- Developments and Novel Approaches in Biomechanics and Metamaterials

ICoNSoM 2019 Conference has been intended to provide an international opportunity for communicating recent developments in various areas of nonlinear solid mechanics. This monograph consists theory, experiments, and applications in mechanics, thermodynamics, and multiphysics simulation in many length scales.

As editors, we intend to thank all authors for their crucial contributions as well as all reviewers for their invaluable time and effort. We delightedly acknowledge Dr. Christoph Baumann (Springer Publisher) for initiating the book project. In addition, we have to thank Dr. Mayra Castro (Senior Editor Applied Sciences; Materials Science; Materials Engineering; Nanotechnology and Nanomedicine) and Mr. Ashok Arumairaj (Production Administrator) giving their support in the process of publication.

Brussels, Rome
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Chapter 8
Integrable Dissipative Dynamical Systems with Three and Four Degrees of Freedom

Maxim V. Shamolin

Abstract In this work, the integrability of some classes of dynamic systems on tangent bundles of three-dimensional manifolds is demonstrated. The corresponding force fields possess the so-called variable dissipation and generalize those considered earlier.

Keywords: Dynamic systems · Tangent bundles

8.1 Introduction

In many problems of dynamics, there appear mechanical systems with three-dimensional manifolds as position spaces. Tangent bundles of such manifolds naturally become phase spaces of such systems. For example, study of a four-dimensional generalized spherical pendulum in a nonconservative force field leads to a dynamic system on the tangent bundle of a three-dimensional sphere, and the metric of special form on it is induced by an additional symmetry group (Bogoyavlenskii, 1986; Bogoyavlenskii and Ivakh, 1985). In this case, dynamic systems describing the motion of such a pendulum possess alternating dissipation and the complete list of first integrals consists of transcendental functions that can be expressed in terms of a finite combination of elementary functions (Bogoyavlenskii and Ivakh, 1985; Dubrovin and Novikov, 1984).

The class of problems about the motion of a point on a three-dimensional surface is also known; the metric on it is induced by the Euclidean metric of the ambient space. In some cases of systems with dissipation, it is also possible to find a complete list of first integrals; the list consists of transcendental functions. The results obtained

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are especially important in the aspect of the presence of just a nonconservative force field in the system (see, e.g., Giorgio and Scerrato, 2017; Baroudi et al, 2019; dell'Isola et al, 2019a,b).

8.2 Equations of Geodesic Lines

It is well known that, in the case of a three-dimensional Riemannian manifold M^3 with coordinates (α, β), $\beta = (\beta_1, \beta_2)$, and affine connection $\Gamma^i_{jk}(x)$ the equations of geodesic lines on the tangent bundle $T_*M^3\{\dot{\alpha}, \dot{\beta}_1, \dot{\beta}_2; \alpha, \beta_1, \beta_2\}$, $\alpha = x^1$, $\beta_1 = x^2$, $\beta_2 = x^3$, $x = (x^1, x^2, x^3)$, have the following form (the derivatives are taken with respect to the natural parameter):

$$\ddot{x}^i + \sum_{j,k=1}^{3} \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0,\ i = 1, 2, 3. \tag{8.1}$$

Let us study the structure of Eqs. (8.1) under a change of coordinates on the tangent bundle T_*M^3. Consider a change of coordinates of the tangent space:

$$\dot{x}^i = \sum_{j=1}^{3} R^{ij}(x) z_j, \tag{8.2}$$

which can be inverted:

$$z_j = \sum_{i=1}^{3} T_{ji}(x)\dot{x}^i,$$

herewith $R^{ij}, T_{ji}, i, j = 1, 2, 3$, are functions of x^1, x^2, x^3, and

$$RT = E,$$

$$R = R^{ij},\ T = T_{ji}.$$

We also call Eqs. (8.2) new kinematic relations, i.e., relations on the tangent bundle T_*M^3.

The following equalities are valid:

$$\dot{z}_j = \sum_{i=1}^{3} \dot{T}_{ji}\dot{x}^i + \sum_{i=1}^{3} T_{ji}\ddot{x}^i,\ \dot{T}_{ji} = \sum_{k=1}^{3} T_{ji,k}\dot{x}^k, \tag{8.3}$$

$$T_{ji,k} = \frac{\partial T_{ji}}{\partial x^k},\ j, i, k = 1, 2, 3.$$

If we substitute Eqs. (8.1) to Eqs. (8.3), we have:

$$\dot{z}_i = \sum_{j,k=1}^{3} T_{ij,k}\dot{x}^j\dot{x}^k - \sum_{j,p,q=1}^{3} T_{ij}\Gamma^j_{pq}\dot{x}^p\dot{x}^q, \tag{8.4}$$

in the last system, one should substitute formulas (8.2) instead of $\dot{x}^i,\ i = 1, 2, 3$.

Furthermore, Eq. (8.4) we can rewrite:

$$\dot{z}_i + \sum_{j,k=1}^{3} Q_{ijk}\dot{x}^j\dot{x}^k|_{(8.2)} = 0, \tag{8.5}$$

$$Q_{ijk}(x) = \sum_{s=1}^{3} T_{is}(x)\Gamma^s_{jk}(x) - T_{ij,k}(x). \tag{8.6}$$

Proposition 1 *System* (8.1) *is equivalent to compound system* (8.2)*,* (8.4) *in a domain where* $\det R(x) \neq 0$.

Therefore, the result of the passage from equations of geodesic lines (8.1) to an equivalent system of equations (8.2), (8.4) depends both on the change of variables (8.2) (i.e., introduced kinematic relations) and on the affine connection $\Gamma^i_{jk}(x)$.

