# On lower- and multi-dimensional pendulum in a nonconservative force fields

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*Abstract:* I showed the integrability of the equations of the plane-parallel motion of a pendulum in a resisting medium, when the first integral, which is the transcendental function of quasi-velocities, was explicitly found for the set of the dynamic equations. In this case, the total interaction of the medium with a rigid body is concentrated on that part of the surface that which has the shape of a one-dimensional plate. Then the problem was generalized to the spatial case, the complete set of transcendental first integrals being found explicitly for the set of dynamic equations. Here already the total interaction of the medium with a rigid body is concentrated on that portion of its surface that has the shape of a flat disk.

## 1. Model assumptions

Let consider the homogeneous flat plate $AB$ symmetrical relative to the plane which perpendicular to the plane of figure and passing through the holder $OD$. The plate is rigidly fixed perpendicular to the tool holder $OD$ located on the cylindrical hinge $O$, and it flows about homogeneous fluid flow (Fig. 1). In this case, the body is a physical pendulum, in which the plate $AB$ and the pivot axis perpendicular to the plane of motion. The medium flow moves from infinity with constant velocity $\mathbf{v} = \mathbf{v}_\infty \neq \mathbf{0}$. Assume that the holder does not create a resistance [1, 2].

I suppose that the total force $\mathbf{S}$ of medium flow interaction is parallel to the holder, and point $N$ of application of this force is determined by at least the angle of attack $\alpha$, which is made by the velocity vector $\mathbf{v}_D$ of the point $D$ with respect to the flow and the holder (Fig. 1, wherein the figure shows the angle of attack equal to $\pi - \alpha$), and also the reduced angular velocity $\omega \cong l\Omega/v_D,\ v_D = |\mathbf{v}_D|$ ($l$ is the length of the holder, $\Omega$ is the algebraic value of a projection of the pendulum angular velocity to the axle hinge). Such conditions arise when one uses the model of streamline flow around plane bodies [5].

The vector $\mathbf{e} = \mathbf{OD}/l$ determines the orientation of the holder. Then $\mathbf{S} = -s(\alpha)v_D^2\mathbf{e}$, where $s(\alpha) = s_1(\alpha)\text{sign}\cos\alpha$, and the resistance coefficient $s_1 \geq 0$ depends only on the angle of attack $\alpha$. By the plate symmetry properties with respect to the point $D$, the function $s(\alpha)$ is even. Let $Dx_1x_2 = Dxy$ be the coordinate system rigidly attached to the body,

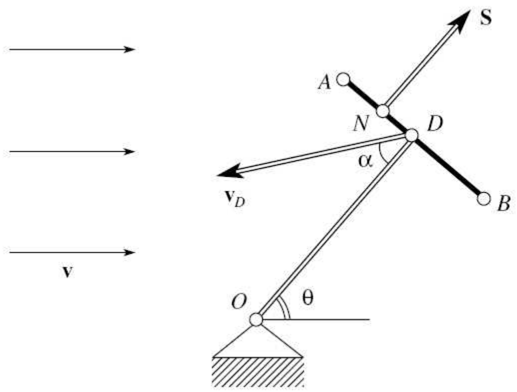


**Figure 1.** Fixed a pendulum on a cylindrical hinge in the stream running medium

herewith, the axis $Dx = Dx_1$ has a direction vector $\mathbf{e}$, and the axis $Dx_2 = Dy$ has the same direction with the vector $\mathbf{DA}$ (Fig. 1). In the same figure it is shown the angle $\theta = \xi$, i.e., the pendulum angle. The space of positions of this physical pendulum is the circle (one-dimensional sphere)

$$\mathbf{S}^1\{\xi \in \mathbf{R}^1 : \ \xi \bmod 2\pi\}, \tag{1}$$

and its phase space is the tangent bundle of a circle

$$T_*\mathbf{S}^1\{(\dot{\xi};\xi) \in \mathbf{R}^2 : \ \xi \bmod 2\pi\}, \tag{2}$$

i.e., two-dimensional cylinder.

To the value $\Omega$, I put in correspondence the skew-symmetric matrix

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\Omega \\ \Omega & 0 \end{pmatrix}, \ \tilde{\Omega} \in \mathrm{so}(2).$$

The distance from the center $D$ of the plate to the center of pressure (the point $N$, Fig. 1) has the form $|\mathbf{r}_N| = r_N = DN\left(\alpha, l\Omega/v_D\right),$ where $\mathbf{r}_N = \{0, x_{2N}\} = \{0, y_N\}$ in system $Dx_1x_2 = Dxy$.

## 2. Set of dynamical equations in Lie algebra so(2)

If $I$ is a central moment of inertia of a rigid body–pendulum then the general equation of motion has the following form:

$$I\dot{\Omega} = DN\left(\alpha, \frac{l\Omega}{v_D}\right) s(\alpha) v_D^2, \tag{3}$$

where $\{-s(\alpha)v_D^2, 0\}$ is the decomposition of the medium interaction force $\mathbf{S}$ in the coordinate system $Dx_1x_2$.

Since the dimension of the Lie algebra so(2) is equal to 1, the single equation (3) is a group equations on so(2), and, simply speaking, the motion equation.

