First Integrals of Variable Dissipation Dynamical Systems in Rigid Body Dynamics

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***Abstract*—The activity is devoted to general aspects of the integrability of dynamical systems with variable dissipation. First, we propose a descriptive characteristic of such systems. The term "variable dissipation" refers to the possibility of alternation of its sign rather than to the value of the dissipation coefficient (therefore, it is more reasonable to use the term "sign-alternating"). Later, we define systems with variable dissipation with zero (nonzero) mean based on the divergence of the vector field of the system, which characterizes the change of the phase volume in the phase space of the system considered.**

I. Introduction

We introduce a class of autonomous dynamical systems with one periodic phase coordinate possessing certain symmetries that are typical for pendulum-type systems. We show that this class of systems can be naturally embedded in the class of systems with variable dissipation with zero mean, i.e., on the average for the period with respect to the periodic coordinate, the dissipation in the system is equal to zero, although in various domains of the phase space, either energy pumping or dissipation can occur, but they balance to each other in a certain sense. We present some examples of pendulum-type systems on lower-dimension manifolds from dynamics of a rigid body in a nonconservative field.

II. Definition of a system with variable dissipation with zero mean

We study systems of ordinary differential equations having a periodic phase coordinate. The systems under study have those symmetries under which, on the average, for the period in the periodic coordinates, their phase volume is preserved. So, for example, the following pendulum-like system with smooth and periodic in α of period T right-hand side $\mathbf{V}(\alpha, \omega)$ of the form

$$\dot\alpha = -\omega + f(\alpha),$$
$$\dot\omega = g(\alpha),\ f(\alpha+T) = f(\alpha),\ g(\alpha+T) = g(\alpha),$$

preserves its phase area on the phase cylinder over the period T:

$$\int_0^T \mathrm{div}\mathbf{V}(\alpha,\omega)d\alpha = \int_0^T f'(\alpha)d\alpha = 0.$$

The system considered is equivalent to the pendulum equation $\ddot\alpha - f'(\alpha)\dot\alpha + g(\alpha) = 0$, in which the integral of the coefficient $f'(\alpha)$ standing by the dissipative term $\dot\alpha$ is equal to zero on the average for the period.

It is seen that the system considered has those symmetries under which it becomes the so-called *zero mean variable dissipation system* in the sense of the following definition (see also [1], [2]).

Consider a smooth autonomous system of the $(n+1)$th order of normal form given on the cylinder $\mathbf{R}^n\{x\} \times \mathbf{S}^1\{\alpha \mod 2\pi\}$, where α is a periodic coordinate of period $T > 0$. The divergence of the right-hand side $\mathbf{V}(x,\alpha)$ (which, in general, is a function of all phase variables and is not identically equal to zero) of this system is denoted by $\mathrm{div}\mathbf{V}(x,\alpha)$. This system is called a zero (nonzero) mean variable dissipation system if the function $\int_0^T \mathrm{div}\mathbf{V}(x,\alpha)d\alpha$ is equal (not equal) to zero identically. Moreover, in some cases (for example, when singularities arise at separate points of the circle $\mathbf{S}^1\{\alpha \mod 2\pi\}$), this integral is understood in the sense of principal value.

III. Systems with Symmetries and Zero Mean Variable Dissipation

Let us consider systems of the form (the dot denotes the derivative in time)

$$\begin{aligned}&\dot\alpha = f_\alpha(\omega, \sin\alpha, \cos\alpha),\\ &\dot\omega_k = f_k(\omega, \sin\alpha, \cos\alpha),\ k = 1,\ldots,n,\end{aligned} \tag{1}$$

given on the set $\mathbf{S}^1\{\alpha \mod 2\pi\}\backslash K \times \mathbf{R}^n\{\omega\}$, $\omega = (\omega_1,\ldots,\omega_n)$, where the smooth functions $f_\lambda(u_1,u_2,u_3)$, $\lambda = \alpha, 1,\ldots,n$, of three variables u_1,u_2,u_3 are as follows:

