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Integrable Systems with Many Degrees of Freedom and with Dissipation

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Abstract—The integrability of certain classes of dynamical systems on the tangent bundle of a multi-dimensional manifold is shown. In this case, the force fields with variable dissipation generalize those considered earlier.

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In many problems of dynamics, there appear mechanical systems with many degrees of freedom and with dissipation (with a position space considered as a multi-dimensional manifold). The tangent bundles of such manifolds become their phase spaces. For example, the study of an n-dimensional generalized spherical pendulum in a nonconservative force field leads to a dynamical system on the tangent bundle of an $(n-1)$-dimensional sphere; a specific metric on it is induced by an additional symmetry group [1, 2]. In this case, the dynamical systems describing the motion of such a pendulum possess alternating dissipation and the complete list of first integrals consists of transcendental functions in the sense of complex analysis; these functions can be expressed in terms of a finite combination of elementary functions.

We also mention the class of problems concerning the motion of a point on a multi-dimensional surface such that its metric is induced by the Euclidean metric of the ambient space. In a number of cases, it is also possible to find the complete list of first integrals for the systems with dissipation when this list consists of transcendental functions. The obtained results are especially important, since a nonconservative force field is present in such a system.

In this paper we show the integrability of certain classes of dynamical systems on the tangent bundle of a multi-dimensional manifold. Similar studies are discussed in [3–5] for the cases when the manifold dimension is equal to 2, 3, and 4. We also show that the force fields with variable dissipation generalize those considered earlier.

1. EQUATIONS OF GEODESIC LINES AFTER CHANGING THE COORDINATES AND THEIR FIRST INTEGRALS

Let us consider an n-dimensional smooth Riemannian manifold M^n with coordinates (α, β), where $\beta = (\beta_1, \dots, \beta_{n-1})$, and with the affine connection $\Gamma^i_{jk}(x)$. As is known, the equations of geodesic lines take the following form on the tangent bundle $T_*M^n\{\dot\alpha, \dot\beta_1, \dots, \dot\beta_{n-1}; \alpha, \beta_1, \dots, \beta_{n-1}\}$, where $\alpha = x^1$, $\beta_1 = x^2$, …, $\beta_{n-1} = x^n$, and $x = (x^1, \dots, x^n)$:

$$\ddot x^i + \sum_{j,k=1}^{n} \Gamma^i_{jk}(x)\dot x^j \dot x^k = 0, \qquad i = 1, \dots, n. \tag{1}$$

Here the differentiation is performed with respect to the natural parameter.

Now we study the structure of Eqs. (1) under a change of coordinates on the tangent bundle T_*M^n. Let us consider the following coordinate change for the tangent space:

$$\dot x^i = \sum_{j=1}^{n} R^{ij}(x) z_j. \tag{2}$$

This coordinate change is dependent on a point x of the above manifold and can be inverted almost everywhere:

$$z_j = \sum_{i=1}^{n} T_{ji}(x)\dot x^i.$$

Here R^{ij} and T_{ji} $(i,j=1,\dots,n)$ are some functions of $x^1,\dots,x^n$ and $RT=E$, where $R=(R^{ij})$ and $T=(T_{ji})$. We call Eqs. (2) the new kinematic relations , i.e., the relations of the tangent bundle T_*M^n.

The following equalities are valid:

$$\dot z_j=\sum_{i=1}^n \dot T_{ji}\dot x^i+\sum_{i=1}^n T_{ji}\ddot x^i,\qquad \dot T_{ji}=\sum_{k=1}^n T_{ji,k}\dot x^k. \tag{3}$$

Here $T_{ji,k}=\partial T_{ji}/\partial x^k$, $j,i,k=1,\dots,n$. Substituting (3) into (1), we get

$$\dot z_i=\sum_{j,k=1}^n T_{ij,k}\dot x^j\dot x^k-\sum_{j,p,q=1}^n T_{ij}\Gamma^j_{pq}\dot x^p\dot x^q. \tag{4}$$

In (4) we should replace $\dot x^i$, $i=1,\dots,n$, by the formulas given in (2).

In other words, the equality expressed by (4) can be rewritten as

$$\dot z_i+\sum_{j,k=1}^n Q_{ijk}\dot x^j\dot x^k|_{(2)}=0,\qquad Q_{ijk}(x)=\sum_{s=1}^n T_{is}(x)\Gamma^s_{jk}(x)-T_{ij,k}(x).$$

Proposition 1. *System* (1) *is equivalent to the system expressed by* (2) *and* (4) *in the domain where* $\det R(x)\neq 0$.

Thus, the transition from Eqs. (1) to the equivalent system (2), (4) is dependent on the coordinate change (2) of the tangent space (i.e., on the new kinematic relations) and on the affine connection $\Gamma^i_{jk}(x)$.

2. AN IMPORTANT PARTICULAR CASE

Further, we consider the following sufficiently general case of specifying the kinematic relations:

$$\begin{aligned}
&\dot\alpha=-z_n,\\
&\dot\beta_1=z_{n-1}f_1(\alpha),\\
&\dot\beta_2=z_{n-2}f_2(\alpha)g_1(\beta_1),\\
&\dot\beta_3=z_{n-3}f_3(\alpha)g_2(\beta_1)h_1(\beta_2),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\dot\beta_{n-1}=z_1f_{n-1}(\alpha)g_{n-2}(\beta_1)h_{n-3}(\beta_2)\cdot\ldots\cdot i_1(\beta_{n-2}).
\end{aligned}\tag{5}$$

