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TRANSCENDENTAL FIRST INTEGRALS OF SOME CLASSES OF DYNAMICAL SYSTEMS

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Abstract. We examine the existence of transcendental first integrals for some classes of systems with symmetries. We obtain sufficient conditions of existence of first integrals of second-order non-autonomous homogeneous systems that are transcendental functions (in the sense of the theory of elementary functions and in the sense of complex analysis) expressed as finite combinations of elementary functions.

1. Preliminary Results

We introduce a class of autonomous dynamical systems with one periodic phase coordinate possessing certain symmetries that are typical for pendulum-type systems. We show that this class of systems can be naturally embedded in the class of systems with variable dissipation with zero mean, i.e., on the average for the period with respect to the periodic coordinate, the dissipation in the system is equal to zero, although in various domains of the phase space, either energy pumping or dissipation can occur, but they balance to each other in a certain sense. We present some examples of pendulum-type systems on lower-dimension manifolds from dynamics of a rigid body in a nonconservative field.

Then we study certain general conditions of the integrability in elementary functions for systems on the two-dimensional plane and the tangent bundles of a one-dimensional sphere (i.e., the two-dimensional cylinder) and a two-dimensional sphere (a four-dimensional manifold). Therefore, we propose an interesting example of a three-dimensional phase portrait of a pendulum-like system which describes the motion of a spherical pendulum in a flowing medium (see also [11, 12, 13, 14, 15]).

For multi-parametric third-order systems, we present sufficient conditions of the existence of first integrals that are expressed through finite combinations of elementary functions.

We deal with three properties that seem, at first glance, to be independent:

- a class of systems with symmetries specified above;
- the fact that this class consists of systems with variable dissipation with zero mean (with respect to the existing periodic variable), which allows us to consider them as almost conservative systems;

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- in certain (although lower-dimensional) cases, these systems have a complete set of first integrals, which, in general, are transcendental (in the sense of complex analysis).

As is well known, the concept of integrability, generally speaking, is quite vague. It is necessary to consider the sense in which it is meant (i.e., a certain criterion that allows one to conclude that trajectories of a dynamical system have an especially “attractive and simple structure”), and in which class of functions first integrals are taken, and so on (see also [1]).

In this activity, we accept an approach in which the class of first integrals consists of elementary transcendental functions. Here the transcendence is meant not only in the sense of the elementary functions (e.g., trigonometric) but in the sense of complex analysis, i.e., as functions of a complex variable possessing essential singular points. In this case these functions must be formally continued in the complex domain (see also [2, 3]).

2. Systems with Symmetries and Variable Dissipation with Zero Mean

We consider systems of the following form (the dot denotes the derivative with respect to time):

$$\begin{aligned} \dot{\alpha} &= f_\alpha(\omega, \sin\alpha, \cos\alpha), \\ \dot{\omega_k} &= f_k(\omega, \sin\alpha, \cos\alpha), \ \ k = 1, \ldots, n, \end{aligned} \tag{2.1}$$

defined on the set

$$\mathbf{S}^1\{\alpha \mod 2\pi\} \backslash K \times \mathbf{R}^n\{\omega\}, \tag{2.2}$$

$\omega = (\omega_1, \ldots, \omega_n)$, where sufficiently smooth functions $f_\lambda(u_1, u_2, u_3)$, $\lambda = \alpha, 1, \ldots, n$, of three variables u_1, u_2, u_3 are as follows:

$$\begin{aligned} f_\lambda(-u_1, -u_2, u_3) &= -f_\lambda(u_1, u_2, u_3), \\ f_\alpha(u_1, u_2, -u_3) &= f_\alpha(u_1, u_2, u_3), \\ f_k(u_1, u_2, -u_3) &= -f_k(u_1, u_2, u_3), \end{aligned} \tag{2.3}$$

while the functions $f_k(u_1, u_2, u_3)$ are defined at $u_3 = 0$ for all $k = 1, \ldots, n$.

The set K is either empty or consists of a finite number of points of the circle $\mathbf{S}^1\{\alpha \mod 2\pi\}$.

The last two variables u_2, and u_3 of the functions $f_\lambda(u_1, u_2, u_3)$ depend on a single parameter α, but they are distinguished in distinct groups for the following reasons. First, they are uniquely expressed through each other not in the whole domain, and, second, the first of them is odd whereas the second is an even function of α, this has a different effect on the symmetry of the system (2.1).

We associate the system (2.1) with the following non-autonomous system:

$$\frac{d\omega_k}{d\alpha} = \frac{f_k(\omega, \sin\alpha, \cos\alpha)}{f_\alpha(\omega, \sin\alpha, \cos\alpha)}, \ \ k = 1, \ldots, n. \tag{2.4}$$

Substituting $\tau = \sin\alpha$ we reduce it to the form

$$\frac{d\omega_k}{d\tau} = \frac{f_k(\omega, \tau, \varphi_k(\tau))}{f_\alpha(\omega, \tau, \varphi_\alpha(\tau))}, \ \ k = 1, \ldots, n, \tag{2.5}$$

$$\varphi_\lambda(-\tau) = \varphi_\lambda(\tau), \ \lambda = \alpha, 1, \dots, n.$$

The last system may have, in particular, an algebraic right-hand side (i.e., the ratio of two polynomials), which sometimes allows one to calculate its first integrals in the explicit form.

Definition 2.1. Consider a smooth autonomous system of the $(n+1)$th order of the normal form defined on the cylinder $\mathbf{R}^n\{x\} \times \mathbf{S}^1\{\alpha \mod T\}$, where α is a periodic coordinate of period $T > 0$ (for simplicity, $T = 2\pi$). Let $\mathbf{V}(x, \alpha)$ is the right-hand side of the system considered (its vector field). We denote the divergence of it (which, generally speaking, is a function of all phase variables and is not identically equal to zero), by $\mathrm{div}\mathbf{V}(x, \alpha)$. Such systems are called *systems with variable dissipation with zero* (respectively, nonzero) *mean* if the function

$$\int_0^T \mathrm{div}\mathbf{V}(x, \alpha) d\alpha \tag{2.6}$$

identically vanishes (respectively, does not vanish). Thus, in some cases (for example, when singularities appear at certain points of the circle $\mathbf{S}^1\{\alpha \mod 2\pi\}$ this integral is understood in the sense of the principal value.

Note that it is quite difficult to define the general notion of a system with variable dissipation with zero (nonzero) mean. The above definition is based on the notion of the divergence (as we know, the divergence of the right-hand side of the normal-form system characterizes the change of the phase volume in the phase space of this system).

The following statement puts this class of systems (2.1) in the class of dynamical systems with a variable dissipation with zero mean. The converse inclusion, generally speaking, does not hold.