$$\begin{aligned}&f_\lambda(-u_1,-u_2,u_3) = -f_\lambda(u_1,u_2,u_3),\\ &f_\alpha(u_1,u_2,-u_3) = f_\alpha(u_1,u_2,u_3),\\ &f_k(u_1,u_2,-u_3) = -f_k(u_1,u_2,u_3),\end{aligned} \tag{2}$$

herewith, the functions $f_k(u_1,u_2,u_3)$ *are defined* for $u_3 = 0$ for any $k = 1,\ldots,n$.

The set K is either empty or consists of finitely many points of the circle $\mathbf{S}^1\{\alpha \mod 2\pi\}$.

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The latter two variables u_2 and u_3 in the functions $f_\lambda(u_1, u_2, u_3)$ depend on one parameter α, but they are allocated into different groups because of the following reasons. First, not in the whole domain, they are one-to-one expressed through each other, and, second, the first of them is an odd function and the second is an even function of α, which influences the symmetries of system (1) in different ways.

To the system (1), let us put in correspondence the following nonautonomous system

$$\frac{d\omega_k}{d\alpha} = \frac{f_k(\omega, \sin\alpha, \cos\alpha)}{f_\alpha(\omega, \sin\alpha, \cos\alpha)},\ k = 1, \ldots, n,$$

which via the substitution $\tau = \sin\alpha$, reduces to the form

$$\frac{d\omega_k}{d\tau} = \frac{f_k(\omega, \tau, \varphi_k(\tau))}{f_\alpha(\omega, \tau, \varphi_\alpha(\tau))},\ k = 1, \ldots, n, \tag{3}$$

$$\varphi_\lambda(-\tau) = \varphi_\lambda(\tau),\ \lambda = \alpha, 1, \ldots, n.$$

The latter system can have in particular an algebraic right-hand side (i.e., it can be the ratio of two polynomials); sometimes this helps us to find its first integrals in explicit form.

The following assertion embeds the class of systems (1) in the class of zero mean variable dissipation systems. The inverse embedding does not hold in general.

Theorem 1: Systems of the form (1) are zero mean variable dissipation dynamical systems.

This theorem is proved by using the certain symmetries (2) of system (1) only, listed above, and uses the periodicity of the right-hand side of the system on α.

The converse assertion is not true in general since we can present a set of dynamical systems on the two-dimensional cylinder, being zero mean variable dissipation systems that have none of the above-listed symmetries.

In this work, we are mainly concerned with the case where the functions $f_\lambda(\omega, \tau, \varphi_k(\tau))$ ($\lambda = \alpha, 1, \ldots, n$) are polynomials in ω, τ.

For example, to the following system

$$\begin{aligned} &\dot{\alpha} = -z_2 + b\sin\alpha, \\ &\dot{z_2} = \sin\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha},\ \dot{z_1} = z_1 z_2\frac{\cos\alpha}{\sin\alpha}, \end{aligned} \tag{4}$$

with parameter b, considered in the three-dimensional domain

$$\mathbf{S}^1\{\alpha \mod 2\pi\}\backslash\{\alpha = 0, \alpha = \pi\} \times \mathbf{R}^2\{z_1, z_2\} \tag{5}$$

(such a system can also be reduced to an equivalent system on the tangent bundle $T_*\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$) and describing the spatial motion of a rigid body in a resisting medium (see [1], [3]), we put in correspondence the following nonautonomous system with algebraic right-hand side: ($\tau = \sin\alpha$):

$$\frac{dz_2}{d\tau} = \frac{\tau - z_1^2/\tau}{-z_2 + b\tau},\ \frac{dz_1}{d\tau} = \frac{z_1 z_2/\tau}{-z_2 + b\tau}. \tag{6}$$

It is seen that system (4) is a zero mean variable dissipation system; in order to achieve complete correspondence with the definition, it suffices to introduce a new phase variable $z_1^* = \ln|z_1|$.