Here $f_k(\alpha)$, $k=1,\dots,n-1$; $g_l(\beta_1)$, $l=1,\dots,n-2$; $h_m(\beta_2)$, $m=1,\dots,n-3$; …, and $i_1(\beta_{n-2})$ are smooth functions on their domains of definition. The coordinates $z_1,\dots,z_n$ are introduced in the tangent space when the following equations of geodesic lines with $n(n-1)$ nonzero connection coefficients are considered [6, 7]:

$$\begin{aligned}
&\ddot\alpha+\Gamma^\alpha_{11}(\alpha,\beta)\dot\beta_1^2+\ldots+\Gamma^\alpha_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2=0,\\
&\ddot\beta_1+2\Gamma^1_{\alpha1}(\alpha,\beta)\dot\alpha\dot\beta_1+\Gamma^1_{22}(\alpha,\beta)\dot\beta_2^2+\ldots+\Gamma^1_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2=0,\\
&\ddot\beta_2+2\Gamma^2_{\alpha2}(\alpha,\beta)\dot\alpha\dot\beta_2+2\Gamma^2_{12}(\alpha,\beta)\dot\beta_1\dot\beta_2+\Gamma^2_{33}(\alpha,\beta)\dot\beta_3^2+\ldots+\Gamma^2_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2=0,\\
&\ddot\beta_3+2\Gamma^3_{\alpha3}(\alpha,\beta)\dot\alpha\dot\beta_3+2\Gamma^3_{13}(\alpha,\beta)\dot\beta_1\dot\beta_3+2\Gamma^3_{23}(\alpha,\beta)\dot\beta_2\dot\beta_3+\Gamma^3_{44}(\alpha,\beta)\dot\beta_4^2\\
&\quad\ldots+\Gamma^3_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2=0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&\ddot\beta_{n-2}+2\Gamma^{n-2}_{\alpha,n-2}(\alpha,\beta)\dot\alpha\dot\beta_{n-2}+2\Gamma^{n-2}_{1,n-2}(\alpha,\beta)\dot\beta_1\dot\beta_{n-2}\\
&\quad\ldots+2\Gamma^{n-2}_{n-3,n-2}(\alpha,\beta)\dot\beta_{n-3}\dot\beta_{n-2}+\Gamma^{n-2}_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2=0,\\
&\ddot\beta_{n-1}+2\Gamma^{n-1}_{\alpha,n-1}(\alpha,\beta)\dot\alpha\dot\beta_{n-1}+2\Gamma^{n-1}_{1,n-1}(\alpha,\beta)\dot\beta_1\dot\beta_{n-1}+\ldots+2\Gamma^{n-1}_{n-2,n-1}(\alpha,\beta)\dot\beta_{n-2}\dot\beta_{n-1}=0.
\end{aligned}\tag{6}$$

Here the other connection coefficients are equal to zero. In particular, these equations can be considered on the multi-dimensional surfaces of revolution.

In the case of the kinematic relations (5), Eqs. (4) take the form

$$
\begin{aligned}
\dot z_1 &= \left[2\Gamma^{n-1}_{\alpha,n-1}(\alpha,\beta) + Df_{n-1}(\alpha)\right] z_1 z_n - \left[2\Gamma^{n-1}_{1,n-1}(\alpha,\beta) + Dg_{n-2}(\beta_1)\right] f_1(\alpha) z_1 z_{n-1}\\
&\quad - \left[2\Gamma^{n-1}_{2,n-1}(\alpha,\beta) + Dh_{n-3}(\beta_2)\right] f_2(\alpha) g_1(\beta_1) z_1 z_{n-2}\\
&\quad \ldots - \left[2\Gamma^{n-1}_{n-2,n-1}(\alpha,\beta) + Di_1(\beta_{n-2})\right] f_{n-2}(\alpha) g_{n-3}(\beta_1) h_{n-4}(\beta_2)\cdot\ldots\cdot r_1(\beta_{n-3}) z_1 z_2,\\
\dot z_2 &= \left[2\Gamma^{n-2}_{\alpha,n-2}(\alpha,\beta) + Df_{n-2}(\alpha)\right] z_2 z_n - \left[2\Gamma^{n-2}_{1,n-2}(\alpha,\beta) + Dg_{n-3}(\beta_1)\right] f_1(\alpha) z_2 z_{n-1}\\
&\quad \ldots - \left[2\Gamma^{n-2}_{n-3,n-2}(\alpha,\beta) + Dr_1(\beta_{n-3})\right] f_{n-3}(\alpha) g_{n-4}(\beta_1) h_{n-5}(\beta_2)\cdot\ldots\cdot s_1(\beta_{n-4}) z_2 z_3\\
&\quad -\Gamma^{n-2}_{n-1,n-1}(\alpha,\beta)\, \frac{f^2_{n-1}(\alpha)}{f_{n-2}(\alpha)}\, \frac{g^2_{n-2}(\beta_1)}{g_{n-3}(\beta_1)}\, \frac{h^2_{n-3}(\beta_2)}{h_{n-4}(\beta_2)}\cdot\ldots\cdot \frac{r^2_2(\beta_{n-3})}{r_1(\beta_{n-3})}\, i^2_1(\beta_{n-2}) z_1^2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
\dot z_{n-1} &= \left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + Df_1(\alpha)\right] z_{n-1} z_n - \Gamma^1_{22}(\alpha,\beta)\, \frac{f^2_2(\alpha)}{f_1(\alpha)}\, g^2_1(\beta_1) z^2_{n-2}\\
&\quad \ldots - \Gamma^1_{n-1,n-1}(\alpha,\beta)\, \frac{f^2_{n-1}(\alpha)}{f_1(\alpha)}\, g^2_{n-2}(\beta_1) h^2_{n-3}(\beta_2)\cdot\ldots\cdot i^2_1(\beta_{n-2}) z_1^2,\\
\dot z_n &= \Gamma^\alpha_{11} f^2_1(\alpha) z^2_{n-1} + \Gamma^\alpha_{22} f^2_2(\alpha) g^2_1(\beta_1) z_2^2 + \ldots + \Gamma^\alpha_{n-1,n-1} f^2_{n-1}(\alpha) g^2_{n-2}(\beta_1) h^2_{n-3}(\beta_2)\cdot\ldots\cdot i^2_1(\beta_{n-2}) z_1^2,
\end{aligned}
\tag{7}
$$

where $DQ(q) = d\ln|Q(q)|/dq$. Equations (6) are equivalent everywhere to the composite system expressed by (5) and (7) on the tangent bundle $T_*M^n\{z_n,\ldots,z_1;\alpha,\beta_1,\ldots,\beta_{n-1}\}$.