Theorem 2.1. *Systems of the form (2.1) are dynamical systems with variable dissipation with zero mean.*

This theorem is proved by using only some of mentioned symmetries (2.3) of the system (2.1), and is based on the periodicity of the right-hand side of the system with respect to α.

Indeed, we calculate the divergence indicated of the vector field of system (2.1):

$$\frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial u_2} \cos\alpha - \frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial u_3} \sin\alpha + \sum_{k=1}^{n} \frac{\partial f_k(\omega, \sin\alpha, \cos\alpha)}{\partial u_1}. \tag{2.7}$$

The next integral of the first two summands in (2.7) vanishes:

$$\int_0^{2\pi} \left\{ \frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial u_2} d\sin\alpha + \frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial u_3} d\cos\alpha \right\} =$$

$$= \int_0^{2\pi} \frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial \alpha} d\alpha = h_\alpha(\omega) \equiv 0, \tag{2.8}$$

since the function $f_\alpha(\omega, \sin\alpha, \cos\alpha)$ is periodic with respect to α.

Further, due to the third equation (2.3) for any $k = 1, \dots, n$, the condition

$$\frac{\partial f_k(\omega, \sin\alpha, \cos\alpha)}{\partial u_1} = \cos\alpha \cdot \frac{\partial g_k(\omega, \sin\alpha)}{\partial u_1}, \tag{2.9}$$

is fulfilled since the certain function $g_k(u_1, u_2)$ is sufficiently smooth for any $k = 1, \ldots, n$.

Then the integral over the period 2π of the right-hand side of Eq. (2.9) yields

$$\int_0^{2\pi} \frac{\partial g_k(\omega, \sin\alpha)}{\partial u_1} d\sin\alpha = h_k(\omega) \equiv 0 \tag{2.10}$$

for any $k = 1, \ldots, n$. Equations (2.8) and (2.10) imply Theorem 2.1.

In this paper, we consider the case where the functions $f_\lambda(\omega, \tau, \varphi_k(\tau))$ ($\lambda = \alpha, 1, \ldots, n$) are polynomials in ω and τ.

Example 2.1. The paper [4] presents pendulum systems on the two-dimensional cylinder $\mathbf{S}^1\{\alpha \mod 2\pi\} \times \mathbf{R}^1\{\omega\}$ with the parameter $b > 0$:

$$\dot\alpha = -\omega + b\sin\alpha, \ \dot\omega = \sin\alpha\cos\alpha, \tag{2.11}$$

and

$$\begin{aligned} \dot\alpha &= -\omega + b\sin\alpha\cos^2\alpha + b\omega^2\sin\alpha, \\ \dot\omega &= \sin\alpha\cos\alpha - b\omega\sin^2\alpha\cos\alpha + b\omega^3\cos\alpha. \end{aligned} \tag{2.12}$$

In the variables (ω, τ) we can assign to these systems equations with the algebraic right-hand sides

$$\frac{d\omega}{d\tau} = \frac{\tau}{-\omega + b\tau}, \tag{2.13}$$

and

$$\frac{d\omega}{d\tau} = \frac{\tau + b\omega[\omega^2 - \tau^2]}{-\omega + b\tau + b\tau[\omega^2 - \tau^2]} \tag{2.14}$$

of the form (2.5), respectively. At the same time, these systems are dynamical systems with variable dissipation with zero mean; this can be easily verified by a direct calculation.

Indeed, the divergences of the right-hand sides are equal to

$$b\cos\alpha, \ b\cos\alpha[4\omega^2 + \cos^2\alpha - 3\sin^2\alpha],$$

respectively; they are contained in the class of systems (2.1).

Moreover, each of them has a first integral that is a transcendental function (in the sense of complex analysis), expressed as a finite combination of elementary functions.

Example 2.2. In the three-dimensional field

$$\{0 < \alpha < \pi\} \times \mathbf{R}^2\{z_1, z_2\} \tag{2.15}$$

we consider the following system with the parameter b (such a system is derived from the system on the tangent bundle of $T_*\mathbf{S}^2\{z_2, z_1; \alpha, \beta\}$ of two-dimensional sphere $\mathbf{S}^2\{\alpha, \beta\}$):

$$\begin{aligned} \dot\alpha &= -z_2 + b\sin\alpha, \\ \dot z_2 &= \sin\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha}, \\ \dot z_1 &= z_1 z_2\frac{\cos\alpha}{\sin\alpha}, \end{aligned} \tag{2.16}$$

(see also [5, 6]). To this system, we assign the following nonautonomous system with the algebraic right-hand side ($\tau = \sin\alpha$):

$$\frac{dz_2}{d\tau} = \frac{\tau - z_1^2/\tau}{-z_2 + b\tau}, \quad \frac{dz_1}{d\tau} = \frac{z_1 z_2/\tau}{-z_2 + b\tau}. \tag{2.17}$$

Let us consider the following system with the parameters b and H_1 (such a system is also separated from the system on the tangent bundle $T_*\mathbf{S}^2\{z_2, z_1; \alpha, \beta\}$ of two-dimensional sphere $\mathbf{S}^2\{\alpha, \beta\}$) in the three-dimensional region (2.15):

$$\begin{aligned} \dot{\alpha} &= -(1 + bH_1)z_2 + b\sin\alpha, \\ \dot{z}_2 &= \sin\alpha\cos\alpha - (1 + bH_1)z_1^2\frac{\cos\alpha}{\sin\alpha} - H_1 z_2\cos\alpha, \\ \dot{z}_1 &= (1 + bH_1)z_1 z_2\frac{\cos\alpha}{\sin\alpha} - H_1 z_1\cos\alpha. \end{aligned} \tag{2.18}$$

To this system, we assign the following nonautonomous system with the algebraic right-hand side ($\tau = \sin\alpha$):

$$\frac{dz_2}{d\tau} = \frac{\tau - (1 + bH_1)z_1^2/\tau - H_1 z_2}{-(1 + bH_1)z_2 + b\tau}, \quad \frac{dz_1}{d\tau} = \frac{(1 + bH_1)z_1 z_2/\tau - H_1 z_1}{-(1 + bH_1)z_2 + b\tau}. \tag{2.19}$$

In these cases, we see that the system (2.16) (or (2.18)) is a system with variable dissipation with zero mean; to fully comply with the definition, it suffices to introduce the new phase variable $z_1^* = \ln|z_1|$.

If we calculate the divergence of the right-hand side of (2.16) in the Cartesian coordinates α, z_1^*, z_2, we find that it is equal to $b\cos\alpha$. At the same time, taking into account (2.15), in the sense of the principal value, we have

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi-\varepsilon} b\cos\alpha + \lim_{\varepsilon\to 0}\int_{\pi+\varepsilon}^{2\pi-\varepsilon} b\cos\alpha = 0.$$

Moreover, it has two first integrals (i.e., a complete list) that are transcendental functions expressed as finite combinations of elementary functions; this becomes possible after assigning the system (generally speaking, non-autonomous) of equations with algebraic (polynomial) right-hand side (2.17) to it.