8.3 A Fairly General Case

Consider next a sufficiently general case of specifying kinematic relations in the following form:

$$\begin{cases} \dot{\alpha} = -z_3, \\ \dot{\beta}_1 = z_2 f_1(\alpha), \\ \dot{\beta}_2 = z_1 f_2(\alpha) g(\beta_1), \end{cases} \tag{8.7}$$

where $f_1(\alpha),\ f_2(\alpha),\ g(\beta_1)$ are smooth functions on their domain of definition. Such coordinates z_1, z_2, z_3 in the tangent space are introduced when the following equations of geodesic lines are considered (Kozlov, 1983; Shamolin, 2015c) (in particular, on surfaces of revolution):

$$\begin{cases} \ddot{\alpha} + \Gamma^\alpha_{11}(\alpha,\beta)\dot{\beta}_1^2 + \Gamma^\alpha_{22}(\alpha,\beta)\dot{\beta}_2^2 = 0, \\ \ddot{\beta}_1 + 2\Gamma^1_{\alpha 1}(\alpha,\beta)\dot{\alpha}\dot{\beta}_1 + \Gamma^1_{22}(\alpha,\beta)\dot{\beta}_2^2 = 0, \\ \ddot{\beta}_2 + 2\Gamma^2_{\alpha 2}(\alpha,\beta)\dot{\alpha}\dot{\beta}_2 + 2\Gamma^2_{12}(\alpha,\beta)\dot{\beta}_1\dot{\beta}_2 = 0, \end{cases} \tag{8.8}$$

i.e., other connection coefficients are zero. In case (8.7), Eqs. (8.4) take the form

$$\begin{cases} \dot{z}_1 = \left[2\Gamma^2_{\alpha 2}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha}\right] z_1 z_3 - \left[2\Gamma^2_{12}(\alpha,\beta) + \frac{d\ln|g(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_1 z_2, \\ \dot{z}_2 = \left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right] z_2 z_3 - \Gamma^1_{22}(\alpha,\beta)\frac{f_2^2(\alpha)}{f_1(\alpha)} g^2(\beta_1) z_1^2, \\ \dot{z}_3 = \Gamma^\alpha_{11} f_1^2(\alpha) z_2^2 + \Gamma^\alpha_{22} f_2^2(\alpha) g^2(\beta_1) z_1^2, \end{cases} \tag{8.9}$$

and Eqs. (8.8) are almost everywhere equivalent to compound system (8.7), (8.9) on the manifold $T_*M^3\{z_3, z_2, z_1; \alpha, \beta_1, \beta_2\}$.

To integrate system (8.7), (8.9) completely, it is necessary to know, generally speaking, five independent first integrals.

Proposition 2 *If the system of equalities*

$$\begin{cases} 2\Gamma^1_{\alpha 1}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha} + \Gamma^\alpha_{11}(\alpha,\beta) f_1^2(\alpha) \equiv 0, \\ 2\Gamma^2_{\alpha 2}(\alpha,\beta) + \\ + \frac{d\ln|f_2(\alpha)|}{d\alpha} + \Gamma^\alpha_{22}(\alpha,\beta) f_2^2(\alpha) g^2(\beta_1) \equiv 0, \\ \left[2\Gamma^2_{12}(\alpha,\beta) + \frac{d\ln|g(\beta_1)|}{d\beta_1}\right] f_1^2(\alpha) + \Gamma^1_{22}(\alpha,\beta) f_2^2(\alpha) g^2(\beta_1) \equiv 0, \end{cases} \tag{8.10}$$

is valid everywhere in its domain of definition, system (8.7)*,* (8.9) *has an analytic first integral of the form*

$$\Phi_1(z_3, z_2, z_1) = z_1^2 + z_2^2 + z_3^2 = C_1^2 = const. \tag{8.11}$$

We suppose that the condition

$$f_1(\alpha) = f_2(\alpha) = f(\alpha), \tag{8.12}$$

is satisfied in Eqs. (8.7); the function $g(\beta_1)$ must satisfy the transformed third equality from (8.10):

$$2\Gamma^2_{12}(\alpha,\beta) + \frac{d\ln|g(\beta_1)|}{d\beta_1} + \Gamma^1_{22}(\alpha,\beta) g^2(\beta_1) \equiv 0. \tag{8.13}$$

Proposition 3 *If properties* (8.12) *and* (8.13) *are valid and the equalities*

$$\Gamma^1_{\alpha 1}(\alpha,\beta) = \Gamma^2_{\alpha 2}(\alpha,\beta) = \Gamma_1(\alpha), \tag{8.14}$$

are satisfied, system (8.7)*,* (8.9) *has a smooth first integral of the following form:*

$$\Phi_2(z_2, z_1; \alpha) = \sqrt{z_1^2 + z_2^2}\, \Phi_0(\alpha) = C_2 = const, \tag{8.15}$$

$$\Phi_0(\alpha) = f(\alpha) \exp\left\{2\int_{\alpha_0}^{\alpha} \Gamma_1(b) db\right\}.$$

Proposition 4 *If property* (8.12) *is valid and the equality*

$$\Gamma^2_{12}(\alpha,\beta) = \Gamma_2(\beta_1), \tag{8.16}$$

and the second equality from (8.14) *(*$\Gamma^2_{\alpha 2}(\alpha,\beta) = \Gamma_1(\alpha)$*) are satisfied, system* (8.7)*,* (8.9) *has a smooth first integral of the following form:*

$$\Phi_3(z_1; \alpha, \beta_1) = z_1 \Phi_0(\alpha) \Phi(\beta_1) = C_3 = const, \tag{8.17}$$

$$\Phi(\beta_1) = g(\beta_1)\exp\left\{2\int_{\beta_{10}}^{\beta_1} \Gamma_2(b)db\right\}.$$

Proposition 5 *If conditions* (8.12), (8.13), (8.14), (8.16) *are satisfied, system* (8.7), (8.9) *has a first integral of the following form:*

$$\Phi_4(z_2, z_1; \beta) = \beta_2 \pm \int_{\beta_{10}}^{\beta_1} \frac{C_3 g(b)}{\sqrt{C_2^2\Phi^2(b) - C_3^2}}\,db = C_4 = const, \tag{8.18}$$

where, after taking integral (8.18)*, one should substitute the left-hand sides of equalities* (8.15), (8.17) *instead of the constants* C_2, C_3*, respectively.*

Under the conditions listed above, system (8.7), (8.9) has a complete set (four) of independent first integrals of the form (8.11), (8.15), (8.17), (8.18).

