Cases of integrability corresponding to the motion of a pendulum in the three-dimensional space

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1. Introduction – We systematize some results on the study of the equations of spatial motion of dynamically symmetric fixed rigid bodies-pendulums located in the nonconservative force fields. The form of these equations is taken from the dynamics of real fixed rigid bodies placed in a homogeneous flow of a medium. In parallel, we study the problem of a spatial motion of a free rigid body also located in the similar force fields. Herewith, this free rigid body is influenced by a nonconservative tracing force; under action of this force, either the magnitude of the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo-constraint, or the center of mass of the body moves rectilinearly and uniformly; this means that there exists a nonconservative couple of forces.

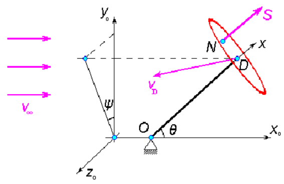

Image 1. Spatial pendulum in a jet flow

2. Preliminary results – Earlier [1], the author already proved the complete integrability of the equations of a plane-parallel motion of a fixed rigid body-pendulum in a homogeneous flow of a medium under the jet flow conditions when the system of dynamical equations possesses a first integral, which is a transcendental function of quasi-velocities. It was assumed that the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a (one-dimensional) plate.

In sequel [2], the planar problem was generalized to the spatial (three-dimensional) case, where the system of dynamical equations has a complete set of transcendental first integrals. It was assumed that the interaction of the homogeneous medium flow with the fixed body (the spherical pendulum) is concentrated on a part of the body surface that has the form of a planar (two-dimensional) disk: Image 1.

In this activity, the results relate to the case where all interaction of the homogeneous flow of a medium with the fixed body is concentrated on that part of the surface of the body, which has the form of a two-dimensional disk, and the action of the force is concentrated in a direction perpendicular to this disk.

3. Model assumptions and equations – Let consider the homogeneous plane circle disk (with the center in the point D), the plane of which perpendicular to the holder OD. The disk is rigidly fixed perpendicular to the tool holder OD located on the spherical hinge O, and it flows about homogeneous fluid flow (Image 1). In this case, the body is a physical (spherical) pendulum. The medium flow moves from infinity with constant $\boldsymbol{v}$. Assume that the holder does not create a resistance.

We suppose that the total force $\boldsymbol{S}$ of medium flow interaction is parallel to the holder, and point N of application of this force is determined by at least the angle of attack α, which is made by the velocity vector $\boldsymbol{v}_D$ of the point D with respect to the flow and the holder OD; the total force is also determined by the angle β, which is made in the plane of the disk D (thus, (v,α,β) are the spherical coordinates of the tip of the vector $\boldsymbol{v}_D$, and also the reduced angular velocity $\omega = l\Omega/vD$ (l is the length of the holder, Ω is the angular velocity of the pendulum). Such conditions arise when one uses the model of streamline flow around spatial bodies [1].

Let $Dx_1x_2x_3 = Dxyz$ be the coordinate system rigidly attached to the body, herewith, the axis $Dx = Dx_1$ has a direction vector, and the axes $Dx_2 = Dy$ and $Dx_3 = Dz$ lie in the plane of the disk D. In the same figure it is shown the angles $\theta = \xi$, $\psi = \eta$, i.e., the angles determining the pendulum position on the sphere.

If diag$\{I_1,I_2,I_2\}$ is the tensor of inertia of the body-pendulum in the coordinate system $Dx_1x_2x_3$ then the general equation of its motion has the following form:

$$I_1\Omega_1^{\bullet}=0\,;\; I_2\Omega_2^{\bullet}+(I_1-I_2)\Omega_1\Omega_3=-z_N s(\alpha)v^2\,;\; I_2\Omega_3^{\bullet}+(I_2-I_1)\Omega_1\Omega_2=y_N s(\alpha)v^2\,, \qquad (1)$$

where $\{-s(\alpha)v_D^2,0,0\}$ is the decomposition of the medium interaction force $\boldsymbol{S}$ in system $Dx_1x_2x_3$. We see, that in the right-hand side of Eqs. (1), first of all, it includes the angles α, β, therefore, this system of equations is not closed. In order to obtain a complete system of equations of motion of the pendulum, it is necessary to attach several sets of kinematic equations to the dynamic equation on the Lie algebra so(3).
We immediately note that the system (1), by the existing dynamic symmetry $I_2 = I_3$, possesses the cyclic first integral $\Omega_1=\Omega_{10}=\text{const}$. Further, we consider the dynamics of our system at zero level: $\Omega_{10}=0$.
In order to obtain a complete system of equations of motion, it needs the set of kinematic equations which relate the velocities of the point D and the over-running medium flow:

$$\boldsymbol{v}_D = v_D\,\boldsymbol{i}_v(\alpha,\beta) = \boldsymbol{\Omega}\times\{l,0,0\} + (-\boldsymbol{v})\,\boldsymbol{i}_v(\xi,\eta),\; \boldsymbol{i}_v(\alpha,\beta) = \{\cos\alpha,\sin\alpha\cos\beta,\,\sin\alpha\sin\beta\}. \qquad (2)$$