The above systems (2.11), (2.12), (2.16), are contained in the class of systems (2.1) and have the variable dissipation with zero mean. Moreover, they possess a complete list of transcendental first integrals expressed as finite combinations of elementary functions (see [7]).

So, to find the first integral of the systems considered, it is convenient to transform systems of the form (2.1) to systems with polynomial right-hand sides (2.5). The integrability of the original system in elementary functions depends on the form of these right-hand sides. So we proceed as follows: we search sufficient conditions for the integrability of systems with polynomial right-hand sides in elementary functions while exploring the systems of the most general form.

3. Nonautonomous Homogeneous Second-Order Systems

Now we discuss the possibilities of integration in elementary functions of the following system of a more general form, including the above systems (2.17) and

(2.19) in three-dimensional phase domains and having a singularity of the type $1/x$:

$$\begin{aligned} \frac{dz}{dx} &= \frac{ax + by + cz + c_1 z^2/x + c_2 zy/x + c_3 y^2/x}{d_1 x + ey + fz}, \\ \frac{dy}{dx} &= \frac{gx + hy + iz + i_1 z^2/x + i_2 zy/x + i_3 y^2/x}{d_1 x + ey + fz}. \end{aligned} \tag{3.1}$$

In other words, we study the existence of first integrals for the class of non autonomous homogeneous second-order systems. A number of results on the subject have already been obtained (see also [8, 9]).

Introducing the substitutions

$$y = ux, \ \ z = vx, \tag{3.2}$$

we see that the system (3.1) can be reduced to the following system:

$$x\frac{dv}{dx} + v = \frac{ax + bux + cvx + c_1 v^2 x + c_2 vux + c_3 u^2 x}{d_1 x + eux + fvx}, \tag{3.3}$$

$$x\frac{du}{dx} + u = \frac{gx + hux + ivx + i_1 v^2 x + i_2 vux + i_3 u^2 x}{d_1 x + eux + fvx}, \tag{3.4}$$

which is equivalent to

$$x\frac{dv}{dx} = \frac{ax + bux + (c - d_1)vx + (c_1 - f)v^2 x + (c_2 - e)vux + c_3 u^2 x}{d_1 x + eux + fvx}, \tag{3.5}$$

$$x\frac{du}{dx} = \frac{gx + (h - d_1)ux + ivx + i_1 v^2 x + (i_2 - f)vux + (i_3 - e)u^2 x}{d_1 x + eux + fvx}. \tag{3.6}$$

To this system, we put in correspondence the following nonautonomous equation with an algebraic right-hand side:

$$\frac{dv}{du} = \frac{a + bu + cv + c_1 v^2 + c_2 vu + c_3 u^2 - v[d_1 + eu + fv]}{g + hu + iv + i_1 v^2 + i_2 vu + i_3 u^2 - u[d_1 + eu + fv]}. \tag{3.7}$$

Integration of the last equation is reduced to integration of the following complete differential equation:

$$\begin{aligned} &[g + hu + iv + i_1 v^2 + i_2 vu + i_3 u^2 - d_1 u - eu^2 - fuv]dv = \\ &= [a + bu + cv + c_1 v^2 + c_2 vu + c_3 u^2 - d_1 v - euv - fv^2]du, \end{aligned} \tag{3.8}$$

or

$$\begin{aligned} &[g + (h - d_1)u + iv + i_1 v^2 + (i_2 - f)uv + (i_3 - e)u^2]dv = \\ &= [a + bu + (c - d_1)v + (c_1 - f)v^2 + (c_2 - e)uv + c_3 u^2]du. \end{aligned} \tag{3.9}$$

Generally speaking, we obtain a 15-parametric family of equations (or (3.9)).

4. Particular Cases of Existence of Rational First Integrals

In the considered cases, the nonautonomous second-order system studied has a complete set of (two) first integrals expressed as finite combinations of elementary functions. Both these two first integrals are, generally speaking, transcendental functions of their arguments in terms of complex analysis. Moreover, one of them is a rational homogeneous function, i.e., the ratio of two polynomials of the same degree:

$$\frac{P_m(x,y,z)}{Q_m(x,y,z)}, \tag{4.1}$$

where $P_m(x,y,z)$ and $Q_m(x,y,z)$ are homogeneous polynomials of degree m.

4.1. Case $m = 2$. I.

4.1.1. *Integration of Eq. (3.9).* We integrate Eq. (3.9) by using an integrating factor (final Jacobi multiplier) of the following form:

$$\varrho(u) = \frac{1}{u^s},\ s = 2. \tag{4.2}$$

Then Eq. (3.9) takes the form

$$\begin{gathered}\left[\frac{g}{u^2} + \frac{h-d_1}{u} + \frac{iv}{u^2} + \frac{i_1 v^2}{u^2} + \frac{(i_2-f)v}{u} + (i_3-e)\right] dv = \\ = \left[\frac{a}{u^2} + \frac{b}{u} + \frac{(c-d_1)v}{u^2} + \frac{(c_1-f)v^2}{u^2} + \frac{(c_2-e)v}{u} + c_3\right] du.\end{gathered} \tag{4.3}$$

A sufficient condition of integrability of the last identity in elementary functions is the set of the following six independent relations:

$$g = 0,\ i = 0,\ i_1 = 0,\ e = c_2,\ h = c,\ i_2 = 2c_1 - f. \tag{4.4}$$

Introduce nine independent parameters $\beta_1, \dots, \beta_9$:

$$\begin{gathered}\beta_1 = a,\ \beta_2 = b,\ \beta_3 = c,\ \beta_4 = c_1,\ \beta_5 = c_2, \\ \beta_6 = c_3,\ \beta_7 = d_1,\ \beta_8 = f,\ \beta_9 = i_3.\end{gathered} \tag{4.5}$$

Thus, Eq. (3.9) under the conditions (4.4) and (4.5) is reduced to the form

$$\frac{dv}{du} = \frac{\beta_1 + \beta_2 u + (\beta_3-\beta_7)v + (\beta_4-\beta_8)v^2 + \beta_6 u^2}{(\beta_3-\beta_7)u + 2(\beta_4-\beta_8)vu + (\beta_9-\beta_5)u^2}, \tag{4.6}$$

and the system (3.5), (3.6), respectively, to the form

$$x\frac{dv}{dx} = \frac{\beta_1 + \beta_2 u + (\beta_3-\beta_7)v + (\beta_4-\beta_8)v^2 + \beta_6 u^2}{\beta_7 + \beta_5 u + \beta_8 v}, \tag{4.7}$$

$$x\frac{du}{dx} = \frac{(\beta_3-\beta_7)u + 2(\beta_4-\beta_8)vu + (\beta_9-\beta_5)u^2}{\beta_7 + \beta_5 u + \beta_8 v}. \tag{4.8}$$

Then Eq. (4.6) is integrated as a finite combination of elementary functions.

