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Preface

Dear Reader,

in this book you will find the Proceedings of the Summer School – Conference "Advanced Problems in Mechanics (APM) 2018". The conference had been started in 1971. The first Summer School was organized by Prof. Ya.G. Panovko and his colleagues. In the early years the main focus of the School was on nonlinear oscillations of mechanical systems with a finite number of degrees of freedom. Since 1994 the Institute for Problems in Mechanical Engineering of the Russian Academy of Sciences organizes the Summer School. The traditional name of "Summer School" has been kept, but the topics covered by the School have been much widened, and the School has been transformed into an international conference. Now it is held under the patronage of the Russian Academy of Sciences. The topics of the conference cover now almost all fields of mechanics, being concentrated around the following main scientific directions:

— aerospace mechanics;
— computational mechanics;
— dynamics of rigid bodies and multibody dynamics;
— fluid and gas;
— mechanical and civil engineering applications;
— mechanics of media with microstructure;
— mechanics of granular media;
— nanomechanics;
— nonlinear dynamics, chaos and vibration;
— molecular and particle dynamics;
— phase transitions;
— solids and structures;
— wave motion.

The Summer School – Conference has two main purposes: to gather specialists from different branches of mechanics to provide a platform for cross-fertilization of ideas, and to give the young scientists a possibility to learn from their colleagues and to present their work. Thus the Scientific Committee encouraged the participation of young researchers, and did its best to gather at the conference leading scientists belonging to various scientific schools of the world.

We believe that the significance of Mechanics as of fundamental and applied science should much increase in the eyes of the world scientific community, and we hope that APM conference makes its contribution into this process.

The Conference is organized by Institute for Problems in Mechanical Engineering of Russian Academy of Sciences (IPME RAS) and Peter the Great St.Petersburg Polytechnic University (SPbPU) under the patronage of Russian Academy of Sciences (RAS), St.Petersburg Scientific Center, Ministry of Education and Science of Russian Federation (project indentificator RFMEFI 60715X0120) and the University of Seville (Universidad de Sevilla). APM 2018 is partially supported by Russian Foundation for Basic Research. Minisymposium in memoriam of Antonio Castellanos Mata is partially sponsored by the Vicerrectorado de Investigacion de la Universidad de Sevilla (Vice-Rectorate for Research, University of Seville, Spain).

We hope that you will find the materials of the conference interesting, and we cordially invite you to participate in the coming APM conferences. You may find the information on the future "Advanced Problems in Mechanics" Schools – Conferences at our website:

`http://apm-conf.spb.ru`

With kind regards,

Co-Chairmen of APM 2018

Dmitri A. Indeitsev, Anton M. Krivtsov

Non-smooth first integrals of dissipative systems with four degrees of freedom

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Abstract

In this study, we show the integrability of certain classes of dynamic systems on the tangent bundle to a four-dimensional manifold. In this case, the force fields have so-called variable dissipation and generalize the cases considered previously.

1 Introduction

In many problems of dynamics, mechanical systems arise with the space of positions — a four-dimensional manifold. Their phase spaces naturally become the tangent bundles to these manifolds. Thus, for example, the study of a five-dimensional generalized spherical pendulum in a nonconservative force field leads to a dynamic system on the tangent bundle to a four-dimensional sphere, while the special metric on it is induced by an additional symmetry group. In this case, the dynamic systems describing the motion of such a pendulum have alternating dissipation and the complete list of first integrals consists of transcendental functions expressed through a finite combination of elementary functions.
We also single out the class of problems on the motion of a point along a four-dimensional surface, the metric on it being induced by the Euclidean metric of a comprehensive space. In a number of cases, the complete list of first integrals consisting of transcendental functions can also be found in systems with dissipation. The results obtained are especially important in the sense of the presence of a precisely nonconservative force field in the system.