8.4 Potential Field of Force

Let us now somewhat modify system (8.7), (8.9) under conditions (8.12), (8.13), (8.14), (8.16), which yields a conservative system. Namely, the presence of the force field is characterized by the coefficient $F(\alpha)$ in the second equation of system (8.19) at $b = 0$. The system under consideration on the tangent bundle $T_*M^3\{z_3, z_2, z_1; \alpha, \beta_1, \beta_2\}$ takes the form

$$\begin{cases} \dot\alpha = -z_3 + b\delta(\alpha), \\ \dot z_3 = F(\alpha) + \Gamma_{11}^{\alpha}(\alpha,\beta)f^2(\alpha)z_2^2 + \Gamma_{22}^{\alpha}(\alpha,\beta)f^2(\alpha)g^2(\beta_1)z_1^2, \\ \dot z_2 = \left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right] z_2z_3 - \Gamma_{22}^{1}(\beta_1)f(\alpha)g^2(\beta_1)z_1^2, \\ \dot z_1 = \left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right] z_1z_3 - \left[2\Gamma_2(\beta_1) + \frac{d\ln|g(\beta_1)|}{d\beta_1}\right] f(\alpha)z_1z_2, \\ \dot\beta_1 = z_2 f(\alpha), \\ \dot\beta_2 = z_1 f(\alpha)g(\beta_1), \end{cases} \tag{8.19}$$

and at $b = 0$ it is almost everywhere equivalent to the following system:

$$\begin{cases} \ddot\alpha + F(\alpha) + \Gamma_{11}^{\alpha}(\alpha,\beta)\dot\beta_1^2 + \Gamma_{22}^{\alpha}(\alpha,\beta)\dot\beta_2^2 = 0, \\ \ddot\beta_1 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_1 + \Gamma_{22}^{1}(\beta_1)\dot\beta_2^2 = 0, \\ \ddot\beta_2 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_2 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_2 = 0. \end{cases}$$

Proposition 6 *If the conditions of Proposition 2 are satisfied, system* (8.19) *at* $b = 0$ *has a smooth first integral of the following form:*

$$\Phi_1(z_3, z_2, z_1; \alpha) = z_1^2 + z_2^2 + z_3^2 + F_1(\alpha) = \tag{8.20}$$

$$= C_1 = const, \; F_1(\alpha) = 2\int_{\alpha_0}^{\alpha} F(a)da.$$

Proposition 7 *If the conditions of Propositions 3, 4 are satisfied, system* (8.19) *at* $b = 0$ *has two smooth first integrals of form* (8.15), (8.17).

Proposition 8 *If the conditions of Proposition 5 are satisfied, system* (8.19) *at* $b = 0$ *has a first integral of form* (8.18).

Under the conditions listed above, system (8.19) at $b = 0$ has a complete set of (four) independent first integrals of form (8.20), (8.15), (8.17), (8.18).

8.5 Force Field with Dissipation

Let us now consider system (8.19) at $b \neq 0$. In doing this, we obtain a system with dissipation. Namely, the presence of dissipation (generally speaking, sign-alternating) is characterized by the coefficient $b\delta(\alpha)$ in the first equation of system (8.19), which is almost everywhere equivalent to the following system:

$$\begin{cases} \ddot{\alpha} - b\dot{\alpha}\delta'(\alpha) + F(\alpha) + \Gamma_{11}^{\alpha}(\alpha,\beta)\dot{\beta}_1^2 + \Gamma_{22}^{\alpha}(\alpha,\beta)\dot{\beta}_2^2 = 0, \\ \ddot{\beta}_1 - b\dot{\beta}_1\delta(\alpha)\left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right] + 2\Gamma_1(\alpha)\dot{\alpha}\dot{\beta}_1 + \Gamma_{22}^1(\beta_1)\dot{\beta}_2^2 = 0, \\ \ddot{\beta}_2 - b\dot{\beta}_2\delta(\alpha)\left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right] + 2\Gamma_1(\alpha)\dot{\alpha}\dot{\beta}_2 + 2\Gamma_2(\beta_1)\dot{\beta}_1\dot{\beta}_2 = 0. \end{cases}$$

Now we pass to integration of the sought six-order system (8.19) under condition (8.13), as well as under the equalities

$$\Gamma_{11}^{\alpha}(\alpha,\beta) = \Gamma_{22}^{\alpha}(\alpha,\beta)g^2(\beta_1) = \Gamma_3(\alpha). \tag{8.21}$$

We also introduce (by analogy with (8.13)) a restriction on the function $f(\alpha)$. It must satisfy the transformed first equality from (8.10):

$$2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha} + \Gamma_3(\alpha)f^2(\alpha) \equiv 0. \tag{8.22}$$