Indeed, integrating the identity (4.3), we obtain the following relation:

$$\begin{gathered}d\left[\frac{(\beta_3-\beta_7)v}{u}\right] + d\left[\frac{(\beta_4-\beta_8)v^2}{u}\right] + d[(\beta_9-\beta_5)v] + d\left[\frac{\beta_1}{u}\right] - \\ - d[\beta_2 \ln|u|] - d[\beta_6 u] = 0,\end{gathered} \tag{4.9}$$

which allows us to obtain the following invariant relation:

$$\begin{gathered}\frac{(\beta_3-\beta_7)v}{u} + \frac{(\beta_4-\beta_8)v^2}{u} + (\beta_9-\beta_5)v + \frac{\beta_1}{u} - \\ - \beta_2 \ln|u| - \beta_6 u = C_1 = \text{const},\end{gathered} \tag{4.10}$$

and then in the coordinates (x, y, z) the first integral of the system (3.1) is

$$\frac{(\beta_4-\beta_8)z^2-\beta_6y^2+(\beta_3-\beta_7)zx+(\beta_9-\beta_5)zy+\beta_1x^2}{yx}- \\ -\beta_2\ln\left|\frac{y}{x}\right|=C_1=\text{const}. \tag{4.11}$$

Thus, we conclude on the integrability in elementary functions of the following (generally speaking, nonconservative) third-order system, which depends on nine parameters:

$$\begin{aligned}\frac{dz}{dx}&=\frac{\beta_1x+\beta_2y+\beta_3z+\beta_4z^2/x+\beta_5zy/x+\beta_6y^2/x}{\beta_7x+\beta_5y+\beta_8z},\\ \frac{dy}{dx}&=\frac{\beta_3y+(2\beta_4-\beta_8)zy/x+\beta_9y^2/x}{\beta_7x+\beta_5y+\beta_8z}.\end{aligned} \tag{4.12}$$

Corollary 4.1. *The third-order system*

$$\begin{aligned}\dot\alpha&=\beta_7\sin\alpha+\beta_5z_1+\beta_8z_2,\\ \dot z_2&=\beta_1\sin\alpha\cos\alpha+\beta_2z_1\cos\alpha+\beta_3z_2\cos\alpha+\\ &+\beta_4z_2^2\frac{\cos\alpha}{\sin\alpha}+\beta_5z_1z_2\frac{\cos\alpha}{\sin\alpha}+\beta_6z_1^2\frac{\cos\alpha}{\sin\alpha},\\ \dot z_1&=\beta_3z_1\cos\alpha+(2\beta_4-\beta_8)z_1z_2\frac{\cos\alpha}{\sin\alpha}+\beta_9z_1^2\frac{\cos\alpha}{\sin\alpha},\end{aligned} \tag{4.13}$$

depending on nine parameters $\beta_1,\dots,\beta_9$, considered on the set

$$\{\alpha\in\mathbf{R}^1:\ 0<\alpha<\pi\}\times\mathbf{R}^2\{z_1,z_2\}, \tag{4.14}$$

has, generally speaking, a transcendental first integral expressed through elementary functions:

$$\frac{(\beta_4-\beta_8)z_2^2-\beta_6z_1^2+(\beta_3-\beta_7)z_2\sin\alpha+(\beta_9-\beta_5)z_2z_1+\beta_1\sin^2\alpha}{z_1\sin\alpha}- \\ -\beta_2\ln\left|\frac{z_1}{\sin\alpha}\right|=C_1=const. \tag{4.15}$$

In particular, the system (4.13):

- with $\beta_1=1,\ \beta_2=\beta_3=\beta_4=\beta_5=\beta_9=0,\ \beta_6=\beta_8=-1$, and $\beta_7=b$ has the form of the system (2.16);
- with $\beta_1=1,\ \beta_2=\beta_4=\beta_5=\beta_9=0,\ \beta_3=-H_1,\ \beta_6=\beta_8=-(1+bH_1)$, and $\beta_7=b$ has the form of the system (2.18).

4.1.2. *Search for an additional invariant relation.* The first integral (4.11) expressed as a finite combination of elementary functions can be used to find an additional first integral of the non-autonomous system (4.12).

We transform the first equation (4.10) as follows:

$$(\beta_4-\beta_8)v^2+[(\beta_9-\beta_5)u+(\beta_3-\beta_7)]\,v+f_1(u)=0, \tag{4.16}$$

where

$$f_1(u)=\beta_1-\beta_6u^2-\beta_2u\ln|u|-C_1u.$$

We can formally find the value of v from the following equations:

$$v_{1,2}(u)=\frac{1}{2(\beta_4-\beta_8)}\left\{(\beta_5-\beta_9)u+(\beta_7-\beta_3)\pm\sqrt{f_2(u)}\right\},\ \beta_4\neq\beta_8, \tag{4.17}$$

where

$$f_2(u) = A_1 + A_2 u + A_3 u^2 + A_4 u \ln|u|,$$

$$A_1 = (\beta_3 - \beta_7)^2 - 4\beta_1(\beta_4 - \beta_8),\ A_2 = 2(\beta_9 - \beta_5)(\beta_3 - \beta_7) + 4C_1(\beta_4 - \beta_8),$$

$$A_3 = (\beta_9 - \beta_5)^2 + 4\beta_6(\beta_4 - \beta_8),\ A_4 = 4\beta_2(\beta_4 - \beta_8),$$

or

$$v_0(u) = -\frac{f_1(u)}{(\beta_9 - \beta_5)u + (\beta_3 - \beta_7)},\ \beta_4 = \beta_8,\ (\beta_9 - \beta_5)u + (\beta_3 - \beta_7) \neq 0. \quad (4.18)$$

Then (in the case $\beta_4 \neq \beta_8$) the quadrature required to find an additional (generally speaking, transcendental) first integral of the system (4.7), (4.8) takes the following form (in this case, Eq. (4.8) is used):

$$\int \frac{dx}{x} = \int \frac{[\beta_7 + \beta_5 u + \beta_8 v_i(u)]du}{(\beta_3 - \beta_7)u + (\beta_9 - \beta_5)u^2 + 2(\beta_4 - \beta_8)uv_i(u)} =$$

$$= \pm \int \frac{[B_1 + B_2 u + B_3\sqrt{f_2(u)}]du}{u\sqrt{f_2(u)}},\ i = 1, 2, 0, \quad (4.19)$$

where

$$B_1 = \beta_7 + \frac{\beta_8(\beta_7 - \beta_3)}{2(\beta_4 - \beta_8)},\ B_2 = \beta_5 + \frac{\beta_8(\beta_5 - \beta_9)}{2(\beta_4 - \beta_8)},\ B_3 = \pm\frac{\beta_8}{2(\beta_4 - \beta_8)}.$$