2 Equations of geodesic lines under a change of coordinates and its first integrals

It is well known that, in the case of a four-dimensional Riemannian manifold M^4 with coordinates (α, β), $\beta = (\beta_1, \beta_2, \beta_3)$, and affine connection $\Gamma^i_{jk}(x)$ the equations of geodesic lines on the tangent bundle $T_*M^4\{\dot\alpha, \dot\beta_1, \dot\beta_2, \dot\beta_3; \alpha, \beta_1, \beta_2, \beta_3\}$, $\alpha = x^1$, $\beta_1 = x^2$, $\beta_2 = x^3$, $\beta_3 = x^4$, $x = (x^1, x^2, x^3, x^4)$, have the following form (the derivatives are taken with respect to the natural parameter):

$$\ddot{x}^i + \sum_{j,k=1}^{4} \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0, \ i = 1, \dots, 4. \tag{1}$$

Let us study the structure of Eqs. (1) under a change of coordinates on the tangent bundle T_*M^4. Consider a change of coordinates of the tangent space:

$$\dot{x}^i = \sum_{j=1}^{4} R^{ij}(x)z_j, \tag{2}$$

which can be inverted:

$$z_j = \sum_{i=1}^{4} T_{ji}(x)\dot{x}^i,$$

here $R^{ij}, T_{ji}, i, j = 1, \dots, 4$, are functions of x^1, x^2, x^3, x^4, and

$$RT = E,$$

where

$$R = (R^{ij}), \ T = (T_{ji}).$$

We also call Eqs. (2) new kinematic relations, i.e., relations on the tangent bundle T_*M^4.
The following equalities are valid:

$$\dot{z}_j = \sum_{i=1}^{4} \dot{T}_{ji}\dot{x}^i + \sum_{i=1}^{4} T_{ji}\ddot{x}^i, \ \dot{T}_{ji} = \sum_{k=1}^{4} T_{ji,k}\dot{x}^k, \tag{3}$$

where

$$T_{ji,k} = \frac{\partial T_{ji}}{\partial x^k}, \ j, i, k = 1, \dots, 4.$$

We also have:

$$\dot{z}_i = \sum_{j,k=1}^{4} T_{ij,k}\dot{x}^j\dot{x}^k - \sum_{j,p,q=1}^{4} T_{ij}\Gamma^j_{pq}\dot{x}^p\dot{x}^q. \tag{4}$$

Otherwise, we can rewrite Eq. (4) in the form

$$\dot{z}_i + \sum_{j,k=1}^{4} Q_{ijk}\dot{x}^j\dot{x}^k|_{(2)} = 0, \tag{5}$$

where

$$Q_{ijk}(x) = \sum_{s=1}^{4} T_{is}(x)\Gamma^s_{jk}(x) - T_{ij,k}(x). \tag{6}$$

Proposition 2.1. *System (1) is equivalent to compound system (2), (4) in a domain where* $\det R(x) \neq 0$.

Therefore, the result of the passage from equations of geodesic lines (1) to an equivalent system of equations (2), (4) depends both on the change of variables (2) (i.e., introduced kinematic relations) and on the affine connection $\Gamma^i_{jk}(x)$.

3 A fairly general case

Consider next a sufficiently general case of specifying kinematic relations in the following form:

$$
\begin{aligned}
&\dot{\alpha} = -z_4,\\
&\dot{\beta}_1 = z_3 f_1(\alpha),\\
&\dot{\beta}_2 = z_2 f_2(\alpha) g_1(\beta_1),\\
&\dot{\beta}_3 = z_1 f_3(\alpha) g_2(\beta_1) h(\beta_2),
\end{aligned}
\tag{7}
$$

where $f_k(\alpha)$, $k = 1, 2, 3$, $g_l(\beta_1)$, $l = 1, 2$, $h(\beta_2)$ are smooth functions on their domain of definition. Such coordinates z_1, z_2, z_3, z_4 in the tangent space are introduced when the following equations of geodesic lines are considered [1, 2, 3] (in particular, on surfaces of revolution):

$$
\begin{cases}
\ddot{\alpha} + \Gamma^{\alpha}_{11}(\alpha,\beta)\dot{\beta}_1^2 + \Gamma^{\alpha}_{22}(\alpha,\beta)\dot{\beta}_2^2 + \Gamma^{\alpha}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0,\\
\ddot{\beta}_1 + 2\Gamma^{1}_{\alpha 1}(\alpha,\beta)\dot{\alpha}\dot{\beta}_1 + \Gamma^{1}_{22}(\alpha,\beta)\dot{\beta}_2^2 + \Gamma^{1}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0,\\
\ddot{\beta}_2 + 2\Gamma^{2}_{\alpha 2}(\alpha,\beta)\dot{\alpha}\dot{\beta}_2 + 2\Gamma^{2}_{12}(\alpha,\beta)\dot{\beta}_1\dot{\beta}_2 + \Gamma^{2}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0,\\
\ddot{\beta}_3 + 2\Gamma^{3}_{\alpha 3}(\alpha,\beta)\dot{\alpha}\dot{\beta}_3 + 2\Gamma^{3}_{13}(\alpha,\beta)\dot{\beta}_1\dot{\beta}_3 + 2\Gamma^{3}_{23}(\alpha,\beta)\dot{\beta}_2\dot{\beta}_3 = 0,
\end{cases}
\tag{8}
$$