To integrate it completely, one should know, generally speaking, five independent first integrals. However, after the following change of variables,

$$z_1, z_2 \to z, z_*, \ z = \sqrt{z_1^2 + z_2^2}, \ z_* = \frac{z_2}{z_1},$$

system (8.19) decomposes as follows:

$$\begin{cases} \dot{\alpha} = -z_3 + b\delta(\alpha), \\ \dot{z}_3 = F(\alpha) + \Gamma_3(\alpha)f^2(\alpha)z^2, \\ \dot{z} = \left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right] zz_3, \end{cases} \tag{8.23}$$

$$\begin{cases} \dot{z}_* = \pm z\sqrt{1+z_*^2}f(\alpha)\left[2\Gamma_2(\beta_1) + \frac{d\ln|g(\beta_1)|}{d\beta_1}\right], \\ \dot{\beta}_1 = \pm \frac{zz_*}{\sqrt{1+z_*^2}}f(\alpha), \end{cases} \tag{8.24}$$

$$\dot{\beta}_2 = \pm \frac{z}{\sqrt{1+z_*^2}}f(\alpha)g(\beta_1). \tag{8.25}$$

It is seen that to integrate system (8.23)–(8.25) completely, it is sufficient to determine two independent first integrals of system (8.23), one integral of system (8.24), and an additional first integral attaching Eq. (8.25) (i.e., four integrals in total).

Theorem 11 *Let the equalities*

$$\Gamma_3(\alpha)f^2(\alpha) = \kappa\frac{d}{d\alpha}\ln|\delta(\alpha)|,\ F(\alpha) = \lambda\frac{d}{d\alpha}\frac{\delta^2(\alpha)}{2} \tag{8.26}$$

be valid for some $\kappa, \lambda \in \mathbf{R}$. *Then system* (8.19) *under equalities* (8.13), (8.21), (8.22) *has a complete set of (four) independent, generally speaking, transcendental first integrals.*

In the general case, the first integrals are written awkwardly. In particular, if $\kappa = -1$, the explicit form of one of first integrals for system (8.23) is as follows:

$$\Theta_1(z_3, z; \alpha) = G_1\left(\frac{z_3}{\delta(\alpha)}, \frac{z}{\delta(\alpha)}\right) = \frac{z_3^2 + z^2 - bz_3\delta(\alpha) + \lambda\delta^2(\alpha)}{z\delta(\alpha)} = C_1 = \text{const.} \tag{8.27}$$

Here, the additional first integral for system (8.23) has the following structural form:

$$\Theta_2(z_3, z; \alpha) = G_2\left(\delta(\alpha), \frac{z_3}{\delta(\alpha)}, \frac{z}{\delta(\alpha)}\right) = C_2 = \text{const.} \tag{8.28}$$

Here, after taking the integral, one should substitute the left-hand side of equality (8.27) for C_1. The right-hand side of this equality is expressed through a finite combination of elementary functions; the left-hand part, depending on the function $\delta(\alpha)$. Therefore, expressing first integrals (8.27), (8.28) through a finite combination of elementary functions depends not only on calculation of quadratures but also on the explicit form of the function $\delta(\alpha)$.

The first integral for system (8.24) has the form

$$\Theta_3(z_*; \beta_1) = \frac{\sqrt{1+z_*^2}}{\Phi(\beta_1)} = C_3 = \text{const}, \tag{8.29}$$

as for the function $\Phi(\beta_1)$, see (8.17). The additional first integral attaching Eq. (8.25) is found by analogy with (8.18):

$$\Theta_4(z_*; \beta) = \beta_2 \pm \int_{\beta_{10}}^{\beta_1} \frac{g(b)}{\sqrt{C_3^2\Phi^2(b) - 1}}\,db = C_4 = \text{const},$$

here, after taking this integral, one should substitute the left-hand side of equality (8.29) for C_3.

8.6 Structure of Transcendental First Integrals

If α is a periodic coordinate with a period of 2π, system (8.23) becomes a dynamic system with variable dissipation with a zero mean (Shamolin, 2015a,b, 2016a). At $b = 0$, it turns into a conservative system having two smooth first integrals of form (8.20), (8.15). By virtue of (8.26),

$$\Phi_1(z_3, z_2, z_1; \alpha) = z_1^2 + z_2^2 + z_3^2 + 2\int_{\alpha_0}^{\alpha} F(a)da \cong z^2 + z_3^2 + \lambda\delta^2(\alpha), \quad (8.30)$$

where "$\cong$" means equality up to an additive constant. At the same time, by virtue of (8.22) and (8.26),

$$\Phi_2(z_2, z_1; \alpha) = \sqrt{z_1^2 + z_2^2}\, f(\alpha) \exp\left\{2\int_{\alpha_0}^{\alpha} \Gamma_1(b)db\right\} \cong z\delta(\alpha) = C_2 = \text{const}, \quad (8.31)$$

where "$\cong$" now means equality up to a multiplicative additive constant.

It is evident that the ratio of the two first integrals (8.30) and (8.31) (or, (8.20) and (8.15)) is also a first integral of system (8.23) for $b = 0$. However, at $b \neq 0$, each of the functions

$$z^2 + z_3^2 - bz_3\delta(\alpha) + \lambda\delta^2(\alpha) \quad (8.32)$$

and (8.31) taken individually is not a first integral of system (8.23). However, the ratio of functions (8.32) and (8.31) is a first integral of system (8.23) (at $\kappa = -1$) for any b.

Generally, for systems with dissipation, transcendence of functions (in the aspect of the presence of essentially singular points) as first integrals is inherited from the existence of attracting and repelling limit sets in the system (Shamolin, 2016b, 2017a).