If $\beta_4 = \beta_8$, then the quadrature takes the form

$$\int \frac{dx}{x} = \int \frac{\{(\beta_7 + \beta_5 u)[(\beta_9 - \beta_5)u + (\beta_3 - \beta_7)] - \beta_8 f_1(u)\}du}{(\beta_3 - \beta_7)u + (\beta_9 - \beta_5)u^2[(\beta_9 - \beta_5)u + (\beta_3 - \beta_7)]}. \quad (4.20)$$

The quadrature needed for the search for an additional (in general, transcendental) first integral of the system (4.7), (4.8) takes the following form (in this case, we use Eq. (4.7)):

$$\int \frac{dx}{x} = \int \frac{[\beta_7 + \beta_5 u(v) + \beta_8 v]dv}{\beta_1 + \beta_2 u(v) + (\beta_3 - \beta_7)v + (\beta_4 - \beta_8)v^2 + \beta_6 u^2(v)}; \quad (4.21)$$

the function $u(v)$ must be obtained as the result of solution of the implicit equation (4.10) with respect to u (which in general is not always obvious).

The integrals (4.19) can be expressed as finite combinations of elementary functions under the following sufficient conditions.

Lemma 4.1. *If $A_4 = 0$, i.e., if*

$$\beta_2 = 0 \quad (4.22)$$

or

$$\beta_4 = \beta_8 \quad (4.23)$$

the indefinite integral in (4.19) is expressed through a finite combination of elementary functions.

We state another important consequence of Lemma 4.1.

Theorem 4.1. *Under the condition (4.22), the system (4.12) (and (4.13)) has a complete set of first integrals expressed as finite combinations of elementary functions.*

4.1.3. *Conditions of absence of the final Jacobi multiplier.* In this section, we use the integrating factor (final Jacobi multiplier) of the form (4.2). In particular, it allows us to find the first integral of the system (2.16). Now we examine the question on the existence of integrating factors (final Jacobi multipliers) of another type, independent of (4.2), that also allows one to integrate Eq. (4.6).

The following two lemmas partially answer this question.

Lemma 4.2. *Equation (4.6) does not have any integrating factor (final Jacobi multipliers) of the form*

$$\varrho = \varrho(u), \tag{4.24}$$

except for the case (4.2).

First, we present a general equation that must satisfy the integrating factor $\varrho(u,v)$ of Eq. (4.6):

$$\begin{gathered}[\beta_1 + \beta_2 u + (\beta_3 - \beta_7)v + (\beta_4 - \beta_8)v^2 + \beta_6 u^2]\varrho(u,v)du = \\ = [(\beta_3 - \beta_7)u + 2(\beta_4 - \beta_8)vu + (\beta_9 - \beta_5)u^2]\varrho(u,v)dv.\end{gathered} \tag{4.25}$$

If $\varrho(u,v)$ is a required integrating factor, the following equality must hold:

$$\begin{gathered}[(\beta_3 - \beta_7) + 2(\beta_4 - \beta_8)v + 2(\beta_9 - \beta_5)u]\varrho(u,v) + \\ + [(\beta_3 - \beta_7)u + 2(\beta_4 - \beta_8)vu + (\beta_9 - \beta_5)u^2]\frac{\partial \varrho(u,v)}{\partial u} = \\ = -[(\beta_3 - \beta_7) + 2(\beta_4 - \beta_8)v]\varrho(u,v) - \\ - [\beta_1 + \beta_2 u + (\beta_3 - \beta_7)v + (\beta_4 - \beta_8)v^2 + \beta_6 u^2]\frac{\partial \varrho(u,v)}{\partial v}.\end{gathered} \tag{4.26}$$

If we search for the integrating factor of the form (4.24), then (4.26) is expressed as follows:

$$[(\beta_3 - \beta_7) + 2(\beta_4 - \beta_8)v + (\beta_9 - \beta_5)u]\left[2\varrho(u) + u\frac{d\varrho(u)}{du}\right] = 0, \tag{4.27}$$

it must hold on the whole of its domain. This implies that the following equation must hold:

$$2\varrho(u) + u\frac{d\varrho(u)}{du} = 0. \tag{4.28}$$

Its general solution has the form

$$\varrho(u) = \frac{C}{u^2}, \ C = \text{const}, \tag{4.29}$$

corresponding to the case (4.2). Lemma 4.2 is proved.

Lemma 4.3. *Equation (4.6) does not have integrating factors (final Jacobi multipliers) of the form*

$$\varrho = \varrho(u,v) = \frac{1}{u^m v^n}, \ m,n \in \mathbf{R}, \tag{4.30}$$

except for the case $m = 2, n = 0$ *(i.e., (4.2)).*

Indeed, the equation (4.6) has a structure for which there exists an integrating factor only of the form (4.29).

4.2. Case $m = 2$**. II.** We integrate Eq. (3.9) with the integrating factor (final Jacobi multiplier) of the form

$$\varrho(u) = \frac{1}{u^s},\ s = 3. \tag{4.31}$$

Then Eq. (3.9) takes the form

$$\begin{gathered}\left[\frac{g}{u^3} + \frac{h-d_1}{u^2} + \frac{iv}{u^3} + \frac{i_1 v^2}{u^3} + \frac{(i_2-f)v}{u^2} + \frac{i_3-e}{u}\right] dv = \\ = \left[\frac{a}{u^3} + \frac{b}{u^2} + \frac{(c-d_1)v}{u^3} + \frac{(c_1-f)v^2}{u^3} + \frac{(c_2-e)v}{u^2} + \frac{c_3}{u}\right] du.\end{gathered} \tag{4.32}$$

The last identity is integrable in elementary functions under the following six independent sufficient relations:

$$g = 0,\ i = 0,\ i_1 = 0,\ h = \frac{c+d_1}{2},\ i_2 = c_1,\ i_3 = c_2. \tag{4.33}$$

We introduce nine independent parameters $\beta_1, \ldots, \beta_9$:

$$\begin{gathered}\beta_1 = a,\ \beta_2 = b,\ \beta_3 = c,\ \beta_4 = c_1,\ \beta_5 = c_2, \\ \beta_6 = c_3,\ \beta_7 = d_1,\ \beta_8 = e,\ \beta_9 = f.\end{gathered} \tag{4.34}$$

Thus, Eq. (3.9) under the conditions (4.33) and (4.34) is reduced to the form

$$\frac{dv}{du} = \frac{\beta_1 + \beta_2 u + (\beta_3-\beta_7)v + (\beta_4-\beta_9)v^2 + (\beta_5-\beta_8)uv + \beta_6 u^2}{(\beta_3-\beta_7)u/2 + (\beta_4-\beta_9)uv + (\beta_5-\beta_8)u^2}, \tag{4.35}$$

and the system (3.5), (3.6), respectively, to the form

$$x\frac{dv}{dx} = \frac{\beta_1 + \beta_2 u + (\beta_3-\beta_7)v + (\beta_4-\beta_9)v^2 + (\beta_5-\beta_8)uv + \beta_6 u^2}{\beta_7 + \beta_8 u + \beta_9 v}, \tag{4.36}$$

$$x\frac{du}{dx} = \frac{(\beta_3-\beta_7)u/2 + (\beta_4-\beta_9)uv + (\beta_5-\beta_8)u^2}{\beta_7 + \beta_8 u + \beta_9 v}. \tag{4.37}$$

Then Eq. (4.35) can be integrated through a finite combination of elementary functions.