i.e., other connection coefficients are zero. In case (7) Eqs. (4) take the form

$$
\begin{aligned}
\dot{z}_1 =& \left[2\Gamma^{3}_{\alpha 3}(\alpha,\beta) + \frac{d\ln|f_3(\alpha)|}{d\alpha}\right] z_1 z_4 - \left[2\Gamma^{3}_{13}(\alpha,\beta) + \frac{d\ln|g_2(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_1 z_3 -\\
&- \left[2\Gamma^{3}_{23}(\alpha,\beta) + \frac{d\ln|h(\beta_2)|}{d\beta_2}\right] f_2(\alpha) g_1(\beta_1) z_1 z_2,\\
\dot{z}_2 =& \left[2\Gamma^{2}_{\alpha 2}(\alpha,\beta) + \frac{d\ln|f_2(\alpha)|}{d\alpha}\right] z_2 z_4 - \left[2\Gamma^{2}_{12}(\alpha,\beta) + \frac{d\ln|g_1(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_2 z_3 -\\
&- \Gamma^{2}_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_2(\alpha)}\frac{g_2^2(\beta_1)}{g_1(\beta_1)} h^2(\beta_2) z_1^2,\\
\dot{z}_3 =& \left[2\Gamma^{1}_{\alpha 1}(\alpha,\beta) + \frac{d\ln|f_1(\alpha)|}{d\alpha}\right] z_3 z_4 - \Gamma^{1}_{22}(\alpha,\beta)\frac{f_2^2(\alpha)}{f_1(\alpha)} g_1^2(\beta_1) z_2^2 -\\
&- \Gamma^{1}_{33}(\alpha,\beta)\frac{f_3^2(\alpha)}{f_1(\alpha)} g_2^2(\beta_1) h^2(\beta_2) z_1^2,\\
\dot{z}_4 =& \Gamma^{\alpha}_{11} f_1^2(\alpha) z_3^2 + \Gamma^{\alpha}_{22} f_2^2(\alpha) g_1^2(\beta_1) z_2^2 + \Gamma^{\alpha}_{33} f_3^2(\alpha) g_2^2(\beta_1) h^2(\beta_2) z_1^2,
\end{aligned}
\tag{9}
$$

and Eqs. (8) are almost everywhere equivalent to compound system (7), (9) on the manifold $T_*M^4\{z_4, z_3, z_2, z_1; \alpha, \beta_1, \beta_2, \beta_3\}$.
To integrate system (7), (9) completely, it is necessary to know, generally speaking, seven independent first integrals.

Proposition 3.1. *If the system of equalities*

$$\begin{cases} 2\Gamma^1_{\alpha 1}(\alpha,\beta) + \dfrac{d\ln|f_1(\alpha)|}{d\alpha} + \Gamma^\alpha_{11}(\alpha,\beta)f_1^2(\alpha) \equiv 0, \\ 2\Gamma^2_{\alpha 2}(\alpha,\beta) + \dfrac{d\ln|f_2(\alpha)|}{d\alpha} + \Gamma^\alpha_{22}(\alpha,\beta)f_2^2(\alpha)g_1^2(\beta_1) \equiv 0, \\ \left[2\Gamma^2_{12}(\alpha,\beta) + \dfrac{d\ln|g_1(\beta_1)|}{d\beta_1}\right] f_1^2(\alpha) + \Gamma^1_{22}(\alpha,\beta)f_2^2(\alpha)g_1^2(\beta_1) \equiv 0, \\ 2\Gamma^3_{\alpha 3}(\alpha,\beta) + \dfrac{d\ln|f_3(\alpha)|}{d\alpha} + \Gamma^\alpha_{33}(\alpha,\beta)f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2) \equiv 0, \\ \left[2\Gamma^3_{13}(\alpha,\beta) + \dfrac{d\ln|g_2(\beta_1)|}{d\beta_1}\right] f_1^2(\alpha) + \Gamma^1_{33}(\alpha,\beta)f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2) \equiv 0, \\ \left[2\Gamma^3_{23}(\alpha,\beta) + \dfrac{d\ln|h(\beta_2)|}{d\beta_2}\right] f_2^2(\alpha)g_1^2(\beta_1) + \Gamma^2_{33}(\alpha,\beta)f_3^2(\alpha)g_2^2(\beta_1)h^2(\beta_2) \equiv 0, \end{cases} \tag{10}$$