8.7 Conclusions

By analogy with low-dimensional cases, we pay special attention to two important cases for the function $f(\alpha)$ defining the metric on a sphere:

$$f(\alpha) = \frac{\cos\alpha}{\sin\alpha}, \quad (8.33)$$

$$f(\alpha) = \frac{1}{\cos\alpha\sin\alpha}. \quad (8.34)$$

Case (8.33) forms a class of systems corresponding to the motion of a dynamically symmetric four-dimensional solid body at zero levels of cyclic integrals, generally speaking, in a nonconservative field of forces (Shamolin, 2017b,c). Case (8.34) forms a class of systems corresponding to the motion of a material point on a three-dimensional sphere also, generally speaking, in a nonconservative field of forces. In particular, at

$$\delta(\alpha) \equiv F(\alpha) \equiv 0,$$

the system under consideration describes a geodesic flow on a three-dimensional sphere. In case (8.33), the system describes the spatial motion of a four-dimensional solid body in the force field under the action of a tracking force (Shamolin, 2017c). In particular, if

$$\delta(\alpha) = \frac{F(\alpha)}{\cos\alpha},$$

and

$$\delta(\alpha) = \sin\alpha,$$

the system also describes a generalized four-dimensional spherical pendulum in a nonconservative force field and has a complete set of transcendental first integrals that can be expressed in terms of a finite combination of elementary functions.

If the function $\delta(\alpha)$ is not periodic, the dissipative system under consideration is a system with variable dissipation with a zero mean (i.e., it is properly dissipative). Nevertheless, an explicit form of transcendental first integrals that can be expressed in terms of a finite combination of elementary functions can be obtained even in this case. This is a new nontrivial case of integrability of dissipative systems in an explicit form.

8.8 Important Example: Case of Four-Dimensional Manifold

We consider the rather general case of introducing the kinematic relations in the form

$$\dot\alpha = -z_4,\ \dot\beta_1 = z_3 f_1(\alpha),\ \dot\beta_2 = z_2 f_2(\alpha) g_1(\beta_1),\ \dot\beta_3 = z_1 f_3(\alpha) g_2(\beta_1) h(\beta_2), \tag{8.35}$$

where $f_k(\alpha)$, $k = 1,2,3$, $g_l(\beta_1)$, $l = 1,2$, $h(\beta_2)$, are smooth functions. Such coordinates z_1, z_2, z_3, z_4 are introduced in the tangent space if the following classes of geodesic equations are considered (in particular, on spheres or more general surfaces of revolution):

$$\begin{cases} \ddot\alpha + \Gamma^\alpha_{11}(\alpha,\beta)\dot\beta_1^2 + \Gamma^\alpha_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^\alpha_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_1 + 2\Gamma^1_{\alpha 1}(\alpha,\beta)\dot\alpha\dot\beta_1 + \Gamma^1_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^1_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_2 + 2\Gamma^2_{\alpha 2}(\alpha,\beta)\dot\alpha\dot\beta_2 + 2\Gamma^2_{12}(\alpha,\beta)\dot\beta_1\dot\beta_2 + \Gamma^2_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_3 + 2\Gamma^3_{\alpha 3}(\alpha,\beta)\dot\alpha\dot\beta_3 + 2\Gamma^3_{13}(\alpha,\beta)\dot\beta_1\dot\beta_3 + 2\Gamma^3_{23}(\alpha,\beta)\dot\beta_2\dot\beta_3 = 0, \end{cases} \tag{8.36}$$

i.e., the remaining connection coefficients vanish. In the case (8.35), we can write (8.4) as

$$\begin{gathered}\dot z_1=\left[2\Gamma^3_{\alpha3}(\alpha,\beta)+\tfrac{d\ln|f_3(\alpha)|}{d\alpha}\right]z_1z_4-\left[2\Gamma^3_{13}(\alpha,\beta)+\tfrac{d\ln|g_2(\beta_1)|}{d\beta_1}\right]f_1(\alpha)z_1z_3-\\ -\left[2\Gamma^3_{23}(\alpha,\beta)+\tfrac{d\ln|h(\beta_2)|}{d\beta_2}\right]f_2(\alpha)g_1(\beta_1)z_1z_2,\\ \dot z_2=\left[2\Gamma^2_{\alpha2}(\alpha,\beta)+\tfrac{d\ln|f_2(\alpha)|}{d\alpha}\right]z_2z_4-\left[2\Gamma^2_{12}(\alpha,\beta)+\tfrac{d\ln|g_1(\beta_1)|}{d\beta_1}\right]f_1(\alpha)z_2z_3-\\ -\Gamma^2_{33}(\alpha,\beta)\tfrac{f_3^2(\alpha)}{f_2(\alpha)}\tfrac{g_2^2(\beta_1)}{g_1(\beta_1)}h^2(\beta_2)z_1^2,\\ \dot z_3=\left[2\Gamma^1_{\alpha1}(\alpha,\beta)+\tfrac{d\ln|f_1(\alpha)|}{d\alpha}\right]z_3z_4-\Gamma^1_{22}(\alpha,\beta)\tfrac{f_2^2(\alpha)}{f_1(\alpha)}g_1^2(\beta_1)z_2^2-\\ -\Gamma^1_{33}(\alpha,\beta)\tfrac{f_3^2(\alpha)}{f_1(\alpha)}g_2^2(\beta_1)h^2(\beta_2)z_1^2,\\ \dot z_4=\Gamma^\alpha_{11}f_1^2(\alpha)z_3^2+\Gamma^\alpha_{22}f_2^2(\alpha)g_1^2(\beta_1)z_2^2+\Gamma^\alpha_{33}f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2)z_1^2,\end{gathered} \tag{8.37}$$

and the system (8.36) is almost everywhere equivalent to the composite system (8.35), (8.37) on the tangent bundle $T_*M^4\{z_4,z_3,z_2,z_1;\alpha,\beta_1,\beta_2,\beta_3\}$.