If we calculate the divergence of the right-hand side of the system (4) in Cartesian coordinates α, z_1^*, z_2, then we shall get that it is equal to $b\cos\alpha$. Herewith, if we consider (5), we shall have in the sense of principal value:

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi-\varepsilon} b\cos\alpha + \lim_{\varepsilon\to 0}\int_{\pi+\varepsilon}^{2\pi-\varepsilon} b\cos\alpha = 0.$$

Moreover, the system (4) has two first integrals (i.e., a complete list) that are transcendental functions and we expressed through a finite combination of elementary functions (see [1], [4]), which, as was mentioned above, becomes possible after putting in correspondence to it a (nonautonomous in general) system of equations with algebraic (polynomial) right-hand (6).

The above-presented system (4), along with the property that they belong to the class of systems (1) and are zero mean variable dissipation systems, also have a complete list of transcendental first integrals expressed through a finite combination of elementary functions.

Therefore, to search for the first integrals of the system considered, it is better to reduce systems of the form (1) to systems (3) with polynomial right-hand sides, on whose form the possibility of integration in elementary functions of the initial system depends. Therefore, we proceed as follows: we seek sufficient conditions for integrability in elementary functions of systems of equations with polynomial right-hand sides studying systems of the most general form in this process.

IV. Systems on Tangent Bundle of Two-Dimensional Sphere

Let consider the following dynamic system

$$\begin{aligned} &\ddot{\theta} + b\dot{\theta}\cos\theta + \sin\theta\cos\theta - \dot{\psi}^2\frac{\sin\theta}{\cos\theta} = 0, \\ &\ddot{\psi} + b\dot{\psi}\cos\theta + \dot{\theta}\dot{\psi}\left[\frac{1+\cos^2\theta}{\sin\theta\cos\theta}\right] = 0 \end{aligned} \tag{7}$$

on tangent bundle $T_*\mathbf{S}^2$ of two-dimensional sphere $\mathbf{S}^2\{\theta, \psi\}$. This system describes the spherical pendulum, placed in the accumulating medium flow (see also [1], [5]). Herewith, both conservative moment is present in the system $\sin\theta\cos\theta$, and the force moment depending on the velocity as linear one with the variable coefficient: $b\begin{pmatrix}\dot{\theta}\\ \dot{\psi}\end{pmatrix}\cos\theta$.

The coefficients remaining in the equations are the coefficients of connectedness, i.e., $\Gamma^\theta_{\psi\psi} = -\frac{\sin\theta}{\cos\theta}$, $\Gamma^\psi_{\theta\psi} = \frac{1+\cos^2\theta}{\sin\theta\cos\theta}$.

The system (7) has an order 3 practically, since the variable ψ is a cyclic, herewith, the derivative $\dot{\psi}$ is present in the system only.

Theorem 2: The equation

$$\dot{\psi} = 0 \tag{8}$$

define the family of integral planes for the system (7).

Furthermore, the equation (8) reduces the system (7) to an equation describing the cylindric pendulum which placed in the accumulating medium flow (see also [1], [6]).

Theorem 3: The system (7) is equivalent to the following system:

$$\begin{gathered}\dot\theta = -z_2 + b\sin\theta,\ \dot{z_2} = \sin\theta\cos\theta - z_1^2\frac{\cos\theta}{\sin\theta},\\ \dot{z_1} = z_1 z_2\frac{\cos\theta}{\sin\theta},\ \dot\psi = z_1\frac{\cos\theta}{\sin\theta},\end{gathered} \tag{9}$$

on the tangent bundle $T_*\mathbf{S}^2\{z_1, z_2, \theta, \psi\}$ of two-dimensional sphere $\mathbf{S}^2\{\theta, \psi\}$.

Moreover, the first three equations of the system (9) form the closed three order system and coincide with the equations of the system (4) (if we denote $\alpha = \theta$). The separation of fourth equation of the system (9) has also occurred by the reason of cyclicity of the variable ψ.

