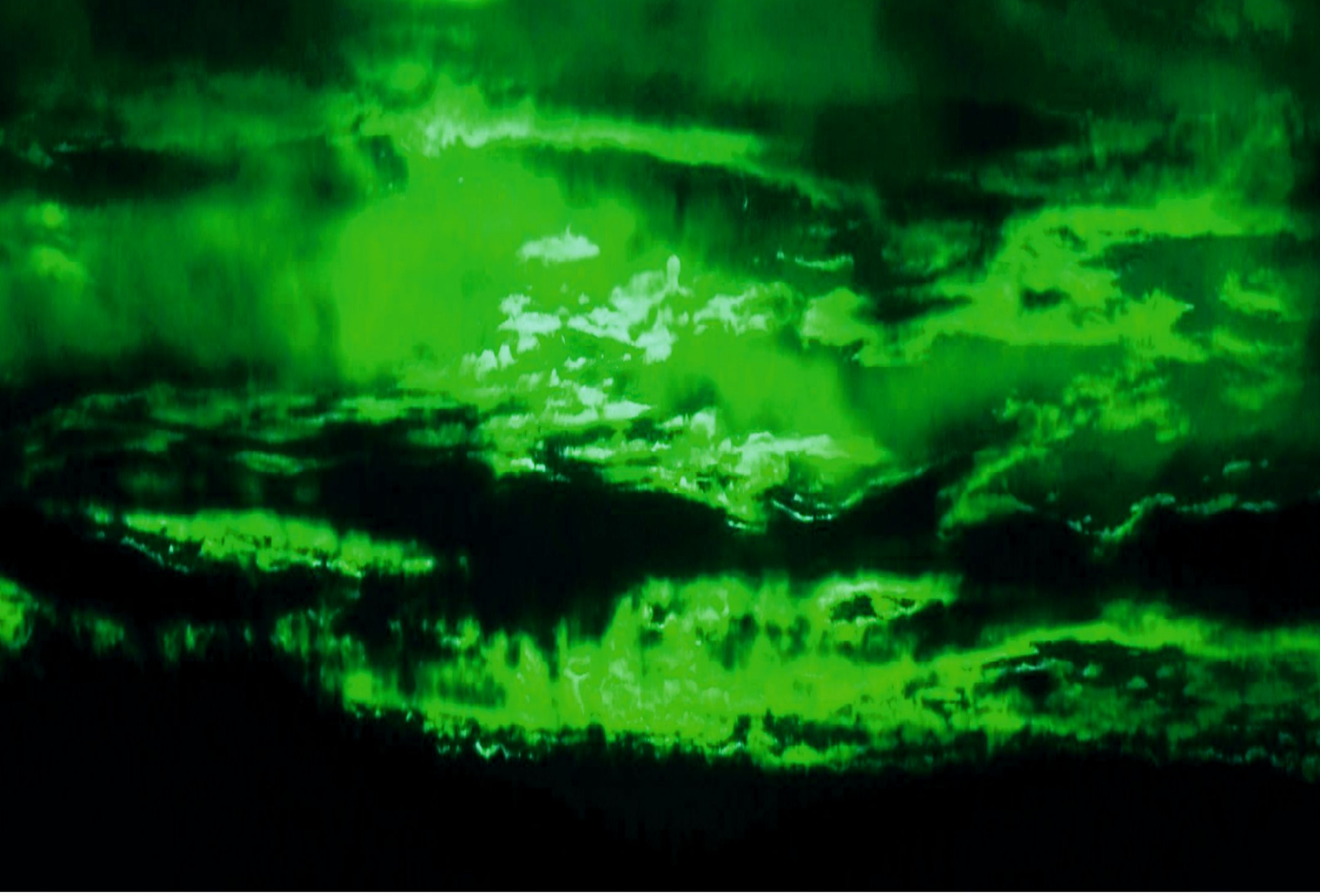
Mathematical and Numerical Aspects of Dynamical System Analysis