Indeed, integrating the identity (4.32), we obtain the relation

$$\begin{gathered}d\left[\frac{(\beta_5-\beta_8)v}{u}\right] + d\left[\frac{(\beta_4-\beta_9)v^2}{2u^2}\right] + d\left[\frac{(\beta_3-\beta_7)v}{2u^2}\right] + d\left[\frac{\beta_1}{2u^2}\right] + d\left[\frac{\beta_2}{u}\right] - \\ - d[\beta_6 \ln|u|] = 0,\end{gathered} \tag{4.38}$$

which implies the following invariant relation:

$$\begin{gathered}\frac{(\beta_5-\beta_8)uv + (\beta_4-\beta_9)v^2/2 + (\beta_3-\beta_7)v/2 + \beta_1/2 + \beta_2 u}{u^2} - \\ - \beta_6 \ln|u| = C_1 = \text{const}.\end{gathered} \tag{4.39}$$

Then in the coordinates (x, y, z) we obtain the first integral in the form

$$\begin{gathered}\frac{(\beta_5-\beta_8)yz + (\beta_4-\beta_9)z^2/2 + (\beta_3-\beta_7)zx/2 + \beta_1 x^2/2 + \beta_2 yx}{y^2} - \\ - \beta_6 \ln\left|\frac{y}{x}\right| = C_1 = \text{const}.\end{gathered} \tag{4.40}$$

Thus, we can conclude on the integrability in elementary functions of the following, generally speaking, nonconservative third-order system, which depends on nine parameters:

$$\begin{aligned} \frac{dz}{dx} &= \frac{\beta_1 x + \beta_2 y + \beta_3 z + \beta_4 z^2/x + \beta_5 zy/x + \beta_6 y^2/x}{\beta_7 x + \beta_8 y + \beta_9 z}, \\ \frac{dy}{dx} &= \frac{(\beta_3 + \beta_7)y/2 + \beta_4 zy/x + \beta_5 y^2/x}{\beta_7 x + \beta_8 y + \beta_9 z}. \end{aligned} \tag{4.41}$$

Corollary 4.2. *On the set*

$$\{\alpha \in \mathbf{R}^1 : \ 0 < \alpha < \pi\} \times \mathbf{R}^2\{z_1, z_2\}, \tag{4.42}$$

consider the following third-order system depending on nine parameters $\beta_1, \dots, \beta_9$*:*

$$\begin{aligned} \dot{\alpha} &= \beta_7 \sin\alpha + \beta_8 z_1 + \beta_9 z_2, \\ \dot{z}_2 &= \beta_1 \sin\alpha\cos\alpha + \beta_2 z_1 \cos\alpha + \beta_3 z_2 \cos\alpha + \\ &+ \beta_4 z_2^2 \frac{\cos\alpha}{\sin\alpha} + \beta_5 z_1 z_2 \frac{\cos\alpha}{\sin\alpha} + \beta_6 z_1^2 \frac{\cos\alpha}{\sin\alpha}, \\ \dot{z}_1 &= ((\beta_3 + \beta_7)/2) z_1 \cos\alpha + \beta_4 z_1 z_2 \frac{\cos\alpha}{\sin\alpha} + \beta_5 z_1^2 \frac{\cos\alpha}{\sin\alpha}. \end{aligned} \tag{4.43}$$

It has a first integral (generally speaking, transcendental) expressed through elementary functions:

$$\frac{(\beta_5 - \beta_8) z_1 z_2 + (\frac{\beta_4 - \beta_9}{2}) z_2^2 + ((\beta_3 - \beta_7)/2) z_2 \sin\alpha + (\beta_1/2)\sin^2\alpha + \beta_2 z_1 \sin\alpha}{z_1^2} - \\ - \beta_6 \ln\left|\frac{z_1}{\sin\alpha}\right| = C_1 = \mathit{const}. \tag{4.44}$$

To find an additional first integral of the non-autonomous system (4.41), we use the first integral (4.40) expressed as a finite combination of elementary functions.

4.3. Case $m = 3$. We integrate Eq. (3.9) with an integrating factor (final Jacobi multiplier) of the following form:

$$\varrho(u) = \frac{1}{u^s}, \ s = 4. \tag{4.45}$$

Then Eq. (3.9) takes the form

$$\begin{aligned} &\left[\frac{g}{u^4} + \frac{h - d_1}{u^3} + \frac{iv}{u^4} + \frac{i_1 v^2}{u^4} + \frac{(i_2 - f)v}{u^3} + \frac{i_3 - e}{u^2}\right] dv = \\ &= \left[\frac{a}{u^4} + \frac{b}{u^3} + \frac{(c - d_1)v}{u^4} + \frac{(c_1 - f)v^2}{u^4} + \frac{(c_2 - e)v}{u^3} + \frac{c_3}{u^2}\right] du. \end{aligned} \tag{4.46}$$

This relations can be integrated in elementary functions under the following six independent relations:

$$g = 0, \ i = 0, \ i_1 = 0, \ h = \frac{c + 2d_1}{3}, \ i_2 = \frac{2c_1 + f}{3}, \ i_3 = \frac{c_2 + e}{2}. \tag{4.47}$$

We introduce nine independent parameters $\beta_1, \dots, \beta_9$:

$$\begin{aligned} &\beta_1 = a, \ \beta_2 = b, \ \beta_3 = c, \ \beta_4 = c_1, \ \beta_5 = c_2, \\ &\beta_6 = c_3, \ \beta_7 = d_1, \ \beta_8 = e, \ \beta_9 = f. \end{aligned} \tag{4.48}$$

Thus, Eq. (3.9) under the conditions (4.47) and (4.48) is reduced to the form

$$\frac{dv}{du}=\frac{\beta_1+\beta_2u+\beta_6u^2+(\beta_3-\beta_7)v+(\beta_4-\beta_9)v^2+(\beta_5-\beta_8)uv}{(\beta_3-\beta_7)u/3+2(\beta_4-\beta_9)uv/3+(\beta_5-\beta_8)u^2/2}, \tag{4.49}$$

and the system (3.5), (3.6), respectively, to the form

$$x\frac{dv}{dx}=\frac{\beta_1+\beta_2u+\beta_6u^2+(\beta_3-\beta_7)v+(\beta_4-\beta_9)v^2+(\beta_5-\beta_8)uv}{\beta_7+\beta_8u+\beta_9v}, \tag{4.50}$$

$$x\frac{du}{dx}=\frac{(\beta_3-\beta_7)u/3+2(\beta_4-\beta_9)uv/3+(\beta_5-\beta_8)u^2/2}{\beta_7+\beta_8u+\beta_9v}. \tag{4.51}$$

Then Eq. (4.49) is integrated through a finite combination of elementary functions.