V. Certain Generalizations

Let us pose the following question: What are the possibilities of integrating in elementary functions the system

$$\begin{aligned}\frac{dz}{dx} &= \frac{ax + by + cz + c_1 z^2/x + c_2 zy/x + c_3 y^2/x}{d_1 x + ey + fz},\\ \frac{dy}{dx} &= \frac{gx + hy + iz + i_1 z^2/x + i_2 zy/x + i_3 y^2/x}{d_1 x + ey + fz},\end{aligned} \tag{10}$$

of a more general form, which includes the system (6) considered above in three-dimensional phase domains and which has a singularity of the form $1/x$?

Previously, a number of results concerning this question were already obtained (see also [1]). Let us present these results and complement them by original arguments.

As previously, introducing the substitutions $y = ux,\ z = vx$, we obtain that system (10) reduces to the following system

$$\begin{gathered}x\frac{dv}{dx} + v =\\ = \frac{ax + bux + cvx + c_1 v^2 x + c_2 vux + c_3 u^2 x}{d_1 x + eux + fvx},\end{gathered} \tag{11}$$

$$\begin{gathered}x\frac{du}{dx} + u =\\ = \frac{gx + hux + ivx + i_1 v^2 x + i_2 vux + i_3 u^2 x}{d_1 x + eux + fvx},\end{gathered} \tag{12}$$

we put in correspondence the following equation with algebraic right-hand side:

$$\begin{gathered}\frac{dv}{du} =\\ = \frac{a + bu + cv + c_1 v^2 + c_2 vu + c_3 u^2 - v[d_1 + eu + fv]}{g + hu + iv + i_1 v^2 + i_2 vu + i_3 u^2 - u[d_1 + eu + fv]}.\end{gathered}$$

The integration of the latter equation reduces to that of the following equation in total differentials:

$$\begin{aligned}&[g + hu + iv + i_1 v^2 + i_2 vu +\\ &+ i_3 u^2 - d_1 u - eu^2 - fuv]dv =\\ &= [a + bu + cv + c_1 v^2 + c_2 vu +\\ &+ c_3 u^2 - d_1 v - euv - fv^2]du.\end{aligned} \tag{13}$$

We have (in general) a 15-parameter family of equations of the form (13). To integrate the latter identity in elementary functions as a homogeneous equation, it suffices to impose seven relations

$$g = 0,\ i = 0,\ i_1 = 0,\ e = c_2,\ h = c,\ i_2 = 2c_1 - f. \tag{14}$$

Introduce nine parameters $\beta_1, \dots, \beta_9$ and consider them as independent parameters:

$$\begin{gathered}\beta_1 = a,\ \beta_2 = b,\ \beta_3 = c,\ \beta_4 = c_1,\ \beta_5 = c_2,\\ \beta_6 = c_3,\ \beta_7 = d_1,\ \beta_8 = f,\ \beta_9 = i_3.\end{gathered} \tag{15}$$

Therefore, under the group of conditions (14), and (15), Eq. (13) reduces to the form

$$\frac{dv}{du} = \frac{\beta_1 + \beta_2 u + (\beta_3 - \beta_7)v + (\beta_4 - \beta_8)v^2 + \beta_6 u^2}{(\beta_3 - \beta_7)u + 2(\beta_4 - \beta_8)vu + (\beta_9 - \beta_5)u^2}, \tag{16}$$

and the system (11), (12), respectively, to the form

$$\begin{gathered}x\frac{dv}{dx} =\\ = \frac{\beta_1 + \beta_2 u + (\beta_3 - \beta_7)v + (\beta_4 - \beta_8)v^2 + \beta_6 u^2}{\beta_7 + \beta_5 u + \beta_8 v},\end{gathered} \tag{17}$$

$$x\frac{du}{dx} = \frac{(\beta_3 - \beta_7)u + 2(\beta_4 - \beta_8)vu + (\beta_9 - \beta_5)u^2}{\beta_7 + \beta_5 u + \beta_8 v}, \tag{18}$$

after that, the equation (16) is integrated in finite combination of elementary functions.