Editors

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DSTA 2017

Mathematical and Numerical Aspects of Dynamical System Analysis

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ISBN 978-83-935312-6-4
Cover design: Marek Kaźmierczak
Technical editor: Marek Kaźmierczak

Printed by:
ARSA Druk i Reklama
90-270 Łódź, ul. Piotrkowska 4
tel./fax (042) 633 02 52
marta@arsa.net.pl
www.arsa.net.pl

PREFACE

This is the fourteenth time when the conference "Dynamical Systems: Theory and Applications" gathers a numerous group of outstanding scientists and engineers, who deal with widely understood problems of theoretical and applied dynamics.

Organization of the conference would not have been possible without a great effort of the staff of the Department of Automation, Biomechanics and Mechatronics. The patronage over the conference has been taken by the Committee of Mechanics of the Polish Academy of Sciences and Ministry of Science and Higher Education of Poland.

It is a great pleasure that our invitation has been accepted by recording in the history of our conference number of people, including good colleagues and friends as well as a large group of researchers and scientists, who decided to participate in the conference for the first time. With proud and satisfaction we welcomed over **180** persons from **31** countries all over the world. They decided to share the results of their research and many years experiences in a discipline of dynamical systems by submitting many very interesting papers.

This year, the DSTA Conference Proceedings were split into three volumes entitled "Dynamical Systems" with respective subtitles: *Vibration, Control and Stability of Dynamical Systems*; *Mathematical and Numerical Aspects of Dynamical System Analysis* and *Engineering Dynamics and Life Sciences*. Additionally, there will be also published two volumes of Springer Proceedings in Mathematics and Statistics entitled "*Dynamical Systems in Theoretical Perspective"* and *"Dynamical Systems in Applications".*

These books include the invited and regular papers covering the following topics:

- asymptotic methods in nonlinear dynamics,
- bifurcation and chaos in dynamical systems,
- control in dynamical systems,
- dynamics in life sciences and bioengineering,
- engineering systems and differential equations,
- non-smooth systems
- mathematical approaches to dynamical systems
- original numerical methods of vibration analysis,
- stability of dynamical systems,
- vibrations of lumped and continuous systems,
- other problems.

Proceedings of the 14th Conference „Dynamical Systems - Theory and Applications" summarize **168** and the Springer Proceedings summarize **80** best papers of university teachers and students, researchers and engineers from all over the world. The papers were chosen by the International Scientific Committee from **370** papers submitted to the conference. The reader thus obtains an overview of the recent developments of dynamical systems and can study the most progressive tendencies in this field of science.

Our previous experience shows that an extensive thematic scope comprising dynamical systems stimulates a wide exchange of opinions among researchers dealing with different branches of dynamics. We think that vivid discussions will influence positively the creativity and will result in effective solutions of many problems of dynamical systems in mechanics and physics, both in terms of theory and applications.

We do hope that DSTA 2017 will contribute to the same extent as all the previous conferences to establishing a new and tightening the already existing relations and scientific and technological cooperation between both Polish and foreign institutions.

On behalf of both
Scientific and Organizing Committees

Chairman
Professor Jan Awrejcewicz

Mathematical modeling of the action of a medium on a conical body

Maxim V. Shamolin

Abstract: We consider a mathematical model of a plane-parallel action of a medium on a rigid body whose surface has a part which is a circular cone. We present a complete system of equations of motion under the quasi-stationarity conditions. The dynamical part of equations of motion form an independent system that possesses an independent second-order subsystem on a two-dimensional cylinder. We obtain an infinite family of phase portraits on the phase cylinder of quasi-velocities corresponding to the presence in the system of only a nonconservative pair of forces.