Generally speaking, for the complete integrability of the system (8.35), (8.37) we need to know seven independent first integrals. However, a less number of first integrals is required in the case under consideration, which will be shown in the study of systems with dissipation below.

Proposition 9 *If the following identities hold everywhere on their domain:*

$$\begin{cases} 2\Gamma^1_{\alpha1}(\alpha,\beta)+\frac{d\ln|f_1(\alpha)|}{d\alpha}+\Gamma^\alpha_{11}(\alpha,\beta)f_1^2(\alpha)\equiv0,\\ 2\Gamma^2_{\alpha2}(\alpha,\beta)+\frac{d\ln|f_2(\alpha)|}{d\alpha}+\Gamma^\alpha_{22}(\alpha,\beta)f_2^2(\alpha)g_1^2(\beta_1)\equiv0,\\ \left[2\Gamma^2_{12}(\alpha,\beta)+\frac{d\ln|g_1(\beta_1)|}{d\beta_1}\right]f_1^2(\alpha)+\Gamma^1_{22}(\alpha,\beta)f_2^2(\alpha)g_1^2(\beta_1)\equiv0,\\ 2\Gamma^3_{\alpha3}(\alpha,\beta)+\frac{d\ln|f_3(\alpha)|}{d\alpha}+\Gamma^\alpha_{33}(\alpha,\beta)f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2)\equiv0,\\ \left[2\Gamma^3_{13}(\alpha,\beta)+\frac{d\ln|g_2(\beta_1)|}{d\beta_1}\right]f_1^2(\alpha)+\Gamma^1_{33}(\alpha,\beta)f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2)\equiv0,\\ \left[2\Gamma^3_{23}(\alpha,\beta)+\frac{d\ln|h(\beta_2)|}{d\beta_2}\right]f_2^2(\alpha)g_1^2(\beta_1)+\Gamma^2_{33}(\alpha,\beta)f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2)\equiv0,\end{cases} \tag{8.38}$$

then the system (8.35), (8.37) *has an analytic first integral of the form*

$$\Phi_1(z_4,\ldots,z_1)=z_1^2+\ldots+z_4^2=C_1^2=const. \tag{8.39}$$

We apply an approach allowing us to find successfully the complete list of first integrals of systems with dissipation by using the solutions to the system (8.38). We assume that in (8.35) the following conditions are satisfied:

$$f_1(\alpha)=f_2(\alpha)=f_3(\alpha)=f(\alpha), \tag{8.40}$$

moreover, $g_l(\beta_1)$, $l=1,2$, $h(\beta_2)$ satisfy the transformed equations from (8.38):

$$\begin{cases} 2\Gamma^2_{12}(\alpha,\beta)+\frac{d\ln|g_1(\beta_1)|}{d\beta_1}+\Gamma^1_{22}(\alpha,\beta)g_1^2(\beta_1)\equiv0,\\ 2\Gamma^3_{13}(\alpha,\beta)+\frac{d\ln|g_2(\beta_1)|}{d\beta_1}+\Gamma^1_{33}(\alpha,\beta)g_2^2(\beta_1)h^2(\beta_2)\equiv0,\\ 2\Gamma^3_{23}(\alpha,\beta)+\frac{d\ln|h(\beta_2)|}{d\beta_2}+\Gamma^2_{33}(\alpha,\beta)h^2(\beta_2)\equiv0.\end{cases} \tag{8.41}$$

Proposition 10 *If* (8.40) *and* (8.41)*, hold and*

$$\Gamma^1_{\alpha 1}(\alpha, \beta) = \Gamma^2_{\alpha 2}(\alpha, \beta) = \Gamma^3_{\alpha 3}(\alpha, \beta) = \Gamma_1(\alpha), \tag{8.42}$$

then the system (8.35)*,* (8.37) *has a smooth first integral of the form*

$$\Phi_2(z_3, z_2, z_1; \alpha) = \sqrt{z_1^2 + z_2^2 + z_3^2}\, \Phi_0(\alpha) = C_2 = const, \tag{8.43}$$

$$\Phi_0(\alpha) = f(\alpha) \exp\left\{2 \int_{\alpha_0}^{\alpha} \Gamma_1(b) db\right\}.$$

Proposition 11 *Under the assumptions of Proposition 10, if*

$$g_1(\beta_1) = g_2(\beta_1) = g(\beta_1), \tag{8.44}$$

$$\Gamma^2_{12}(\alpha, \beta) = \Gamma^3_{13}(\alpha, \beta) = \Gamma_2(\beta_1), \tag{8.45}$$

then the system (8.35)*,* (8.37) *has a smooth first integral of the form*

$$\Phi_3(z_2, z_1; \alpha, \beta_1) = \sqrt{z_1^2 + z_2^2}\Phi_0(\alpha)\Psi_1(\beta_1) = C_3 = const, \tag{8.46}$$

where

$$\Psi_1(\beta_1) = g(\beta_1) \exp\left\{2 \int_{\beta_{10}}^{\beta_1} \Gamma_2(b) db\right\}.$$

The following two assertions are proved in the same way as Propositions 10 and 11.