Indeed, integrating the identity (13), we obtain the relation

$$\begin{gathered}d\left[\frac{(\beta_3 - \beta_7)v}{u}\right] + d\left[\frac{(\beta_4 - \beta_8)v^2}{u}\right] + d[(\beta_9 - \beta_5)v] + d\left[\frac{\beta_1}{u}\right] -\\ - d[\beta_2 \ln|u|] - d[\beta_6 u] = 0,\end{gathered}$$

which for the beginning allows us to obtain the following invariant relation:

$$\begin{gathered}\frac{(\beta_3 - \beta_7)v}{u} + \frac{(\beta_4 - \beta_8)v^2}{u} + (\beta_9 - \beta_5)v + \frac{\beta_1}{u} -\\ - \beta_2 \ln|u| - \beta_6 u = C_1 = \text{const},\end{gathered} \tag{19}$$

and in the coordinates (x, y, z), it allows us to obtain the first integral in the form

$$\begin{gathered}\frac{(\beta_4 - \beta_8)z^2 - \beta_6 y^2 + (\beta_3 - \beta_7)zx + (\beta_9 - \beta_5)zy + \beta_1 x^2}{yx} -\\ - \beta_2 \ln\left|\frac{y}{x}\right| = \text{const}.\end{gathered} \tag{20}$$

Therefore, we can make a conclusion about the integrability in elementary functions of the following, in general, nonconservative third-order system depending on nine parameters:

$$\begin{aligned}\frac{dz}{dx} &=\\ &= \frac{\beta_1 x + \beta_2 y + \beta_3 z + \beta_4 z^2/x + \beta_5 zy/x + \beta_6 y^2/x}{\beta_7 x + \beta_5 y + \beta_8 z},\\ \frac{dy}{dx} &= \frac{\beta_3 y + (2\beta_4 - \beta_8)zy/x + \beta_9 y^2/x}{\beta_7 x + \beta_5 y + \beta_8 z}.\end{aligned}$$

Corollary. The third-order system

$$\begin{gathered}\dot{\alpha} = \beta_7 \sin\alpha + \beta_5 z_1 + \beta_8 z_2,\\ \dot{z_2} = \beta_1 \sin\alpha\cos\alpha + \beta_2 z_1 \cos\alpha + \beta_3 z_2 \cos\alpha + \\ + \beta_4 z_2^2 \frac{\cos\alpha}{\sin\alpha} + \beta_5 z_1 z_2 \frac{\cos\alpha}{\sin\alpha} + \beta_6 z_1^2 \frac{\cos\alpha}{\sin\alpha},\\ \dot{z_1} = \beta_3 z_1 \cos\alpha + (2\beta_4 - \beta_8) z_1 z_2 \frac{\cos\alpha}{\sin\alpha} + \beta_9 z_1^2 \frac{\cos\alpha}{\sin\alpha},\end{gathered} \quad (21)$$

on the set

$$\mathbf{S}^1\{\alpha \mod 2\pi\}\backslash\{\alpha = 0, \alpha = \pi\} \times \mathbf{R}^2\{z_1, z_2\},$$

which depends on nine parameters, has a first integral, which is in general transcendental and expressed through elementary functions:

$$\frac{A}{z_1 \sin\alpha} - \beta_2 \ln\left|\frac{z_1}{\sin\alpha}\right| = \text{const}, \quad (22)$$

$$A = (\beta_4 - \beta_8) z_2^2 - \beta_6 z_1^2 +$$

$$+ (\beta_3 - \beta_7) z_2 \sin\alpha + (\beta_9 - \beta_5) z_2 z_1 + \beta_1 \sin^2\alpha^2.$$

In particular, for $\beta_1 = 1$, $\beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_9 = 0$, $\beta_6 = \beta_8 = -1$, $\beta_7 = b$ system (21) reduces to system (4).