In order to solve this system, it is necessary to know $2n-1$ independent first integrals. Below we show that, in our case, it is necessary to know a less number of first integrals.

Proposition 2. *Let the following system of $n(n-1)/2$ equalities be valid everywhere on its domain of definition:*

$$
\begin{aligned}
&2\Gamma^1_{\alpha 1}(\alpha,\beta) + Df_1(\alpha) + \Gamma^\alpha_{11}(\alpha,\beta) f^2_1(\alpha) \equiv 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&2\Gamma^{n-1}_{\alpha,n-1}(\alpha,\beta) + Df_{n-1}(\alpha) + \Gamma^\alpha_{n-1,n-1}(\alpha,\beta) f^2_{n-1}(\alpha) g^2_{n-2}(\beta_1) h^2_{n-3}(\beta_2)\ldots i^2_1(\beta_{n-2}) \equiv 0,\\
&\left[2\Gamma^2_{12}(\alpha,\beta) + Dg_1(\beta_1)\right] f^2_1(\alpha) + \Gamma^1_{22}(\alpha,\beta) f^2_2(\alpha) g^2_1(\beta_1) \equiv 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&\left[2\Gamma^{n-1}_{1,n-1}(\alpha,\beta) + Dg_{n-2}(\beta_1)\right] f^2_1(\alpha) + \Gamma^1_{n-1,n-1}(\alpha,\beta) f^2_{n-1}(\alpha) g^2_{n-2}(\beta_1) h^2_{n-3}(\beta_2)\cdot\ldots\cdot i^2_1(\beta_{n-2}) \equiv 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&\left[2\Gamma^{n-1}_{n-2,n-1}(\alpha,\beta) + Di_1(\beta_{n-2})\right] f^2_{n-2}(\alpha) g^2_{n-3}(\beta_1) h^2_{n-4}(\beta_2)\cdot\ldots\cdot r^2_1(\beta_{n-3})\\
&\quad + \Gamma^{n-2}_{n-1,n-1}(\alpha,\beta) f^2_{n-1}(\alpha) g^2_{n-2}(\beta_1) h^2_{n-3}(\beta_2)\cdot\ldots\cdot i^2_1(\beta_{n-2}) \equiv 0.
\end{aligned}
\tag{8}
$$

Then, the system expressed by (5) *and* (7) *has the analytic first integral*

$$
\Phi_1(z_n,\ldots,z_1) = z_1^2 + \ldots + z_n^2 = C_1^2 = \text{const.} \tag{9}
$$

At first glance, it is not necessary to solve the complicated quasilinear equations (8) to prove the existence of the first integral (9). The system of these equations contains some partial differential equations that degenerate into ordinary differential equations. It is possible to prove a theorem concerning the existence of the solution $f_k(\alpha)$, $k = 1,\ldots,n-1$; $g_l(\beta_1)$, $l = 1,\ldots,n-2$; $h_m(\beta_2)$, $m = 1,\ldots,n-3$; …, and $i_1(\beta_{n-2})$ to Eqs. (8) in order to reveal the existence of the analytic first integral (9) for system (5) and (7) of the geodesic equations (6). In our further study of dynamical systems with dissipation, however, it is not necessary to use the complete group of conditions (8). Nevertheless, below we assume that the following conditions should be valid in Eqs. (5):

$$
f_1(\alpha) = \ldots = f_{n-1}(\alpha) = f(\alpha). \tag{10}
$$

In addition, we also assume that the functions $g_l(\beta_1)$, $l = 1,\ldots,n-2$; $h_m(\beta_2)$, $m = 1,\ldots,n-3$; …, and

$i_1(\beta_{n-2})$ should satisfy the following modified equations given in (8):

$$\begin{aligned}
&2\Gamma^2_{12}(\alpha,\beta) + Dg_1(\beta_1) + \Gamma^1_{22}(\alpha,\beta)g_1^2(\beta_1) \equiv 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&2\Gamma^{n-1}_{1,n-1}(\alpha,\beta) + Dg_{n-2}(\beta_1) + \Gamma^1_{n-1,n-1}(\alpha,\beta)g^2_{n-2}(\beta_1)h^2_{n-3}(\beta_2)\cdot\ldots\cdot i_1^2(\beta_{n-2}) \equiv 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&2\Gamma^{n-1}_{n-2,n-1}(\alpha,\beta) + Di_1(\beta_{n-2})g^2_{n-3}(\beta_1)h^2_{n-4}(\beta_2)\cdot\ldots\cdot r_1^2(\beta_{n-3})\\
&\quad + \Gamma^{n-2}_{n-1,n-1}(\alpha,\beta)g^2_{n-2}(\beta_1)h^2_{n-3}(\beta_2)\cdot\ldots\cdot i_1^2(\beta_{n-2}) \equiv 0.
\end{aligned} \tag{11}$$

Thus, the functions $g_l(\beta_1)$, $l = 1,\ldots,n-2$; $h_m(\beta_2)$, $m = 1,\ldots,n-3$; …, and $i_1(\beta_{n-2})$ are dependent on the connection coefficients through the system expressed by (11). The restrictions imposed on the function $f(\alpha)$ are discussed below.