Indeed, integrating the identity (4.46), we obtain the relation

$$d\left[\frac{(\beta_3-\beta_7)v}{3u^3}\right]+d\left[\frac{(\beta_4-\beta_9)v^2}{3u^3}\right]+d\left[\frac{(\beta_5-\beta_8)v}{2u^2}\right]+d\left[\frac{\beta_1}{3u^3}\right]+d\left[\frac{\beta_2}{2u^2}\right]+ \\ +d\left[\frac{\beta_6}{u}\right]=0, \tag{4.52}$$

then the invariant relation

$$\frac{((\beta_3-\beta_7)/3)v+((\beta_4-\beta_9)/3)v^2+((\beta_5-\beta_8)/2)uv+\beta_1/3+\beta_2u/2+\beta_6u^2}{u^3}= \\ =C_1=\text{const}, \tag{4.53}$$

and finally, in the coordinates (x, y, z), the first integral in the following form:

$$\frac{(\frac{\beta_3-\beta_7}{3})zx^2+(\frac{\beta_4-\beta_9}{3})z^2x+\frac{\beta_5-\beta_8}{2}yzx+\beta_1x^3/3+\beta_2yx^2/2+\beta_6y^2x}{y^3}= \\ =C_1=\text{const}. \tag{4.54}$$

Thus, we can conclude on the integrability in elementary functions of the following (generally speaking, nonconservative) third-order system, which depends on nine parameters:

$$\begin{aligned} \frac{dz}{dx}&=\frac{\beta_1x+\beta_2y+\beta_3z+\beta_4z^2/x+\beta_5zy/x+\beta_6y^2/x}{\beta_7x+\beta_8y+\beta_9z}, \\ \frac{dy}{dx}&=\frac{(\beta_3+2\beta_7)y/3+((2\beta_4+\beta_9)/3)zy/x+((\beta_5+\beta_8)/2)y^2/x}{\beta_7x+\beta_8y+\beta_9z}. \end{aligned} \tag{4.55}$$

Corollary 4.3. *Consider on the set*

$$\{\alpha\in\mathbf{R}^1:\ 0<\alpha<\pi\}\times\mathbf{R}^2\{z_1,z_2\}, \tag{4.56}$$

the following third-order system

$$\begin{aligned} &\dot\alpha=\beta_7\sin\alpha+\beta_8z_1+\beta_9z_2, \\ &\dot z_2=\beta_1\sin\alpha\cos\alpha+\beta_2z_1\cos\alpha+\beta_3z_2\cos\alpha+ \\ &+\beta_4z_2^2\frac{\cos\alpha}{\sin\alpha}+\beta_5z_1z_2\frac{\cos\alpha}{\sin\alpha}+\beta_6z_1^2\frac{\cos\alpha}{\sin\alpha}, \\ &\dot z_1=\frac{\beta_3+2\beta_7}{3}z_1\cos\alpha+\frac{2\beta_4+\beta_9}{3}z_1z_2\frac{\cos\alpha}{\sin\alpha}+((\beta_5+\beta_8)/2)z_1^2\frac{\cos\alpha}{\sin\alpha}, \end{aligned} \tag{4.57}$$

depending on 9 parameters $\beta_1, \ldots, \beta_9$.

It has a first integral (generally speaking, transcendental) expressed through elementary functions:

$$\frac{P_3(z_2, z_1, \sin\alpha)}{z_1^3} = C_1 = const, \tag{4.58}$$

where

$$P_3(z_2, z_1, \sin\alpha) =$$
$$= ((\beta_3 - \beta_7)/3)z_2 \sin^2\alpha + ((\beta_4 - \beta_9)/3)z_2^2 \sin\alpha + ((\beta_5 - \beta_8)/2)z_1 z_2 \sin\alpha +$$
$$+(\beta_1/3)\sin^3\alpha + (\beta_2/2)z_1 \sin^2\alpha + \beta_6 z_1^2 \sin\alpha$$

is a homogeneous polynomial of 3rd degree of the variables $(z_2, z_1, \sin\alpha)$.

To find an additional first integral of the non-autonomous system (4.55), we use the first integral (4.54) expressed as a finite combination of elementary functions.

5. Particular Cases of the Existence of Transcendental First Integrals

We integrate Eq. (3.9) with an integrating factor (final Jacobi multiplier) of the following form:

$$\varrho(u) = \frac{1}{u^s}, \ s > 1, \ s \neq 2, s \neq 3. \tag{5.1}$$

Then Eq. (3.9) takes the form

$$\begin{gathered}\left[\frac{g}{u^s} + \frac{h - d_1}{u^{s-1}} + \frac{iv}{u^s} + \frac{i_1 v^2}{u^s} + \frac{(i_2 - f)v}{u^{s-1}} + \frac{i_3 - e}{u^{s-2}}\right] dv = \\ = \left[\frac{a}{u^s} + \frac{b}{u^{s-1}} + \frac{(c - d_1)v}{u^s} + \frac{(c_1 - f)v^2}{u^s} + \frac{(c_2 - e)v}{u^{s-1}} + \frac{c_3}{u^{s-2}}\right] du.\end{gathered} \tag{5.2}$$

The sufficient condition of integrability of the last identity in elementary functions is the following six independent relations:

$$\begin{gathered} g = 0, \ i = 0, \ i_1 = 0, \ h = \tfrac{c+(s-2)d_1}{s-1}, \\ i_2 = \tfrac{2c_1+(s-3)f}{s-1}, \ i_3 = \tfrac{c_2+(s-3)e}{s-2}. \end{gathered} \tag{5.3}$$

Introduce nine independent parameters $\beta_1, \ldots, \beta_9$:

$$\begin{gathered} \beta_1 = a, \ \beta_2 = b, \ \beta_3 = c, \ \beta_4 = c_1, \ \beta_5 = c_2, \\ \beta_6 = c_3, \ \beta_7 = d_1, \ \beta_8 = e, \ \beta_9 = f. \end{gathered} \tag{5.4}$$