1. Introduction

In [5, 6], we present a qualitative analysis of plane-parallel and spatial problems on the motion of realistic rigid bodies in a resistive medium. We construct a nonlinear model of the influence of the medium on the rigid body, in which the dependence of the arm of force on the reduced angular velocity of the body is taken into account; in this case, the moment of force is also a function of the angle of attack. Experiments on the motion of homogeneous circular cylinders in water show that these circumstances must be taken into account in modeling. In the study of plane and spatial models of interaction of a rigid body with a medium (both in the presence or absence of an additional tracking force), we find sufficient conditions of stability of one of the key regimes of motion, rectilinear translational motion. We show that under certain conditions, stable or unstable auto-oscillation regimes in the system can appear.

In [7, 8], a mathematical model of the effect of a medium on a homogeneous rigid body whose outer surface includes a circumferential cone is considered. The complete system of equations of motion under quasi-stationarity conditions is given. In the dynamic part forming an independent third-order system, an independent second-order subsystem is distinguished. A new two-parameter family of phase portraits on the phase cylinder of quasi-velocities is obtained.

2. Statement of the Problem

Consider a plane-parallel motion of a homogeneous rigid body of mass m with the cone-shaped front part interacting with a flow of medium under conditions of jet circumfluence of flow-separated circumfluence (see Fig. 1).

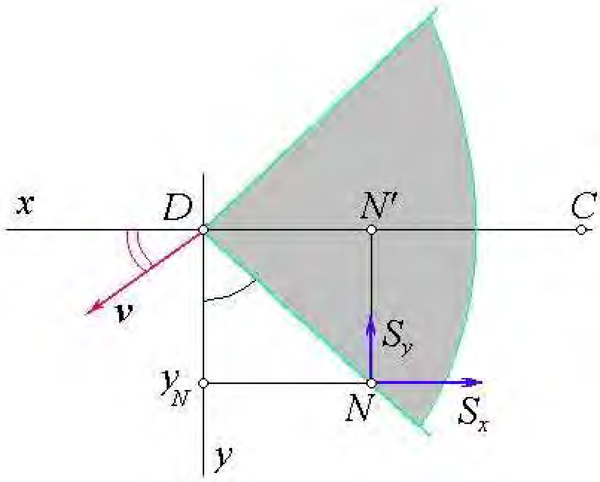

Figure 1. Action of a medium on a rigid body

For simplicity, we assume that the coordinate y_N of the application point N of the action force of the medium is determined by a single parameter, namely, by the angle of attack α, i.e., the angle between the velocity vector of the point D and the symmetry axis Dx:

$$y_N = R(\alpha). \tag{1}$$

We represent the forces of frontal and side resistance (see Fig. 1) as quadratic functions of the speed of the point D:

$$\mathbf{S}_x = -s(\alpha)v^2\mathbf{e}_x,\ \mathbf{S}_y = -b(\alpha)v^2\mathbf{e}_y,\ |\mathbf{v}_D| = v. \tag{2}$$

Thus, we triple of functions $R(\alpha), s(\alpha)$, and $b(\alpha)$ determines the action of a medium on a rigid body under the quasi-stationarity conditions (see [1–3]). In this case, the conical shape of the surface of the body and the hypothesis on the quasi-static action of the medium allow one to determine the complete scheme of forces that contains all characteristics of the system. In the sequel, the analysis of systems constructed is performed by well-known methods of qualitative theory and new methods developed especially for systems with variable dissipation (see [7, 8]).

3. Dynamical Part of Equations of Motion

Taking into account the conditions (1) and (2), we can rewrite the dynamical part of equations of motion in the following form:

$$\dot{v}\cos\alpha - \dot{\alpha}v\sin\alpha + \Omega v\sin\alpha + \sigma\Omega^2 = -\frac{s(\alpha)}{m}v^2, \tag{3}$$

$$\dot{v}\sin\alpha + \dot{\alpha}v\cos\alpha - \Omega v\cos\alpha + \sigma\dot{\Omega} = -\frac{b(\alpha)}{m}v^2, \tag{4}$$

$$I\dot{\Omega} = -F(\alpha)s(\alpha)v^2 + \sigma b(\alpha)v^2 - h\Omega v, \tag{5}$$

where I is the central moment of inertia of the body, m is the mass of a body, $\sigma = CD$, C is the center of mass, $F(\alpha) = R(\alpha)s(\alpha)$, and the coefficient $h > 0$ characterizes an additional moment depending on the angular velocity (see [6]). Note that the the dependence of forces and moments on the angular velocity in such problems is not a priori obvious.