Proposition 3. *If the properties expressed by* (10) *and* (11) *are valid and the equalities*

$$\Gamma^1_{\alpha 1}(\alpha,\beta) = \ldots = \Gamma^{n-1}_{\alpha,n-1}(\alpha,\beta) = \Gamma_1(\alpha) \tag{12}$$

are satisfied, then the system expressed by (5) *and* (7) *has the smooth first integral*

$$\Phi_2(z_{n-1},\ldots,z_1;\alpha) = \sqrt{z_1^2+\ldots+z_{n-1}^2}\,\Phi_0(\alpha) = C_2 = \text{const}, \tag{13}$$

$$\Phi_0(\alpha) = f(\alpha)\exp\left\{2\int\limits_{\alpha_0}^{\alpha}\Gamma_1(b)\,db\right\}.$$

Proposition 4. *If the conditions of Proposition* 3 *are fulfilled and the equalities*

$$g_1(\beta_1) = \ldots = g_{n-2}(\beta_1) = g(\beta_1), \tag{14}$$

$$\Gamma^2_{12}(\alpha,\beta) = \ldots = \Gamma^{n-1}_{1,n-1}(\alpha,\beta) = \Gamma_2(\beta_1) \tag{15}$$

are valid, then the system expressed by (5) *and* (7) *has the smooth first integral*

$$\Phi_3(z_{n-2},\ldots,z_1;\alpha,\beta_1) = \sqrt{z_1^2+\ldots+z_{n-2}^2}\,\Phi_0(\alpha)\Psi_1(\beta_1) = C_3 = \text{const}, \tag{16}$$

$$\Psi_1(\beta_1) = g(\beta_1)\exp\left\{2\int\limits_{\beta_{10}}^{\beta_1}\Gamma_2(b)\,db\right\}.$$

By induction, we repeat the above reasoning and come to the following proposition.

Proposition 5. *If the conditions of Propositions* 3 *and* 4 *are fulfilled and the equality*

$$\Gamma^{n-1}_{n-2,n-1}(\alpha,\beta) = \Gamma_{n-1}(\beta_{n-2}) \tag{17}$$

is valid, then the system expressed by (5) *and* (7) *has the smooth first integral*

$$\Phi_n(z_1;\alpha,\beta_1,\ldots,\beta_{n-2}) = z_1\Phi_0(\alpha)\Psi_1(\beta_1)\cdot\ldots\cdot\Psi_{n-2}(\beta_{n-2}) = C_n = \text{const}, \tag{18}$$

$$\Psi_{n-2}(\beta_{n-2}) = i(\beta_{n-2})\exp\left\{2\int\limits_{\beta_{20}}^{\beta_2}\Gamma_3(b)\,db\right\},\qquad i(\beta_{n-2}) = i_1(\beta_{n-2}).$$

Proposition 6. *If the conditions of Propositions* 3, 4, *and* 5 *are fulfilled, then the system expressed by* (5) *and* (7) *has the first integral*

$$\Phi_{n+1}(z_{n-2},\ldots,z_1;\alpha,\beta) = \beta_{n-1} \pm \int\limits_{\beta_{n-20}}^{\beta_{n-2}} \frac{C_n i(b)}{\sqrt{C^2_{n-1}\Phi^2_{n-2}(b) - C_n^2}}\,db = C_{n+1} = \text{const}. \tag{19}$$

The first integrals expressed by (9), (13), (16), (18), and (19) represent the complete list of the independent first integrals of the system expressed by (5) and (7) if the above-mentioned conditions are fulfilled. Below we show that this list contains the $n+1$ first integrals rather than the $2n-1$ first integrals.

The question of the smoothness of the first integral (19) is not simple. In principle, this integral can be expressed in terms of a finite combination of elementary functions or can be a rational function. Since the dynamical system under consideration has no asymptotic limit sets, the function expressed by (19) cannot be a transcendental function from the standpoint of complex analysis. Indeed, this function has no essential singular points. From the standpoint of the elementary function analysis, however, this function can be transcendental [8].

3. EQUATIONS OF MOTION IN THE POTENTIAL FORCE FIELD AND THEIR FIRST INTEGRALS

Now we modify the system expressed by (5) and (7) under the conditions given in (10)–(12), (14), (15), and (17). As a result, we come to a conservative system, where the existence of the force field is characterized by the sufficiently smooth coefficient $F(\alpha)$ in the second equation of the following system given on the tangent bundle $T_*M^n\{z_n, \ldots, z_1; \alpha, \beta_1, \ldots, \beta_{n-1}\}$:

$$\begin{aligned}
&\dot\alpha = -z_n,\\
&\dot z_n = F(\alpha) + \Gamma^\alpha_{11}(\alpha,\beta) f^2(\alpha) z_{n-1}^2 + \Gamma^\alpha_{22}(\alpha,\beta) f^2(\alpha) g^2(\beta_1) z_2^2\\
&\quad \ldots + \Gamma^\alpha_{n-1,n-1}(\alpha,\beta) f^2(\alpha) g^2(\beta_1) h^2(\beta_2)\cdot\ldots\cdot i^2(\beta_{n-2}) z_1^2,\\
&\dot z_{n-1} = [2\Gamma_1(\alpha) + Df(\alpha)]\, z_{n-1} z_n - \Gamma^1_{22}(\alpha,\beta) f(\alpha) g^2(\beta_1) z_{n-2}^2\\
&\quad \ldots - \Gamma^1_{n-1,n-1}(\alpha,\beta) f(\alpha) g^2(\beta_1) h^2(\beta_2)\cdot\ldots\cdot i^2(\beta_{n-2}) z_1^2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\dot z_2 = [2\Gamma_1(\alpha) + Df(\alpha)]\, z_2 z_n - [2\Gamma_2(\beta_1) + Dg(\beta_1)]\, f(\alpha) z_2 z_{n-1}\\
&\quad \ldots - [2\Gamma_{n-2}(\beta_{n-3}) + Dr(\beta_{n-3})]\, f(\alpha) g(\beta_1) h(\beta_2)\cdot\ldots\cdot s(\beta_{n-4}) z_2 z_3\\
&\quad - \Gamma^{n-2}_{n-1,n-1}(\alpha,\beta) f(\alpha) g(\beta_1) h(\beta_2)\cdot\ldots\cdot r(\beta_{n-3}) i^2(\beta_{n-2}) z_1^2,\\
&\dot z_1 = [2\Gamma_1(\alpha) + Df(\alpha)]\, z_1 z_n - [2\Gamma_2(\beta_1) + Dg(\beta_1)]\, f_1(\alpha) z_1 z_{n-1}\\
&\quad - [2\Gamma_3(\beta_2) + Dh(\beta_2)]\, f(\alpha) g(\beta_1) z_1 z_{n-2}\\
&\quad \ldots - [2\Gamma_{n-1}(\beta_{n-2}) + Di(\beta_{n-2})]\, f(\alpha) g(\beta_1) h(\beta_2)\cdot\ldots\cdot r(\beta_{n-3}) z_1 z_2,\\
&\dot\beta_1 = z_{n-1} f(\alpha), \qquad \dot\beta_2 = z_{n-2} f(\alpha) g(\beta_1), \qquad \dot\beta_3 = z_{n-3} f(\alpha) g(\beta_1) h(\beta_2),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\dot\beta_{n-1} = z_1 f(\alpha) g(\beta_1) h(\beta_2)\cdot\ldots\cdot i(\beta_{n-2}).
\end{aligned}\tag{20}$$

This system is equivalent almost everywhere to the following system:

$$\begin{aligned}
&\ddot\alpha + F(\alpha) + \Gamma^\alpha_{11}(\alpha,\beta)\dot\beta_1^2 + \ldots + \Gamma^\alpha_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2 = 0,\\
&\ddot\beta_1 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_1 + \Gamma^1_{22}(\alpha,\beta)\dot\beta_2^2 + \ldots + \Gamma^1_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2 = 0,\\
&\ddot\beta_2 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_2 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_2 + \Gamma^2_{33}(\alpha,\beta)\dot\beta_3^2 + \ldots + \Gamma^2_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2 = 0,\\
&\ddot\beta_3 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_3 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_3 + 2\Gamma_3(\beta_2)\dot\beta_2\dot\beta_3\\
&\quad + \Gamma^3_{44}(\alpha,\beta)\dot\beta_4^2 + \ldots + \Gamma^3_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2 = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ddot\beta_{n-2} + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_{n-2} + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_{n-2}\\
&\quad \ldots + 2\Gamma_{n-2}(\beta_{n-3})\dot\beta_{n-3}\dot\beta_{n-2} + \Gamma^{n-2}_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2 = 0,\\
&\ddot\beta_{n-1} + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_{n-1} + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_{n-1} + \ldots + 2\Gamma_{n-1}(\beta_{n-2})\dot\beta_{n-2}\dot\beta_{n-1} = 0.
\end{aligned}$$

Proposition 7. *If the conditions of Proposition* 2 *are fulfilled, then the system expressed by* (20) *has the smooth first integral*

$$\Phi_1(z_n, \ldots, z_1; \alpha) = z_1^2 + \ldots + z_n^2 + F_1(\alpha) = C_1 = \text{const}, \qquad F_1(\alpha) = 2\int_{\alpha_0}^{\alpha} F(b)\, db. \tag{21}$$

Proposition 8. *If the conditions of Propositions* 3, 4, *and* 5 *are fulfilled, then the system expressed by* (20) *has the smooth first integrals* (13), (16), *and* (18).

Proposition 9. *If the conditions of Proposition* 6 *are fulfilled, then the system expressed by* (20) *has the first integral* (19).

The first integrals (21), (13), (16), (18), and (19) represent the complete list of independent first integrals of system (20) if the above-mentioned conditions are fulfilled. Below we show that this list contains the $n+1$ first integrals rather than the $2n-1$ first integrals.

The question of the smoothness of the first integral (19) is still not simple. Since the dynamical system under consideration has no asymptotic limit sets even when a conservative force field exists, the function expressed by (19) cannot be a transcendental function from the standpoint of complex analysis: this function has no essential singular points. From the standpoint of the elementary function analysis, however, this function can be transcendental [8].

4. EQUATIONS OF MOTION IN A FORCE FIELD WITH DISSIPATION AND THEIR FIRST INTEGRALS

In order to obtain a system with dissipation, we complicate the system expressed by (20). The presence of alternating dissipation is characterized by the sufficiently smooth coefficient $b\delta(\alpha)$ in the first equation of the following system:

$$
\begin{aligned}
&\dot\alpha = -z_n + b\delta(\alpha),\\
&\dot z_n = F(\alpha) + \Gamma^{\alpha}_{11}(\alpha,\beta)f^2(\alpha)z_{n-1}^2 + \Gamma^{\alpha}_{22}(\alpha,\beta)f^2(\alpha)g^2(\beta_1)z_2^2\\
&\quad \ldots + \Gamma^{\alpha}_{n-1,n-1}(\alpha,\beta)f^2(\alpha)g^2(\beta_1)h^2(\beta_2)\cdot\ldots\cdot i^2(\beta_{n-2})z_1^2,\\
&\dot z_{n-1} = [2\Gamma_1(\alpha) + Df(\alpha)]\,z_{n-1}z_n - \Gamma^{1}_{22}(\alpha,\beta)f(\alpha)g^2(\beta_1)z_{n-2}^2\\
&\quad \ldots - \Gamma^{1}_{n-1,n-1}(\alpha,\beta)f(\alpha)g^2(\beta_1)h^2(\beta_2)\cdot\ldots\cdot i^2(\beta_{n-2})z_1^2,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\dot z_2 = [2\Gamma_1(\alpha) + Df(\alpha)]\,z_2z_n - [2\Gamma_2(\beta_1) + Dg(\beta_1)]\,f(\alpha)z_2z_{n-1}\\
&\quad \ldots - [2\Gamma_{n-2}(\beta_{n-3}) + Dr(\beta_{n-3})]\,f(\alpha)g(\beta_1)h(\beta_2)\cdot\ldots\cdot s(\beta_{n-4})z_2z_3\\
&\quad - \Gamma^{n-2}_{n-1,n-1}(\alpha,\beta)f(\alpha)g(\beta_1)h(\beta_2)\cdot\ldots\cdot r(\beta_{n-3})i^2(\beta_{n-2})z_1^2,\\
&\dot z_1 = [2\Gamma_1(\alpha) + Df(\alpha)]\,z_1z_n - [2\Gamma_2(\beta_1) + Dg(\beta_1)]\,f_1(\alpha)z_1z_{n-1}\\
&\quad - [2\Gamma_3(\beta_2) + Dh(\beta_2)]\,f(\alpha)g(\beta_1)z_1z_{n-2}\\
&\quad \ldots - [2\Gamma_{n-1}(\beta_{n-2}) + Di(\beta_{n-2})]\,f(\alpha)g(\beta_1)h(\beta_2)\cdot\ldots\cdot r(\beta_{n-3})z_1z_2,\\
&\dot\beta_1 = z_{n-1}f(\alpha),\qquad \dot\beta_2 = z_{n-2}f(\alpha)g(\beta_1),\qquad \dot\beta_3 = z_{n-3}f(\alpha)g(\beta_1)h(\beta_2),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\dot\beta_{n-1} = z_1f(\alpha)g(\beta_1)h(\beta_2)\cdot\ldots\cdot i(\beta_{n-2}).
\end{aligned}
\tag{22}
$$

This system is equivalent almost everywhere to the following system:

$$
\begin{aligned}
&\ddot\alpha - b\dot\alpha\delta'(\alpha) + F(\alpha) + \Gamma^{\alpha}_{11}(\alpha,\beta)\dot\beta_1^2 + \ldots + \Gamma^{\alpha}_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2 = 0,\\
&\ddot\beta_1 - b\dot\beta_1\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_1 + \Gamma^{1}_{22}(\alpha,\beta)\dot\beta_2^2 + \ldots + \Gamma^{1}_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2 = 0,\\
&\ddot\beta_2 - b\dot\beta_2\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_2 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_2\\
&\quad + \Gamma^{2}_{33}(\alpha,\beta)\dot\beta_3^2 + \ldots + \Gamma^{2}_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2 = 0,\\
&\ddot\beta_3 - b\dot\beta_3\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_3 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_3 + 2\Gamma_3(\beta_2)\dot\beta_2\dot\beta_3\\
&\quad + \Gamma^{3}_{44}(\alpha,\beta)\dot\beta_4^2 + \ldots + \Gamma^{3}_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2 = 0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ddot\beta_{n-2} - b\dot\beta_{n-2}\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_{n-2} + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_{n-2}\\
&\quad \ldots + 2\Gamma_{n-2}(\beta_{n-3})\dot\beta_{n-3}\dot\beta_{n-2} + \Gamma^{n-2}_{n-1,n-1}(\alpha,\beta)\dot\beta_{n-1}^2 = 0,\\
&\ddot\beta_{n-1} - b\dot\beta_{n-1}\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_{n-1} + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_{n-1}\\
&\quad \ldots + 2\Gamma_{n-1}(\beta_{n-2})\dot\beta_{n-2}\dot\beta_{n-1} = 0.
\end{aligned}
$$

Here $W(\alpha) = 2\Gamma_1(\alpha) + Df(\alpha)$.

Now we solve the sought system (22) of order $2n$ when the following equalities are valid:

$$\Gamma_{11}^{\alpha}(\alpha,\beta) \equiv \Gamma_{22}^{\alpha}(\alpha,\beta)g^2(\beta_1) \equiv \ldots \equiv \Gamma_{n-1,n-1}^{\alpha}(\alpha,\beta)g^2(\beta_1)h^2(\beta_2)\ldots = \Gamma_n(\alpha). \tag{23}$$

As was done in the case of Eqs. (11), we impose the following restriction on the function $f(\alpha)$: this function should satisfy the modified first equality of (8) written in the form

$$2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha} + \Gamma_n(\alpha)f^2(\alpha) \equiv 0. \tag{24}$$