Proposition 12 *Under the assumptions of Propositions 10 and 11, if*

$$\Gamma^3_{23}(\alpha, \beta) = \Gamma_3(\beta_2), \tag{8.47}$$

then the system (8.35)*,* (8.37) *has a smooth first integral of the form*

$$\Phi_4(z_1; \alpha, \beta_1, \beta_2) = z_1\Phi_0(\alpha)\Psi_1(\beta_1)\Psi_2(\beta_2) = C_4 = const, \tag{8.48}$$

where

$$\Psi_2(\beta_2) = h(\beta_2) \exp\left\{2 \int_{\beta_{20}}^{\beta_2} \Gamma_3(b) db\right\}.$$

Proposition 13 *Under the assumptions of Propositions 10, 11, and 12, the system* (8.35)*,* (8.37) *has a first integral of the form*

$$\Phi_5(z_2, z_1; \alpha, \beta) = \beta_3 \pm \int_{\beta_{20}}^{\beta_2} \frac{C_4 h(b)}{\sqrt{C_3^2 \Phi_2^2(b) - C_4^2}} db = C_5 = const. \tag{8.49}$$

The first integrals (8.39), (8.43), (8.46), (8.48), (8.49) form the complete list of independent first integrals of the system (8.35), (8.37) under the above conditions (the fact that only five (instead of seven) first integrals are included into the complete list will be justified below).

8.8.1 Equations of Motion in a Potential Force Field and First Integrals

Modifying the system (8.35), (8.37) under the conditions (8.40)–(8.42), (8.44), (8.45), (8.47), we obtain a conservative system. Namely, the presence of a force field is characterized by a sufficiently smooth coefficient $F(\alpha)$ in the second equation of the system (8.50). The system under consideration on the tangent bundle $T_*M^4\{z_4,z_3,z_2,z_1;\alpha,\beta_1,\beta_2,\beta_3\}$ takes the form

$$\begin{cases} \dot\alpha = -z_4, \\ \dot z_4 = F(\alpha) + \Gamma^\alpha_{11} f_1^2(\alpha) z_3^2 + \Gamma^\alpha_{22} f_2^2(\alpha) g_1^2(\beta_1) z_2^2 + \Gamma^\alpha_{33} f_3^2(\alpha) g_2^2(\beta_1) h^2(\beta_2) z_1^2, \\ \dot z_3 = \left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right] z_3 z_4 - \Gamma^1_{22}(\alpha,\beta)\frac{f_2^2(\alpha)}{f_1(\alpha)} g_1^2(\beta_1) z_2^2 - \\ -\Gamma^1_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_1(\alpha)} g_2^2(\beta_1) h^2(\beta_2) z_1^2, \\ \dot z_2 = \left[2\Gamma^2_{\alpha 2}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha}\right] z_2 z_4 - \left[2\Gamma^2_{12}(\alpha,\beta) + \frac{d\ln|g_1(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_2 z_3 - \\ -\Gamma^2_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_2(\alpha)}\frac{g_2^2(\beta_1)}{g_1(\beta_1)} h^2(\beta_2) z_1^2, \\ \dot z_1 = \left[2\Gamma^3_{\alpha 3}(\alpha,\beta) + \frac{d\ln|f_3(\alpha)|}{d\alpha}\right] z_1 z_4 - \left[2\Gamma^3_{13}(\alpha,\beta) + \frac{d\ln|g_2(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_1 z_3 - \\ - \left[2\Gamma^3_{23}(\alpha,\beta) + \frac{d\ln|h(\beta_2)|}{d\beta_2}\right] f_2(\alpha) g_1(\beta_1) z_1 z_2, \\ \dot\beta_1 = z_3 f(\alpha), \\ \dot\beta_2 = z_2 f(\alpha) g(\beta_1), \\ \dot\beta_3 = z_1 f(\alpha) g(\beta_1) h(\beta_2), \end{cases} \tag{8.50}$$

which is almost everywhere equivalent to the system

$$\begin{cases} \ddot\alpha + F(\alpha) + \Gamma^\alpha_{11}(\alpha,\beta)\dot\beta_1^2 + \Gamma^\alpha_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^\alpha_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_1 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_1 + \Gamma^1_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^1_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_2 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_2 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_2 + \Gamma^2_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_3 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_3 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_3 + 2\Gamma_3(\beta_2)\dot\beta_2\dot\beta_3 = 0. \end{cases}$$

Proposition 14 *Under the assumptions of Proposition 9, the system* (8.50) *has a smooth first integral of the form*

$$\Phi_1(z_4,\ldots,z_1;\alpha) = z_1^2 + \ldots + z_4^2 + F_1(\alpha) = C_1 = const,\ F_1(\alpha) = 2\int_{\alpha_0}^{\alpha} F(a)da. \tag{8.51}$$

The following two assertions are proved in the same way as Propositions 10–13.

Proposition 15 *Under the assumptions of Propositions 10, 11, 12, the system* (8.50) *has three smooth first integrals of the form* (8.43), (8.46), (8.48).

Proposition 16 *Under the assumptions of Proposition 13, the system* (8.50) *has a first integral of the form* (8.49).

The first integrals (8.51), (8.43), (8.46), (8.48), (8.49) form the complete list of independent first integrals of the system (8.50) under the above conditions (we will show below that the complete list consists of five (not seven) first integrals).