The first two equations are obtained from the theorem on the motion of the center of mass and the third from the theorem on the change of the kinetic moments in the König axes. Similar results without side forces were used earlier in [5].

Since the kinetic energy of the body and generalized forces and moments are independent of the location of the body on the plane, the position coordinates in the system are cyclic. This allow one to consider the system of dynamical equations (3)–(5) as an independent system. The system of kinematic equations

$$\dot{\varphi} = \Omega,\ \dot{x_0} = v\cos(\alpha - \varphi),\ \dot{y_0} = v\sin(\alpha - \varphi),$$

where the variables φ, x_0, and y_0 define the location of the body on the plane, together with the system (3)–(5) for a complete system for the study of the motion in the force field constructed above.

To obtain the form of the functions $R(\alpha), s(\alpha), and b(\alpha)$, one need experimental information about properties of jet circumfluence (see also [8]).

Classes of considered dynamical functions are quite wide: they consist of sufficiently smooths, 2π-periodic functions ($R(\alpha)$ and $b(\alpha)$ are odd and $s(\alpha)$ is even) satisfying the following conditions:

$R(\alpha), b(\alpha) > 0$ for $\alpha \in (0,\pi)$, $R'(0), b'(0) > 0$, $R'(\pi), b'(\pi) < 0$ (class of functions $\{R\}, \{b\}$); $s(\alpha) > 0$ for $\alpha \in (0, \pi/2)$, $s(\alpha) < 0$ for $\alpha \in (\pi/2, \pi)$, $s(0) > 0$, $s'(\pi/2) < 0$ (class of functions $\{s\}$).

The functions R, b, and s change sign if one replace α by $\alpha + \pi$.

In particular, the analytical functions

$$R(\alpha) = A\sin\alpha,\ b(\alpha) = b_1\sin\alpha, \tag{6}$$

$$s(\alpha) = B\cos\alpha;\ A, B, b_1 > 0, \tag{7}$$

are typical representatives of classes described; two of them (namely, R and s) correspond to the functions of action of medium obtained by Chaplygin (see [2]) in the study of a plane-parallel circumfluence of an infinite flat plate by a homogeneous flow.

We explain the necessity of the wide classes of the functions $\{R\}, \{b\}$, and $\{s\}$. Geometric characteristics of cone-shape bodies may be quite different, which allows to classify three dynamical functions into several classes. As was noted above, these functional classes are constrained by sufficiently weak conditions and therefor these classes are sufficiently wide: they contain admissible functions for all bodies and all motions.

Therefore, for the study the circumfluence of a conical body by a medium, we use classes of dynamical systems determined by triples of dynamical functions, which considerably complicates the global nonlinear analysis.

In some cases, without loss of generality (see [2,3,6]), we will consider the representations (6) and (7) for the functions $R(\alpha), s(\alpha)$, and $b(\alpha)$ that determine the action of a medium.

4. Reduction of order

Equations (3)and (4) can be transformed to the form

$$\dot{v} + \sigma\Omega^2\cos\alpha + \sigma\dot{\Omega}\sin\alpha = -\frac{s(\alpha)}{m}v^2\cos\alpha - \frac{b(\alpha)}{m}v^2\sin\alpha, \tag{8}$$

$$\dot{\alpha}v - \Omega v + \sigma\dot{\Omega}\cos\alpha - \sigma\Omega^2\sin\alpha = -\frac{b(\alpha)}{m}v^2\cos\alpha + \frac{s(\alpha)}{m}v^2\sin\alpha. \tag{9}$$