is valid everywhere in its domain of definition, system (7), (9) has an analytic first integral of the form

$$\Phi_1(z_4,\ldots,z_1) = z_1^2 + \ldots + z_4^2 = C_1^2 = const. \tag{11}$$

One can prove a special existence theorem for the solution $f_k(\alpha)$, $k = 1,2,3$, $g_l(\beta_1)$, $l = 1,2$, $h(\beta_2)$ of system (10) for the presence of analytic integral (11) for system (7), (9) of equations of geodesic lines. Below, however, we do not need all conditions (10) in studying dynamic systems with dissipation. Nevertheless, in what follows, we suppose that the condition

$$f_1(\alpha) = f_2(\alpha) = f_3(\alpha) = f(\alpha), \tag{12}$$

is satisfied in Eqs. (7); the functions $g_l(\beta_1)$, $l = 1,2$, $h(\beta_2)$ must satisfy the transformed third equality from (10):

$$\begin{cases} 2\Gamma^2_{12}(\alpha,\beta) + \dfrac{d\ln|g_1(\beta_1)|}{d\beta_1} + \Gamma^1_{22}(\alpha,\beta)g_1^2(\beta_1) \equiv 0, \\ 2\Gamma^3_{13}(\alpha,\beta) + \dfrac{d\ln|g_2(\beta_1)|}{d\beta_1} + \Gamma^1_{33}(\alpha,\beta)g_2^2(\beta_1)h^2(\beta_2) \equiv 0, \\ 2\Gamma^3_{23}(\alpha,\beta) + \dfrac{d\ln|h(\beta_2)|}{d\beta_2} + \Gamma^2_{33}(\alpha,\beta)h^2(\beta_2) \equiv 0. \end{cases} \tag{13}$$

Thus, the functions $g_l(\beta_1)$, $l = 1,2$, $h(\beta_2)$ depend on the connection coefficients; as for restrictions on the function $f(\alpha)$ they are given below.

Proposition 3.2. *If properties (12) and (13) are valid, and the equalities*

$$\Gamma^1_{\alpha 1}(\alpha,\beta) = \Gamma^2_{\alpha 2}(\alpha,\beta) = \Gamma^3_{\alpha 3}(\alpha,\beta) = \Gamma_1(\alpha), \tag{14}$$

are satisfied, system (7), (9) has a smooth first integral of the following form:

$$\Phi_2(z_3,z_2,z_1;\alpha) = \sqrt{z_1^2+z_2^2+z_3^2}\;\Phi_0(\alpha) = C_2 = const, \tag{15}$$

$$\Phi_0(\alpha) = f(\alpha)\exp\left\{2\int_{\alpha_0}^{\alpha}\Gamma_1(b)db\right\}.$$

Proposition 3.3. *If the properties in proposition 3.2 are valid, and also*

$$g_1(\beta_1) = g_2(\beta_1) = g(\beta_1), \tag{16}$$

herewith the equalities

$$\Gamma^2_{12}(\alpha, \beta) = \Gamma^3_{13}(\alpha, \beta) = \Gamma_2(\beta_1), \tag{17}$$

are valid, that system (7), (9) has a smooth first integral of the following form:

$$\Phi_3(z_2, z_1; \alpha, \beta_1) = \sqrt{z_1^2 + z_2^2}\Phi_0(\alpha)\Psi_1(\beta_1) = C_3 = const, \tag{18}$$

$$\Psi_1(\beta_1) = g(\beta_1) \exp\left\{2\int_{\beta_{10}}^{\beta_1} \Gamma_2(b)db\right\}.$$

Proposition 3.4. *If the properties in propositions 3.2, 3.3 are valid, herewith the equality*

$$\Gamma^3_{23}(\alpha, \beta) = \Gamma_3(\beta_2), \tag{19}$$

are valid, that system (7), (9) has a smooth first integral of the following form:

$$\Phi_4(z_1; \alpha, \beta_1, \beta_2) = z_1\Phi_0(\alpha)\Psi_1(\beta_1)\Psi_2(\beta_2) = C_4 = const, \tag{20}$$

$$\Psi_2(\beta_2) = h(\beta_2) \exp\left\{2\int_{\beta_{20}}^{\beta_2} \Gamma_3(b)db\right\}.$$