In order to solve the system expressed by (22), we should know the $2n-1$ independent first integrals. However, using the change of variables

$$w_n = z_n, \quad w_{n-1} = \sqrt{z_1^2+\ldots+z_{n-1}^2}, \quad w_{n-2} = \frac{z_2}{z_1},$$

$$w_{n-3} = \frac{z_3}{\sqrt{z_1^2+z_2^2}}, \ \ldots, \ w_1 = \frac{z_{n-1}}{\sqrt{z_1^2+\ldots+z_2^{n-2}}},$$

we split this system as follows:

$$\begin{aligned}
\dot\alpha &= -w_n + b\delta(\alpha),\\
\dot w_n &= F(\alpha) + \Gamma_n(\alpha)f^2(\alpha)w_{n-1}^2,\\
\dot w_{n-1} &= \left[2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha}\right]w_{n-1}w_n;
\end{aligned} \tag{25}$$

$$\begin{aligned}
\dot w_s &= \pm w_{n-1}\sqrt{1+w_s^2}f(\alpha)\ldots\left[2\Gamma_{s+1}(\beta_s) + Dj(\beta_s)\right],\\
\dot\beta_s &= \pm\frac{w_s w_{n-1}}{\sqrt{1+w_s^2}}f(\alpha)\ldots, \qquad s = 1,\ldots,n-2;
\end{aligned} \tag{26}$$

$$\dot\beta_{n-1} = \pm\frac{w_{n-1}}{\sqrt{1+w_{n-2}^2}}f(\alpha)g(\beta_1)h(\beta_2)\cdot\ldots\cdot i(\beta_{n-2}). \tag{27}$$

In (26), by the symbol "$\ldots$" we denote the identical terms, whereas the function $j(\beta_s)$ is one of the functions $g, h, \ldots$ and is dependent on the corresponding angle β_s.

In order to solve the system expressed by (25)–(27), it is sufficient to know the two independent first integrals of system (25), the $n-2$ first integrals of system (26), and an additional first integral to "attach" Eq. (27). Thus, we should know $n+1$ first integrals in total.

Theorem. *Let the equalities*

$$\Gamma_n(\alpha)f^2(\alpha) = \kappa\frac{d}{d\alpha}\ln|\delta(\alpha)|, \qquad F(\alpha) = \lambda\frac{d}{d\alpha}\frac{\delta^2(\alpha)}{2} \tag{28}$$

be valid for κ, $\lambda \in \mathbf{R}$. *Under the conditions given by* (23) *and* (24), *then, the system expressed by* (22) *has a complete set of* $(n+1)$ *independent transcendental first integrals.*

To begin with, we relate the third-order system (25) to the nonautonomous second-order system

$$\frac{dw_n}{d\alpha} = \frac{F(\alpha) + \Gamma_n(\alpha)f^2(\alpha)w_{n-1}^2}{-w_n + b\delta(\alpha)}, \qquad \frac{dw_{n-1}}{d\alpha} = \frac{\left[2\Gamma_1(\alpha) + Df(\alpha)\right]w_{n-1}w_n}{-w_n + b\delta(\alpha)}. \tag{29}$$

Introducing the variables $w_n = u_n\delta(\alpha)$ and $w_{n-1} = u_{n-1}\delta(\alpha)$, we represent the system expressed by (29) as

$$\begin{aligned}
\delta(\alpha)\frac{du_n}{d\alpha} &= \frac{F_3(\alpha) + \Gamma_n(\alpha)f^2(\alpha)\delta(\alpha)u_{n-1}^2 + \delta'(\alpha)u_n^2 - b\delta'(\alpha)u_n}{-u_n+b},\\
\delta(\alpha)\frac{du_{n-1}}{d\alpha} &= \frac{-\Gamma_n(\alpha)f^2(\alpha)\delta(\alpha)u_{n-1}u_n + \delta'(\alpha)u_{n-1}u_n - b\delta'(\alpha)u_n}{-u_n+b},\\
F_3(\alpha) &= \frac{F(\alpha)}{\delta(\alpha)}.
\end{aligned} \tag{30}$$

Taking into account (28), we reduce (30) to the first-order equation

$$\frac{du_n}{du_{n-1}} = \frac{\lambda + \kappa u_{n-1}^2 + u_n^2 - bu_n}{(1-\kappa)u_{n-1}u_n - bu_n}. \tag{31}$$

Equation (31) is an Abel-type equation [9]. In particular, this equation has the following first integral for $\kappa = -1$:

$$\frac{u_n^2 + u_{n-1}^2 - bu_n + \lambda}{u_{n-1}} = C_1 = \text{const.} \tag{32}$$

Using the previous variables, this integral can be represented as

$$\Theta_1(w_n, w_{n-1}; \alpha) = G_1\left(\frac{w_n}{\delta(\alpha)}, \frac{w_{n-1}}{\delta(\alpha)}\right) = \frac{w_n^2 + w_{n-1}^2 - bw_n\delta(\alpha) + \lambda\delta^2(\alpha)}{w_{n-1}\delta(\alpha)} = C_1 = \text{const.} \tag{33}$$

Further, we find the additional first integral of the third-order system (25) for $\kappa = -1$. To accomplish this, we transform the invariant relation (32) for $u_{n-1} \neq 0$ as follows:

$$\left(u_n - \frac{b}{2}\right)^2 + \left(u_{n-1} - \frac{C_1}{2}\right)^2 = \frac{b^2 + C_1^2}{4} - \lambda. \tag{34}$$

The parameters of this invariant relation satisfy the condition

$$b^2 + C_1^2 - 4\lambda \geq 0. \tag{35}$$

Then, the phase space of system (21) can be splitted into the family of surfaces given by (34).

By virtue of (32), thus, the first equation of (30) take the following form for $\kappa = -1$:

$$\frac{\delta(\alpha)}{\delta'(\alpha)}\frac{du_n}{d\alpha} = \frac{2(\lambda - bu_n + u_4^2) - C_1U_1(C_1, u_n)}{-u_n + b}, \qquad U_1(C_1, u_n) = \frac{1}{2}\{C_1 \pm \sqrt{C_1^2 - 4(u_n^2 - bu_n + \lambda)}\}.$$

Here the integration constant C_1 is chosen using (35).