Introducing the new differentiation by the formula

$$<\cdot>= d/dt = vd/dq = v <'>,$$

where q is the path travelled by the point D, we have $\Omega = \omega v$, $\dot{\Omega} = v(\omega' v + \omega v')$. Then the dynamical part of the equations of motion takes the following form:

$$v' = v\Psi_1(\alpha, \omega), \tag{10}$$

$$\alpha' = \omega + \frac{\sigma}{I}\psi(\alpha, \omega)\cos\alpha + \sigma\omega^2\sin\alpha + \frac{s(\alpha)}{m}\sin\alpha - \frac{b(\alpha)}{m}\cos\alpha, \tag{11}$$

$$\omega' = -\frac{1}{I}\psi(\alpha, \omega) - \omega\Psi_1(\alpha, \omega), \tag{12}$$

where

$$\psi(\alpha, \omega) = F(\alpha) - \sigma b(\alpha) + h\omega,$$

$$\Psi_1(\alpha, \omega) = \frac{\sigma}{I}\psi(\alpha, \omega)\sin\alpha - \sigma\omega^2\cos\alpha - \frac{s(\alpha)}{m}\cos\alpha - \frac{b(\alpha)}{m}\sin\alpha.$$

We introduce the dimensionless parameters and the differentiation of the form

$$q = Q\sigma,\ \bar{\omega} = \omega\sigma,\ \beta_1 = \frac{\sigma^2 AB}{I},\ \beta_2 = \frac{\sigma^3 b_1}{I},\ \beta_3 = \frac{\sigma h}{I},\ \beta_4 = \frac{B\sigma}{m},\ \beta_5 = \frac{b_1\sigma}{m}.$$

In the sequel, we omit the bar in the notation of the dimensionless variable $\bar{\omega}$ and denote the derivative with respect to the dimensionless variable Q by $'$. In the cases (6) and (7), we can rewrite the system (11), (12) as follows:

$$\alpha' = \omega + \beta_1 \sin\alpha\cos^2\alpha - \beta_2 \sin\alpha\cos\alpha + \beta_3\omega\cos\alpha +$$

$$+\omega^2\sin\alpha + \beta_4 \sin\alpha\cos\alpha - \beta_5\sin\alpha\cos\alpha, \tag{13}$$

$$\omega' = -\beta_1\sin\alpha\cos\alpha + \beta_2\sin\alpha - \beta_3\omega + \omega^3\cos\alpha - \beta_1\omega\sin^2\alpha\cos\alpha +$$

$$+\beta_2\omega^2\sin\alpha - \beta_3\omega^2\sin\alpha + \beta_4\omega\cos^2\alpha + \beta_5\omega\sin^2\alpha. \tag{14}$$

The dimensionless parameters $\beta_k, k = 1, \ldots, 5$, have the following sense:

β_1 is the parameter of the moment of the frontal resistance force;

β_2 is the parameter of the moment of the lateral force;

β_3 is the parameter of the additional damping moment;

β_4 is the parameter of the frontal resistance force;

β_5 is the parameter of the moment of the lateral force.

Thus, we have a five-parameter family of systems (13), (14) on the two-dimensional phase cylinder

$$\{(\alpha, \omega) \in \mathbf{R}^2 : \alpha \bmod 2\pi\}.$$

5. Regime of Rectilinear Translational Deceleration and Its Stability

Among all possible motions of a body, there exists a key regime — a rectilinear translational deceleration: the body moves translationally with zero attack angle and speeds of all points of the body decrease (see also [6]). The key regime corresponds to the trivial solution of the system (13), (14).

Under the stability of the key regime, we understand the stability of angular oscillations of a body about its longitudinal axis with respect to perturbations of the attack angle and the angular velocity. From the point of view of the theory of stability, this type of stability is treated as the stability with respect to a part of variables.