Proposition 3.5. *If the properties in propositions 3.2, 3.3, 3.4 are valid, that system (7), (9) has a first integral of the following form:*

$$\Phi_5(z_2, z_1; \alpha, \beta) = \beta_3 \pm \int_{\beta_{20}}^{\beta_2} \frac{C_4 h(b)}{\sqrt{C_3^2\Phi_2^2(b) - C_4^2}}\, db = C_5 = const. \tag{21}$$

Under the conditions listed above, system (7), (9) has a complete set (five) of independent first integrals of the form (11), (15), (18), (20), and (21).

4 Equations of motion on the tangent bundle of a three-dimensional manifold in a potential field of force and its first integrals

Let us now somewhat modify system (7), (9) under conditions (12)–(14), (16), (17), and (19), which yields a conservative system. Namely, the presence of the force field is characterized by the coefficient $F(\alpha)$ in the second equation of system (22). The system under consideration on the tangent bundle $T_*M^4\{z_4, z_3, z_2, z_1; \alpha, \beta_1, \beta_2, \beta_3\}$ takes the form

$$\begin{cases} \dot\alpha = -z_4, \\ \dot z_4 = F(\alpha) + \Gamma^\alpha_{11} f_1^2(\alpha) z_3^2 + \Gamma^\alpha_{22} f_2^2(\alpha) g_1^2(\beta_1) z_2^2 + \Gamma^\alpha_{33} f_3^2(\alpha) g_2^2(\beta_1) h^2(\beta_2) z_1^2, \\ \dot z_3 = \left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + \dfrac{d\ln|f_1(\alpha)|}{d\alpha}\right] z_3 z_4 - \Gamma^1_{22}(\alpha,\beta)\dfrac{f_2^2(\alpha)}{f_1(\alpha)} g_1^2(\beta_1) z_2^2 - \\ - \Gamma^1_{33}(\alpha,\beta)\dfrac{f_3^2(\alpha)}{f_1(\alpha)} g_2^2(\beta_1) h^2(\beta_2) z_1^2, \\ \dot z_2 = \left[2\Gamma^2_{\alpha 2}(\alpha,\beta) + \dfrac{d\ln|f_2(\alpha)|}{d\alpha}\right] z_2 z_4 - \left[2\Gamma^2_{12}(\alpha,\beta) + \dfrac{d\ln|g_1(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_2 z_3 - \\ - \Gamma^2_{33}(\alpha,\beta)\dfrac{f_3^2(\alpha)}{f_2(\alpha)}\dfrac{g_2^2(\beta_1)}{g_1(\beta_1)} h^2(\beta_2) z_1^2, \\ \dot z_1 = \left[2\Gamma^3_{\alpha 3}(\alpha,\beta) + \dfrac{d\ln|f_3(\alpha)|}{d\alpha}\right] z_1 z_4 - \left[2\Gamma^3_{13}(\alpha,\beta) + \dfrac{d\ln|g_2(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_1 z_3 - \\ - \left[2\Gamma^3_{23}(\alpha,\beta) + \dfrac{d\ln|h(\beta_2)|}{d\beta_2}\right] f_2(\alpha) g_1(\beta_1) z_1 z_2, \\ \dot\beta_1 = z_3 f(\alpha), \\ \dot\beta_2 = z_2 f(\alpha) g(\beta_1), \\ \dot\beta_3 = z_1 f(\alpha) g(\beta_1) h(\beta_2), \end{cases} \tag{22}$$

and it is almost everywhere equivalent to the following system:

$$\begin{cases} \ddot\alpha + F(\alpha) + \Gamma^\alpha_{11}(\alpha,\beta)\dot\beta_1^2 + \Gamma^\alpha_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^\alpha_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_1 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_1 + \Gamma^1_{22}(\alpha,\beta)\dot\beta_2^2 + \Gamma^1_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_2 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_2 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_2 + \Gamma^2_{33}(\alpha,\beta)\dot\beta_3^2 = 0, \\ \ddot\beta_3 + 2\Gamma_1(\alpha)\dot\alpha\dot\beta_3 + 2\Gamma_2(\beta_1)\dot\beta_1\dot\beta_3 + 2\Gamma_3(\beta_2)\dot\beta_2\dot\beta_3 = 0. \end{cases}$$