Then, the additional first integral for the system expressed by (25) take the following structural form:

$$\Theta_2(w_n, w_{n-1}; \alpha) = G_2\left(\delta(\alpha), \frac{w_n}{\delta(\alpha)}, \frac{w_{n-1}}{\delta(\alpha)}\right) = C_2 = \text{const.} \tag{36}$$

For $\kappa = -1$, this first integral can be found from the quadrature

$$\ln|g(\alpha)| = \int \frac{(b - u_n)\,du_n}{2(\lambda - bu_n + u_n^2) - C_1\{C_1 \pm \sqrt{C_1^2 - 4(u_n^2 - bu_n + \lambda)}\}/2}, \qquad u_n = \frac{w_n}{\delta(\alpha)}.$$

After evaluating this integral, we can use the equality expressed by (33) instead of C_1. The right-hand side of this equality can be expressed in terms of a combination of elementary functions, whereas the left-hand side is dependent on the function $\delta(\alpha)$. Hence, the expression of the first integrals (33) and (36) in terms of a combination of elementary functions depends on the quadratures and on the explicit form of the function $\delta(\alpha)$.

The first integrals of system (26) can be written as

$$\Theta_{s+2}(w_s; \beta_s) = \frac{\sqrt{1 + w_s^2}}{\Psi_s(\beta_s)} = C_{s+2} = \text{const}, \qquad s = 1, \ldots, n-2, \tag{37}$$

where the functions $\Psi_s(\beta_s)$, $s = 1, \ldots, n-2$, are clarified by (16) and (18)). The additional first integral “attaching” Eq. (27) can be found in a similar way as was done for (19):

$$\Theta_{n+1}(w_2, w_1; \alpha, \beta) = \beta_{n-1} \pm \int\limits_{\beta_{n-20}}^{\beta_n-2} \frac{C_n i(b)}{\sqrt{C_{n-1}^2\Psi_{n-2}^2(b) - C_n^2}}\,db = C_{n+1} = \text{const.}$$

After evaluating the last integral, the left-hand sides of (37) can be substituted instead of C_{n-1} and C_n.

5. STRUCTURE OF THE FIRST INTEGRALS FOR THE SYSTEMS WITH DISSIPATION

If α is a periodic coordinate with a period of 2π, then the system expressed by (25) becomes a dynamical system possessing a variable dissipation with zero mean [1–5]. If $b = 0$, then this system becomes a conservative system with the two smooth first integrals (21) and (13). Be virtue of (28), we have

$$\Phi_1(z_n, \ldots, z_1; \alpha) = z_1^2 + \ldots + z_n^2 + 2\int\limits_{\alpha_0}^{\alpha} F(b)\, db \cong w_n^2 + w_{n-1}^2 + \lambda\delta^2(\alpha), \tag{38}$$

where the symbol "$\cong$" indicates the equality up to an additive constant. Taking into account (24) and (28), we conclude that

$$\Phi_2(z_{n-1}, \ldots, z_1; \alpha) = \sqrt{z_1^2 + \ldots + z_{n-1}^2}\; f(\alpha) \exp\left\{2\int\limits_{\alpha_0}^{\alpha} \Gamma_1(b)\, db\right\} \cong w_{n-1}\delta(\alpha) = C_2 = \text{const}, \tag{39}$$

where the symbol "$\cong$" indicates the equality up to an multiplicative constant.

Obviously, the ratio of the two first integrals (38) and (39) (or (21) and (13)) is also a first integral of system (25) for $b = 0$. However, if $b \neq 0$, then each of the functions

$$w_n^2 + w_{n-1}^2 - bw_n\delta(\alpha) + \lambda\delta^2(\alpha) \tag{40}$$

and (39) taken separately is not a first integral of system (25). However, the ratio of the functions (40) and (39) is a first integral of system (25) for any b when $\kappa = -1$.

For systems with dissipation, in general, the transcendence of functions (understood in the sense of the existence of essentially singular points) as first integrals is caused by some attracting and repelling limit sets existing in the system under study [10].

6. SOME APPLICATIONS

By analogy with low-dimensional cases, we consider the following important cases for the function $f(\alpha)$ defining the metric on a sphere:

$$f(\alpha) = \frac{\cos\alpha}{\sin\alpha}, \tag{41}$$

$$f(\alpha) = \frac{1}{\cos\alpha\sin\alpha}. \tag{42}$$

In the case of (41), the class of systems corresponding to the motion of a dynamically symmetric $(n+1)$-dimensional rigid body is formed at zero levels of cyclic integrals in a nonconservative field of forces [11]. In the case of (42), the class of systems corresponding to the motion of a material point on an n-dimensional sphere is formed also in a nonconservative field of forces. In particular, if $\delta(\alpha) \equiv F(\alpha) \equiv 0$, then the system under consideration describes a geodesic flow on an n-dimensional sphere. In the case of (41), if $\delta(\alpha) = F(\alpha)/\cos\alpha$, then the system describes the motion of an $(n+1)$-dimensional rigid body in the force field $F(\alpha)$ under the action of a tracking force [11]. In particular, if $F(\alpha) = \sin\alpha\cos\alpha$ and $\delta(\alpha) = \sin\alpha$, then this system also describes a generalized $(n+1)$-dimensional spherical pendulum in a nonconservative force field and has a complete list of transcendental first integrals that can be expressed in terms of a finite combination of elementary functions [2].

If the function $\delta(\alpha)$ is not periodic, then the dissipation system under consideration is a system with variable dissipation with zero mean; in other words, this system is properly dissipative and can be considered as a system with accelerating forces. Nevertheless, an explicit form of transcendental first integrals that could be expressed in terms of elementary functions can also be obtained in this case. This fact is a new nontrivial case of integrability of dissipation systems in an explicit form.

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