To examine this stability, we linearize system (13), (14) at the origin:

$$\alpha' = \omega + \beta_1\alpha - \beta_2\alpha + \beta_3\omega + \beta_4\alpha - \beta_5\alpha, \tag{15}$$

$$\omega' = -\beta_1\alpha + \beta_2\alpha - \beta_3\omega + \beta_4\omega. \tag{16}$$

The matrix of this system has the form

$$A = \begin{pmatrix} \beta_1 - \beta_2 + \beta_4 - \beta_5 & 1 + \beta_3 \\ -\beta_1 + \beta_2 & -\beta_3 + \beta_4 \end{pmatrix}, \tag{17}$$

which leads to the characteristic equation

$$\lambda^2 - \mathrm{tr}A \cdot \lambda + \det A = 0, \tag{18}$$

where

$$\mathrm{tr}A = \beta_1 - \beta_2 - \beta_3 + 2\beta_4 - \beta_5. \tag{19}$$

Clearly, the conditions

$$\mathrm{tr}A < 0, \ \det A > 0$$

provide the asymptotic stability of the trivial solution of the system (13), (14).

Relation (19) implies that the presence of a frontal resistance force (and its moment) in the system makes the rectilinear translational deceleration more nonstable. In other words, increasing the coefficients β_1 and β_4 leads to increasing $\mathrm{tr}A$. Conversely, the presence in the system of a lateral resistance force (and its moment) and an additional damping moment makes the rectilinear translational deceleration more stable. In other words, increasing the coefficients β_2, β_3, and β_5 leads to decreasing $\mathrm{tr}A$.

6. Two-parameter Family of Phase Portrait

Consider the case where the system contains two force couples: a couple of frontal resistance forces and a couple of lateral forces (these couples can be added). Namely, we assume that the following conditions hold:

$$\beta_3 = \beta_4 = \beta_5 = 0. \tag{20}$$

In this case, the system (13), (14) becomes

$$\alpha' = \omega + \omega^2 \sin\alpha + \beta_1 \sin\alpha \cos^2\alpha - \beta_2 \sin\alpha \cos\alpha, \tag{21}$$

$$\omega' = -\beta_1 \sin\alpha \cos\alpha + \beta_2 \sin\alpha + \omega^3 \cos\alpha - \beta_1 \omega \sin^2\alpha \cos\alpha +$$

$$+\beta_2 \omega^2 \sin\alpha. \tag{22}$$

Then the system (21), (22) possesses a two-parameter family of phase portraits (see Figs. 2–7, change $\Omega \leftrightarrow \omega$). This family differs from families obtained earlier (see [5]).

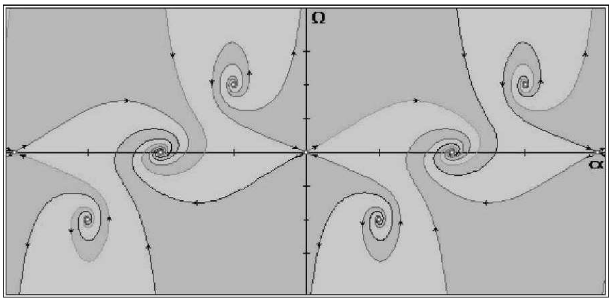

Figure 2.

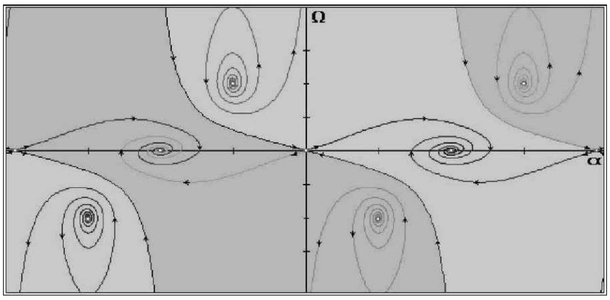

Figure 3.

7. On the Possibility of the Stability of the Key Regime

Under the condition (20), the characteristic equation (18) has the form

$$\lambda^2 + (\beta_2 - \beta_1)\lambda + \beta_1 - \beta_2 = 0; \tag{23}$$

this shows that in the domain of the parameters specified above the stability cannot be achieved. For example, for

$$\beta_1 < \beta_2 \tag{24}$$

the trivial solution of the system is nonstable due to the saddle point. Therefore, by an appropriate choice of the corresponding initial conditions, one can obtain a conditionally stable solution. Indeed, one can take initial conditions near stable separatrices in a neighborhood of the origin and calculate the eigenvectors in the linear approximation.