To find the additional first integral of the nonautonomous system (10), we use the found first integral (20), which is expressed through a finite combination of elementary functions.

For the beginning let transform the relation (19) as follows:

$$(\beta_4 - \beta_8) v^2 + [(\beta_9 - \beta_5) u + (\beta_3 - \beta_7)]\, v + f_1(u) = 0, \quad (23)$$

where

$$f_1(u) = \beta_1 - \beta_6 u^2 - \beta_2 u \ln|u| - C_1 u.$$

Herewith, the value v formally can be found from the equality

$$v_{1,2}(u) =$$

$$= \frac{1}{2(\beta_4 - \beta_8)} \left\{ (\beta_5 - \beta_9) u + (\beta_7 - \beta_3) \pm \sqrt{f_2(u)} \right\}, \quad (24)$$

where

$$f_2(u) = A_1 + A_2 u + A_3 u^2 + A_4 u \ln|u|,$$

$$A_1 = (\beta_3 - \beta_7)^2 - 4\beta_1(\beta_4 - \beta_8),$$

$$A_2 = 2(\beta_9 - \beta_5)(\beta_3 - \beta_7) + 4C_1(\beta_4 - \beta_8),$$

$$A_3 = (\beta_9 - \beta_5)^2 + 4\beta_6(\beta_4 - \beta_8), \;\; A_4 = 4\beta_2(\beta_4 - \beta_8).$$

Then the quadrature studied for the search of additional (in general) transcendental first integral (for example, of the system (17), (18), herewith, the equation (18) is used) has the following form

$$\int \frac{dx}{x} =$$

$$= \int \frac{[\beta_7 + \beta_5 u + \beta_8 v_{1,2}(u)] du}{(\beta_3 - \beta_7) u + (\beta_9 - \beta_5) u^2 + 2(\beta_4 - \beta_8) u v_{1,2}(u)} =$$

$$= \int \frac{[B_1 + B_2 u + B_3 \sqrt{f_2(u)}] du}{B_4 u \sqrt{f_2(u)}}, \quad (25)$$

$B_k = \text{const}$, $k = 1, \ldots, 4$. And the quadrature studied for the search of additional (in general) transcendental first integral (for example, of the system (17), (18), herewith, the equation (17) is used) has the following form

$$\int \frac{dx}{x} = \int \frac{[\beta_7 + \beta_5 u(v) + \beta_8 v] dv}{B},$$

$$B = \beta_1 + \beta_2 u(v) + (\beta_3 - \beta_7) v + (\beta_4 - \beta_8) v^2 + \beta_6 u^2(v),$$

herewith, the function $u(v)$ should be obtain as a result of resolving of implicit equation (19) respectively to u (that, in general case, is not always evident).

Theorem 4: The indefinite integral in (25) is expressed through the finite combination of elementary functions for $A_4 = 0$, i.e., either

$$\beta_2 = 0 \quad (26)$$

or $\beta_4 = \beta_8$.

Theorem 5: The system (21) under assumption of necessary conditions of theorem 4 (the property (26) holds in given case), has the compete set of first integrals, which expressing through the finite combination of elementary functions.

Therefore, the dynamical systems considered in this work refer to zero mean variable dissipation systems in the existing periodic coordinate, and such systems often have a complete list of first integrals expressed through elementary functions.

VI. Conclusion

So, it was shown above the certain cases of complete integrability in spatial dynamics of a rigid body motion in a nonconservative field. Herewith, we dealt with with three properties, which are independent for the first glance: (i) the distinguished class of systems (1) with the symmetries above; (ii) the fact that this class of systems consists of systems with zero mean variable dissipation (in the variable α), which allows us to consider them as "almost" conservative systems; (iii) in certain (although lower-dimensional) cases, these systems have the complete tuple of first integrals, which are transcendental (from the viewpoint of complex analysis).

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