Thus, Eq. (3.9) under the conditions (5.3) and (5.4) is reduced to the form

$$\frac{dv}{du} = \frac{\beta_1 + \beta_2 u + \beta_6 u^2 + (\beta_3 - \beta_7)v + (\beta_4 - \beta_9)v^2 + (\beta_5 - \beta_8)uv}{(\beta_3 - \beta_7)u/(s-1) + 2(\beta_4 - \beta_9)uv/(s-1) + (\beta_5 - \beta_8)u^2/(s-2)}, \tag{5.5}$$

and the system (3.5), (3.6), respectively, to the form

$$x\frac{dv}{dx} = \frac{\beta_1 + \beta_2 u + \beta_6 u^2 + (\beta_3 - \beta_7)v + (\beta_4 - \beta_9)v^2 + (\beta_5 - \beta_8)uv}{\beta_7 + \beta_8 u + \beta_9 v}, \tag{5.6}$$

$$x\frac{du}{dx} = \frac{(\beta_3 - \beta_7)u/(s-1) + 2(\beta_4 - \beta_9)uv/(s-1) + (\beta_5 - \beta_8)u^2/(s-2)}{\beta_7 + \beta_8 u + \beta_9 v}. \tag{5.7}$$

Then Eq. (5.5) is integrated through a finite combination of elementary functions.

Indeed, integrating the identity (5.2), we obtain the relation

$$
\begin{gathered}
d\left[\frac{(\beta_3-\beta_7)v}{(s-1)u^{s-1}}\right]+d\left[\frac{(\beta_4-\beta_9)v^2}{(s-1)u^{s-1}}\right]+d\left[\frac{(\beta_5-\beta_8)v}{(s-2)u^{s-2}}\right]+\\
+d\left[\frac{\beta_1}{(s-1)u^{s-1}}\right]+d\left[\frac{\beta_2}{(s-2)u^{s-2}}\right]+d\left[\frac{\beta_6}{(s-3)u^{s-3}}\right]=0,
\end{gathered}
\tag{5.8}
$$

which implies the invariant relation

$$
\frac{\frac{\beta_3-\beta_7}{s-1}v+\frac{\beta_4-\beta_9}{s-1}v^2+\frac{\beta_5-\beta_8}{s-2}uv+\frac{\beta_1}{s-1}+\frac{\beta_2}{s-2}u+\frac{\beta_6}{s-3}u^2}{u^{s-1}}=C_1=\text{const},
\tag{5.9}
$$

and then, in the coordinates (x, y, z), the first integral of the following form:

$$
\frac{A}{y^{s-1}}=C_1=\text{const},
\tag{5.10}
$$

$$
\begin{gathered}
A=\frac{\beta_3-\beta_7}{s-1}zx^{s-2}+\frac{\beta_4-\beta_9}{s-1}z^2x^{s-3}+\frac{\beta_5-\beta_8}{s-2}yzx^{s-3}+\\
+\frac{\beta_1}{s-1}x^{s-1}+\frac{\beta_2}{s-2}yx^{s-2}+\frac{\beta_6}{s-3}y^2x^{s-3}.
\end{gathered}
$$

Thus, we can conclude on the integrability in elementary functions of the following (generally speaking, nonconservative) third-order system, which depends on nine parameters:

$$
\begin{aligned}
\frac{dz}{dx}&=\frac{\beta_1x+\beta_2y+\beta_3z+\beta_4z^2/x+\beta_5zy/x+\beta_6y^2/x}{\beta_7x+\beta_8y+\beta_9z},\\
\frac{dy}{dx}&=\frac{\frac{\beta_3+(s-2)\beta_7}{s-1}y+\frac{2\beta_4+(s-3)\beta_9}{s-1}zy/x+\frac{\beta_5+(s-3)\beta_8}{s-2}y^2/x}{\beta_7x+\beta_8y+\beta_9z}.
\end{aligned}
\tag{5.11}
$$

Corollary 5.1. *Consider on the set*

$$
\{\alpha\in\mathbf{R}^1:\ 0<\alpha<\pi\}\times\mathbf{R}^2\{z_1,z_2\},
\tag{5.12}
$$

the third-order system

$$
\begin{aligned}
\dot\alpha&=\beta_7\sin\alpha+\beta_8z_1+\beta_9z_2,\\
\dot z_2&=\beta_1\sin\alpha\cos\alpha+\beta_2z_1\cos\alpha+\beta_3z_2\cos\alpha+\\
&+\beta_4z_2^2\frac{\cos\alpha}{\sin\alpha}+\beta_5z_1z_2\frac{\cos\alpha}{\sin\alpha}+\beta_6z_1^2\frac{\cos\alpha}{\sin\alpha},\\
\dot z_1&=\frac{\beta_3+(s-2)\beta_7}{s-1}z_1\cos\alpha+\frac{2\beta_4+(s-3)\beta_9}{s-1}z_1z_2\frac{\cos\alpha}{\sin\alpha}+\\
&+\frac{\beta_5+(s-3)\beta_8}{s-2}z_1^2\frac{\cos\alpha}{\sin\alpha},
\end{aligned}
\tag{5.13}
$$

depending on nine parameters $\beta_1,\dots,\beta_9$.

It has a first integral (generally speaking, transcendental) expressed through elementary functions:

$$
\frac{P_{s-1}(z_2,z_1,\sin\alpha)}{z_1^{s-1}}=C_1=const,
\tag{5.14}
$$

where

$$
P_{s-1}(z_2,z_1,\sin\alpha)=
$$

$$= \frac{\beta_3 - \beta_7}{s-1} z_2 \sin^{s-2}\alpha + \frac{\beta_4 - \beta_9}{s-1} z_2^2 \sin^{s-3}\alpha + \frac{\beta_5 - \beta_8}{s-2} z_1 z_2 \sin^{s-3}\alpha +$$
$$+ \frac{\beta_1}{s-1} \sin^{s-1}\alpha + \frac{\beta_2}{s-2} z_1 \sin^{s-2}\alpha + \frac{\beta_6}{s-3} z_1^2 \sin^{s-3}\alpha$$

is a homogeneous function of degree $s-1$ *of the variables* $(z_2, z_1, \sin\alpha)$.

To find an additional first integral of the nonautonomous system (5.11), we use the first integral (5.10) expressed as a finite combination of elementary functions.

6. Conclusion

The dynamical systems considered in this paper are systems with variable dissipation with zero mean with respect to their periodic coordinate. Moreover, such systems often possess a complete list of first integrals expressed through elementary functions.

We also presented a method of reduction of systems with right-hand sides containing polynomial of trigonometric functions to systems with polynomial right-hand sides, which allows one to find first integrals (or prove their absence) for systems of a more general form, not only those having specified symmetries (see also [10]).

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