Another type of nonstability of the trivial solution of the system occurs under the condition

$$\beta_1 > \beta_2. \tag{25}$$

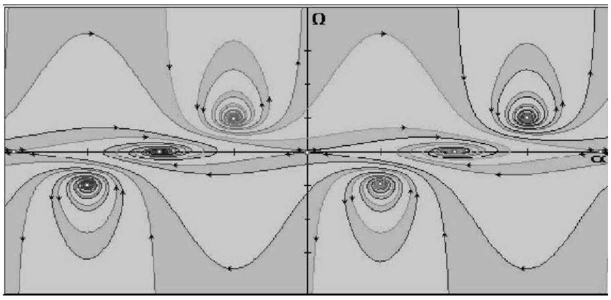

Figure 4.

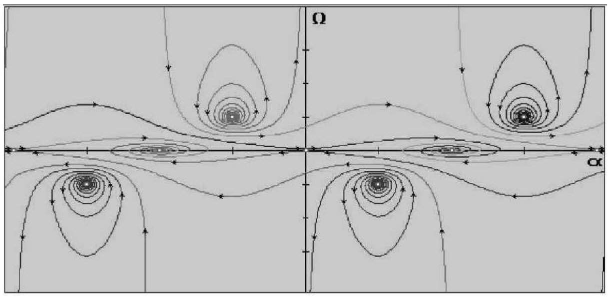

Figure 5.

In this case, the origin is a repelling singular point, and no choice of initial conditions leads to a stable solution.

Families of portraits obtained earlier (see [4,5,8]) deal with the case where the asymptotic stability of the origin can be achieved. The family of portraits obtained in the present paper deals with the case of the conditional stability, which can be achieved by an appropriate choice of initial conditions. In Figs. 2–6, we present the cases corresponding to the inequality (24) (a saddle at the origin), whereas in Fig. 7 the case corresponding to the inequality (25) (a repealing point at the origin) is illustrated.

8. Conclusions

We consider a mathematical model of a plane-parallel action of a medium on a rigid body whose surface has a part which is a circular cone. We present a complete system of equations of motion under the quasi-stationarity conditions. The dynamical part of equations of motion form an independent system that possesses an independent second-order subsystem on a two-dimensional cylinder. We obtain an infinite family of phase portraits on the phase cylinder

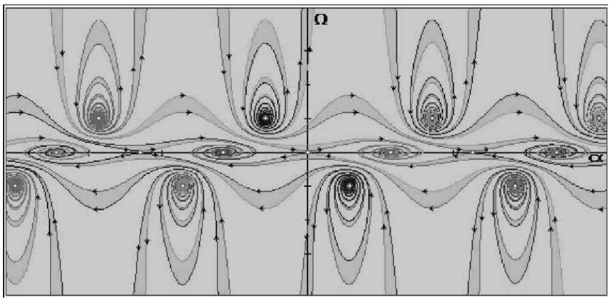

Figure 6.

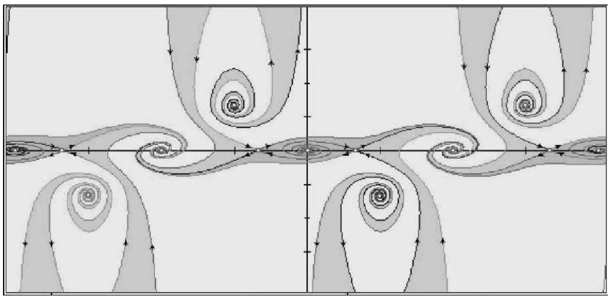

Figure 7.

of quasi-velocities corresponding to the presence in the system only of a nonconservative pair of forces.

Acknowledgments

This work was supported by the Russian Foundation for Basic Research, project no. 15-01-00848-a.

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