Proposition 4.1. *If the conditions of Proposition 3.1 are satisfied, system (22) has a smooth first integral of the following form:*

$$\Phi_1(z_4,\ldots,z_1;\alpha) = z_1^2 + \ldots + z_4^2 + F_1(\alpha) = C_1 = const, \; F_1(\alpha) = 2\int_{\alpha_0}^{\alpha} F(a)da. \tag{23}$$

Proposition 4.2. *If the conditions of Propositions 3.2, 3.3, and 3.4 are satisfied, system (22) has three smooth first integrals of form (15), (18), and (20).*

Proposition 4.3. *If the conditions of Proposition 3.5 are satisfied, system (22) has first integral of form (21).*

Under the conditions listed above, system (22) has a complete set of (five) independent first integrals of form (23), (15), (18), (20), and (21).

5 Equations of motion on the tangent bundle of a two-dimensional manifold in a force field with dissipation and its first integrals

Let us now consider system (24). In doing this, we obtain a system with dissipation. Namely, the presence of dissipation (generally speaking, sign-alternating) is characterized by the coefficient $b\delta(\alpha)$ in the first equation of system (24):

$$\begin{cases} \dot{\alpha} = -z_4 + b\delta(\alpha), \\ \dot{z}_4 = F(\alpha) + \Gamma^{\alpha}_{11} f_1^2(\alpha) z_3^2 + \Gamma^{\alpha}_{22} f_2^2(\alpha) g_1^2(\beta_1) z_2^2 + \Gamma^{\alpha}_{33} f_3^2(\alpha) g_2^2(\beta_1) h^2(\beta_2) z_1^2, \\ \dot{z}_3 = \left[2\Gamma^1_{\alpha 1}(\alpha,\beta) + \dfrac{d\ln|f_1(\alpha)|}{d\alpha}\right] z_3 z_4 - \Gamma^1_{22}(\alpha,\beta)\dfrac{f_2^2(\alpha)}{f_1(\alpha)} g_1^2(\beta_1) z_2^2 - \\ - \Gamma^1_{33}(\alpha,\beta)\dfrac{f_3^2(\alpha)}{f_1(\alpha)} g_2^2(\beta_1) h^2(\beta_2) z_1^2, \\ \dot{z}_2 = \left[2\Gamma^2_{\alpha 2}(\alpha,\beta) + \dfrac{d\ln|f_2(\alpha)|}{d\alpha}\right] z_2 z_4 - \left[2\Gamma^2_{12}(\alpha,\beta) + \dfrac{d\ln|g_1(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_2 z_3 - \\ - \Gamma^2_{33}(\alpha,\beta)\dfrac{f_3^2(\alpha)}{f_2(\alpha)}\dfrac{g_2^2(\beta_1)}{g_1(\beta_1)} h^2(\beta_2) z_1^2, \\ \dot{z}_1 = \left[2\Gamma^3_{\alpha 3}(\alpha,\beta) + \dfrac{d\ln|f_3(\alpha)|}{d\alpha}\right] z_1 z_4 - \left[2\Gamma^3_{13}(\alpha,\beta) + \dfrac{d\ln|g_2(\beta_1)|}{d\beta_1}\right] f_1(\alpha) z_1 z_3 - \\ - \left[2\Gamma^3_{23}(\alpha,\beta) + \dfrac{d\ln|h(\beta_2)|}{d\beta_2}\right] f_2(\alpha) g_1(\beta_1) z_1 z_2, \\ \dot{\beta}_1 = z_3 f(\alpha), \\ \dot{\beta}_2 = z_2 f(\alpha) g(\beta_1), \\ \dot{\beta}_3 = z_1 f(\alpha) g(\beta_1) h(\beta_2), \end{cases} \tag{24}$$