We also have the second set of kinematic equations:

$$\Omega_2=-\xi^{\bullet}\sin\eta-\eta^{\bullet}tg\xi\cos\eta,\;\Omega_3=\xi^{\bullet}\cos\eta-\eta^{\bullet}tg\xi\sin\eta\,. \qquad (3)$$

And now we see that three sets of the relations (1)–(3) form the closed system of equations.

4. Theorem – We take the function $\boldsymbol{r}_N$ as follows (the disk is given by the equation $x_{1N} = 0$): $\boldsymbol{r}_N = \{0,x_{2N},x_{3N}\} = R(\alpha)\boldsymbol{i}_N(\alpha,\beta)$, $\boldsymbol{i}_N(\alpha,\beta) = \boldsymbol{i}_v(\pi/2,\beta)$. Thus, the equalities $x_{2N} = R(\alpha)\cos\beta$, $x_{3N} = R(\alpha)\sin\beta$ hold and show that for the considered system, the moment of the nonconservative forces is independent of the angular velocity (it depends only on the angles α, β).
And so, for the construction of the force field, we use the pair of dynamical functions $R(\alpha)$, $s(\alpha)$; the information about them is of a qualitative nature. Similarly to the choice of the Chaplygin analytical functions (see [1, 2]), we take the dynamical functions s and R as follows: $R(\alpha) = A\sin\alpha$, $s(\alpha) = B\cos\alpha$, A, $B > 0$.
Theorem 1. *The simultaneous equations (1)–(3), under conditions above can be reduced to the dynamical system on the tangent bundle of the two-dimensional sphere.*
Indeed, if we introduce the dimensionless parameter and the differentiation by the formulas $b_* = ln_0$, $n_0^2 = AB/I_2$, <•> = n_0v <'>, then the obtained equations have the following form:

$$\theta^{\bullet\bullet}+b\theta^{\bullet}\cos\theta+\sin\theta\cos\theta-\psi^{\bullet 2}\frac{\sin\theta}{\cos\theta}=0,\;\psi^{\bullet\bullet}+b\psi^{\bullet}\cos\theta+\theta^{\bullet}\psi^{\bullet}\left[\frac{1+\cos^2\theta}{\cos\theta\sin\theta}\right]=0\,. \qquad (4)$$

The phase pattern of the reduced system (4) ($\xi\leftrightarrow\theta$, $\eta\leftrightarrow\psi$) is shown in Image 2.

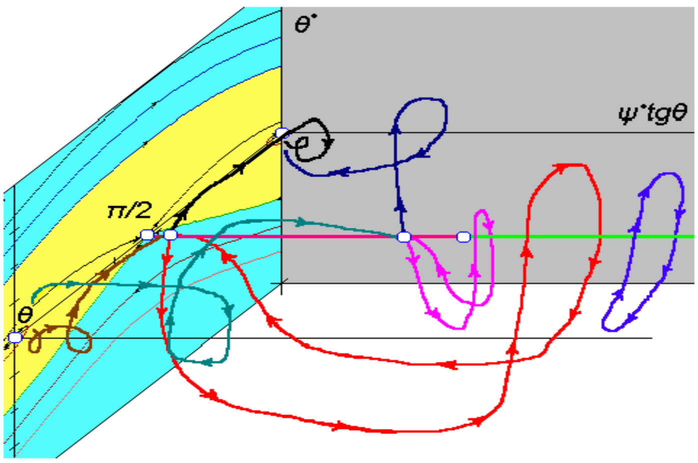

Image 2. Phase pattern of a pendulum in a jet flow

5. Conclusions – Our system possesses the full set of transcendental first integrals expressing through the finite combination of elementary functions.

Furthermore, we have the following topological and mechanical analogies in the sense explained above.

(1) A motion of a fixed physical pendulum on a spherical hinge in a flowing medium (nonconservative force fields under assumption of additional dependence of the moment of the forces on the angular velocity).

(2) A spatial free motion of a rigid body in a nonconservative force field under a tracing force (in the presence of a nonintegrable constraint under assumption of additional dependence of the moment of the forces on the angular velocity).

(3) A spatial composite motion of a rigid body rotating about its center of mass, which moves rectilinearly and uniformly, in a nonconservative force field under assumption of additional dependence of the moment of the forces on the angular velocity.

On more general topological analogues, see also [1, 2].

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7. References

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