8.8.2 Equations of Motion in a Force Field with Dissipation and First Integrals

We consider a more complicated system of the form (8.50) with dissipation. Namely, the presence of dissipation (generally speaking, alternating) is characterized by a sufficiently smooth coefficient $b\delta(\alpha)$ in the first equation of the system

$$\begin{cases} \dot\alpha = -z_4 + b\delta(\alpha), \\ \dot z_4 = F(\alpha) + \Gamma^\alpha_{11} f_1^2(\alpha) z_3^2 + \Gamma^\alpha_{22} f_2^2(\alpha) g_1^2(\beta_1) z_2^2 + \Gamma^\alpha_{33} f_3^2(\alpha) g_2^2(\beta_1) h^2(\beta_2) z_1^2, \\ \dot z_3 = \left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right] z_3 z_4 - \Gamma^1_{22}(\alpha,\beta)\frac{f_2^2(\alpha)}{f_1(\alpha)} g_1^2(\beta_1) z_2^2 - \\ \quad -\Gamma^1_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_1(\alpha)} g_2^2(\beta_1) h^2(\beta_2) z_1^2, \\ \dot z_2 = \left[2\Gamma^2_{\alpha 2}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha}\right] z_2 z_4 - \left[2\Gamma^2_{12}(\alpha,\beta) + \frac{d\ln|g_1(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_2 z_3 - \\ \quad -\Gamma^2_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_2(\alpha)}\frac{g_2^2(\beta_1)}{g_1(\beta_1)} h^2(\beta_2) z_1^2, \\ \dot z_1 = \left[2\Gamma^3_{\alpha 3}(\alpha,\beta) + \frac{d\ln|f_3(\alpha)|}{d\alpha}\right] z_1 z_4 - \left[2\Gamma^3_{13}(\alpha,\beta) + \frac{d\ln|g_2(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_1 z_3 - \\ \quad - \left[2\Gamma^3_{23}(\alpha,\beta) + \frac{d\ln|h(\beta_2)|}{d\beta_2}\right] f_2(\alpha) g_1(\beta_1) z_1 z_2, \\ \dot\beta_1 = z_3 f(\alpha), \\ \dot\beta_2 = z_2 f(\alpha) g(\beta_1), \\ \dot\beta_3 = z_1 f(\alpha) g(\beta_1) h(\beta_2), \end{cases} \tag{8.52}$$

which is almost everywhere equivalent to the system

$$\begin{cases} \ddot\alpha - b\dot\alpha\delta'(\alpha) + F(\alpha) + \Gamma^\alpha_{11}(\alpha,\beta)\dot\beta_1^2 + \Gamma^\alpha_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^\alpha_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_1 - b\dot\beta_1\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_1 + \Gamma^1_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^1_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_2 - b\dot\beta_2\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_2 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_2 + \Gamma^2_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_3 - b\dot\beta_3\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_3 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_3 + 2\Gamma_3(\beta_2)\dot\beta_2\dot\beta_3 = 0, \end{cases}$$

$$W(\alpha) = \left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right].$$

We proceed by integrating the eighth order system (8.52) under the conditions (8.40), (8.41), (8.44), provided that the following identities hold:

$$\Gamma^\alpha_{11}(\alpha,\beta) = \Gamma^\alpha_{22}(\alpha,\beta)g^2(\beta_1) = \Gamma^\alpha_{33}(\alpha,\beta)g^2(\beta_1)h^2(\beta_2) = \Gamma_4(\alpha). \tag{8.53}$$

We also impose the condition on the function $f(\alpha)$: it should satisfy the transformed first identity in (8.38):

$$2\Gamma_1(\alpha) + \frac{d \ln |f(\alpha)|}{d\alpha} + \Gamma_4(\alpha) f^2(\alpha) \equiv 0. \tag{8.54}$$

In general, for the complete integrability of the system (8.52) it is necessary to know seven independent first integrals. However, after the change of variables

$$w_4 = z_4,\ w_3 = \sqrt{z_1^2 + z_2^2 + z_3^2},\ w_2 = \frac{z_2}{z_1},\ w_1 = \frac{z_3}{\sqrt{z_1^2 + z_2^2}},$$

the system (8.52) splits:

$$\begin{cases} \dot{\alpha} = -w_4 + b\delta(\alpha), \\ \dot{w}_4 = F(\alpha) + \Gamma_4(\alpha) f^2(\alpha) w_3^2, \\ \dot{w}_3 = \left[2\Gamma_1(\alpha) + \frac{d \ln |f(\alpha)|}{d\alpha}\right] w_3 w_4, \end{cases} \tag{8.55}$$

$$\begin{cases} \dot{w}_2 = \pm w_3 \sqrt{1 + w_2^2} f(\alpha) g(\beta_1) \left[2\Gamma_3(\beta_2) + \frac{d \ln |h(\beta_2)|}{d\beta_2}\right], \\ \dot{\beta}_2 = \pm \frac{w_2 w_3}{\sqrt{1 + w_2^2}} f(\alpha) g(\beta_1), \end{cases} \tag{8.56}$$

$$\begin{cases} \dot{w}_1 = \pm w_3 \sqrt{1 + w_1^2} f(\alpha) \left[2\Gamma_2(\beta_1) + \frac{d \ln |g(\beta_1)|}{d\beta_1}\right], \\ \dot{\beta}_1 = \pm \frac{w_1 w_3}{\sqrt{1 + w_1^2}} f(\alpha), \end{cases} \tag{8.57}$$

$$\dot{\beta}_3 = \pm \frac{w_3}{\sqrt{1 + w_2^2}} f(\alpha) g(\beta_1) h(\beta_2). \tag{8.58}$$

It is clear that for the complete integrability of the system (8.55)–(8.58) it suffices to have two independent first integrals of the system (8.55), one first integral of the system (8.56), one first integral of the system (8.57) (exchanging independent variables), and the additional first integral, “binding” Equation (8.58) (i.e., five first integrals in total).

Theorem 8.1. *Let for some* $\kappa, \lambda \in \mathbf{R}$

$$\Gamma_4(\alpha) f^2(\alpha) = \kappa \frac{d}{d\alpha} \ln |\delta(\alpha)|,\ F(\alpha) = \lambda \frac{d}{d\alpha} \frac{\delta^2(\alpha)}{2}. \tag{8.59}$$

Then the system (8.52) *under the conditions* (8.40), (8.41), (8.44), (8.53), (8.54) *possesses the complete list of (five) independent, generally speaking, transcendental first integrals.*

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