which is almost everywhere equivalent to the following system

$$\begin{cases} \ddot{\alpha} - b\dot{\alpha}\delta'(\alpha) + F(\alpha) + \Gamma^{\alpha}_{11}(\alpha,\beta)\dot{\beta}_1^2 + \Gamma^{\alpha}_{22}(\alpha,\beta)\dot{\beta}_2^2 + \Gamma^{\alpha}_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0, \\ \ddot{\beta}_1 - b\dot{\beta}_1\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot{\alpha}\dot{\beta}_1 + \Gamma^1_{22}(\alpha,\beta)\dot{\beta}_2^2 + \Gamma^1_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0, \\ \ddot{\beta}_2 - b\dot{\beta}_2\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot{\alpha}\dot{\beta}_2 + 2\Gamma_2(\beta_1)\dot{\beta}_1\dot{\beta}_2 + \Gamma^2_{33}(\alpha,\beta)\dot{\beta}_3^2 = 0, \\ \ddot{\beta}_3 - b\dot{\beta}_3\delta(\alpha)W(\alpha) + 2\Gamma_1(\alpha)\dot{\alpha}\dot{\beta}_3 + 2\Gamma_2(\beta_1)\dot{\beta}_1\dot{\beta}_3 + 2\Gamma_3(\beta_2)\dot{\beta}_2\dot{\beta}_3 = 0, \\ W(\alpha) = 2\Gamma^1_{\alpha 1}(\alpha,\beta) + \dfrac{d\ln|f_1(\alpha)|}{d\alpha}. \end{cases}$$

Now we pass to integration of the sought six-order system (24) under condition (13), as well as under the equalities

$$\Gamma^{\alpha}_{11}(\alpha,\beta) = \Gamma^{\alpha}_{22}(\alpha,\beta) g^2(\beta_1) = \Gamma^{\alpha}_{33}(\alpha,\beta) g^2(\beta_1) h^2(\beta_2) = \Gamma_4(\alpha), \tag{25}$$

hold.

We also introduce (by analogy with (13)) a restriction on the function $f(\alpha)$. It must satisfy the transformed first equality from (10):

$$2\Gamma_1(\alpha) + \frac{d\ln|f(\alpha)|}{d\alpha} + \Gamma_4(\alpha)f^2(\alpha) \equiv 0. \tag{26}$$

To integrate it completely, one should know, generally speaking, seven independent first integrals. However, after the following change of variables,

$$w_4 = z_4,\ w_3 = \sqrt{z_1^2+z_2^2+z_3^2},\ w_2 = \frac{z_2}{z_1},\ w_1 = \frac{z_3}{\sqrt{z_1^2+z_2^2}},$$

system (24) decomposes as follows:

$$\begin{cases} \dot\alpha = -w_4 + b\delta(\alpha), \\ \dot w_4 = F(\alpha) + \Gamma_4(\alpha)f^2(\alpha)w_3^2, \\ \dot w_3 = \left[2\Gamma_1(\alpha) + \dfrac{d\ln|f(\alpha)|}{d\alpha}\right] w_3 w_4, \end{cases} \tag{27}$$

$$\begin{cases} \dot w_2 = \pm w_3\sqrt{1+w_2^2}f(\alpha)g(\beta_1)\left[2\Gamma_3(\beta_2) + \dfrac{d\ln|h(\beta_2)|}{d\beta_2}\right], \\ \dot\beta_2 = \pm\dfrac{w_2w_3}{\sqrt{1+w_2^2}}f(\alpha)g(\beta_1), \end{cases} \tag{28}$$

$$\begin{cases} \dot w_1 = \pm w_3\sqrt{1+w_1^2}f(\alpha)\left[2\Gamma_2(\beta_1) + \dfrac{d\ln|g(\beta_1)|}{d\beta_1}\right], \\ \dot\beta_1 = \pm\dfrac{w_1w_3}{\sqrt{1+w_1^2}}f(\alpha), \end{cases} \tag{29}$$

$$\dot\beta_3 = \pm\frac{w_3}{\sqrt{1+w_2^2}}f(\alpha)g(\beta_1)h(\beta_2). \tag{30}$$

It is seen that to integrate system (27)–(30) completely, it is sufficient to determine two independent first integrals of system (27), by one integral of systems (28) and (29), and an additional first integral "attaching" Eq. (30) (i.e., five integrals in total).

Theorem 5.1. *Let the equalities*

$$\Gamma_4(\alpha)f^2(\alpha) = \kappa\frac{d}{d\alpha}\ln|\delta(\alpha)|,\ F(\alpha) = \lambda\frac{d}{d\alpha}\frac{\delta^2(\alpha)}{2} \tag{31}$$

be valid for some $\kappa, \lambda \in \mathbf{R}$. Then system (24) under equalities (12), (13), (16), (25), and (26) has a complete set of (five) independent, generally speaking, transcendental first integrals.

6 Conclusions

By analogy with low-dimensional cases, we pay special attention to two important cases for the function $f(\alpha)$ defining the metric on a sphere:

$$f(\alpha) = \frac{\cos\alpha}{\sin\alpha}, \tag{32}$$

$$f(\alpha) = \frac{1}{\cos\alpha \sin\alpha}. \tag{33}$$

Case (32) forms a class of systems corresponding to the motion of a dynamically symmetric five-dimensional solid body at zero levels of cyclic integrals, generally speaking, in a nonconservative field of forces [3, 4, 5]. Case (33) forms a class of systems corresponding to the motion of a material point on a four-dimensional sphere also, generally speaking, in a nonconservative field of forces. In particular, at $\delta(\alpha) \equiv F(\alpha) \equiv 0$ the system under consideration describes a geodesic flow on a four-dimensional sphere. In case (32), if

$$\delta(\alpha) = \frac{F(\alpha)}{\cos\alpha},$$

the system describes the spatial motion of a five-dimensional solid body in the force field $F(\alpha)$ under the action of a tracking force [6, 7, 8]. In particular, if

$$F(\alpha) = \sin\alpha\cos\alpha,\ \delta(\alpha) = \sin\alpha,$$

the system also describes a generalized five-dimensional spherical pendulum in a nonconservative force field and has a complete set of transcendental first integrals that can be expressed in terms of a finite combination of elementary functions [9, 10, 11].

If the function $\delta(\alpha)$ is not periodic, the dissipative system under consideration is a system with variable dissipation with a zero mean (i.e., it is properly dissipative). Nevertheless, an explicit form of transcendental first integrals that can be expressed in terms of a finite combination of elementary functions can be obtained even in this case. This is a new nontrivial case of integrability of dissipative systems in an explicit form [12].

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References

[1] M. V. Shamolin, "Comparison of complete integrability cases in Dynamics of a two-, three-, and four-dimensional rigid body in a nonconservative field", Journal of Mathematical Sciences, 187:3 (2012), 346–359.

[2] M. V. Shamolin, "Some questions of qualitative theory in dynamics of systems with the variable dissipation", Journal of Mathematical Sciences, 189:2 (2013), 314–323.

[3] M. V. Shamolin, "Variety of Integrable Cases in Dynamics of Low- and Multi-Dimensional Rigid Bodies in Nonconservative Force Fields", Journal of Mathematical Sciences, 204:4 (2015), 379–530.

[4] M. V. Shamolin, "Classification of Integrable Cases in the Dynamics of a Four-Dimensional Rigid Body in a Nonconservative Field in the Presence of a Tracking Force", Journal of Mathematical Sciences, 204:6 (2015), 808–870.

[5] M. V. Shamolin, "Some Classes of Integrable Problems in Spatial Dynamics of a Rigid Body in a Nonconservative Force Field", Journal of Mathematical Sciences, 210:3 (2015), 292–330.

[6] M. V. Shamolin, "Integrable Cases in the Dynamics of a Multi-dimensional Rigid Body in a Nonconservative Force Field in the Presence of a Tracking Force", Journal of Mathematical Sciences, 214:6 (2016), 865–891.

[7] M. V. Shamolin, "Integrable Systems with Variable Dissipation on the Tangent Bundle of a Sphere", Journal of Mathematical Sciences, 219:2 (2016), 321–335.

[8] M. V. Shamolin, "New Cases of Integrability of Equations of Motion of a Rigid Body in the n-Dimensional Space", Journal of Mathematical Sciences, 221:2 (2017), 205–259.

[9] M. V. Shamolin, "Some Problems of Qualitative Analysis in the Modeling of the Motion of Rigid Bodies in Resistive Media", Journal of Mathematical Sciences, 221:2 (2017), 260–296.

[10] V. V. Trofimov, "Euler equations on Borel subalgebras of semisimple Lie algebras," *Izv. Akad. Nauk SSSR, Ser. Mat.,* **43**, No. 3 (1979), 714–732.

[11] V. V. Trofimov, "Finite-dimensional representations of Lie algebras and completely integrable systems,"*Mat. Sb.,* **111**, No. 4 (1980), 610–621.

[12] V. V. Trofimov and M. V. Shamolin, "Geometric and dynamical invariants of integrable Hamiltonian and dissipative systems", Journal of Mathematical Sciences, 180:4 (2012), 365–530.

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