I can see, that in the right-hand side of Eq. (3), first of all, it includes the angle of attack, therefore, this equation is not closed. In order to obtain a complete system of equations of motion of the pendulum, it is necessary to attach several sets of kinematic equations to the dynamic equation on the Lie algebra so(2).

### 3. First set of kinematic equations

In order to obtain a complete system of equations of motion, it needs the set of kinematic equations which relate the velocities of the point $D$ (i.e., the formal center of the plate $AB$) and the over-running medium flow:

$$\mathbf{v}_D = v_D \cdot \mathbf{i}_v(\alpha) = \tilde{\Omega} \begin{pmatrix} l \\ 0 \end{pmatrix} + (-v_\infty)\mathbf{i}_v(-\xi), \tag{4}$$

$$\mathbf{i}_v(\alpha) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}. \tag{5}$$

The equation (4) expresses the theorem of addition of velocities in projections on the related coordinate system $Dx_1x_2$.

Indeed, the left-hand side of Eq. (4) is the velocity of the point $D$ of the pendulum with respect to the flow in the projections on the related with the pendulum coordinate system $Dx_1x_2$. Herewith, the vector $\mathbf{i}_v(\alpha)$ is the unit vector along the axis of the vector $\mathbf{v}_D$. The vector $\mathbf{i}_v(\alpha)$ is the image of the unit vector along the axis $Dx_1$, rotated around the vertical (the axis $Dx_3$) by the angle $\alpha$ and has the decomposition (5).

The right-hand side of the Eq. (4) is the sum of the velocities of the point $D$ when you rotate the pendulum (the first term), and the motion of the flow (the second term). In this case, in the first term, I have the coordinates of the vector $\mathbf{OD} = \{l, 0\}$ in the coordinate system $Dx_1x_2$.

I explain the second term of the right-hand side of Eq. (4) in more detail. I have in it the coordinates of the vector $(-\mathbf{v}_\infty) = \{-v_\infty, 0\}$ in the immovable space. In order to describe it in the projections on the related coordinate system $Dx_1x_2$, I need to make a (reverse) rotation of the pendulum at the angle $(-\xi)$ that is algebraically equivalent to multiplying the value $(-v_\infty)$ on the vector $\mathbf{i}_v(-\xi)$.

Thus, the first set of kinematic equations (4) has the following form in our case:

$$\begin{aligned} &v_D \cos\alpha = -v_\infty \cos\xi, \\ &v_D \sin\alpha = l\Omega + v_\infty \sin\xi. \end{aligned} \tag{6}$$

## 4. Second set of kinematic equations

I also need a set of kinematic equations which relate the angular velocity tensor $\tilde{\Omega}$ and coordinates $\dot{\xi}, \xi$ of the phase space (2) of pendulum studied, i.e., the tangent bundle $T_*\mathbf{S}^1\{\dot{\xi};\xi\}$.

I draw the reasoning style allowing arbitrary dimension. The desired equations are obtained from the following two sets of relations. Since the motion of the body takes place in a Euclidean space $\mathbf{E}^n, n = 2$ formally, at the beginning, I express the tuple consisting of a phase variable $\Omega$, through new variable $z_1$ (from the tuple $z$):

$$\Omega = z_1. \tag{7}$$

Then I substitute the following relationship instead of the variable $z$:

$$z_1 = \dot{\xi}. \tag{8}$$

Thus, two sets of Eqs. (7) and (8) give the second set of kinematic equations:

$$\Omega = \dot{\xi}. \tag{9}$$

I see that three sets of the relations (3), (6), and (9) form the closed system of equations. These three sets of equations include the following two functions: $r_N = DN(\alpha, l\Omega/v_D)$, $s(\alpha)$. In this case, the function $s$ is considered to be dependent only on $\alpha$, and the function $r_N = DN$ may depend on, along with the angle $\alpha$, generally speaking, the reduced angular velocity $\omega \cong l\Omega/v_D$.

## 5. Problem on free body motion under assumption of tracing force

Parallel to the present problem of the motion of the fixed body, we study the plane-parallel motion of the free symmetric rigid body with the frontal plane butt-end (one-dimensional plate $AB$) in the resistance force fields under the quasi-stationarity conditions with the same model of medium interaction (Fig. 2).

If $(v, \alpha)$ are the polar coordinates of the velocity vector of the certain characteristic point $D$ of the rigid body ($D$ is the center of the plate $AB$), $\Omega$ is the value of its angular velocity, $I, m$ are characteristics of inertia and mass, then the dynamical part of the equations of motion in which the tangent forces of the interaction of the body with the medium are

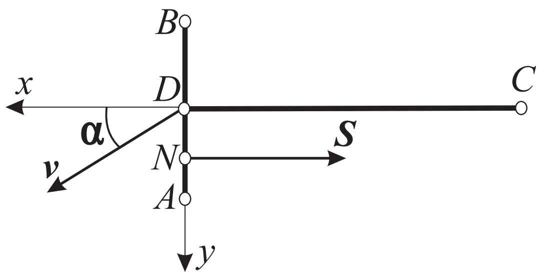


**Figure 2.** Plane-parallel motion of the free symmetric rigid body in a resisting medium

absent, has the form

$$\begin{aligned} \dot v\cos\alpha - \dot\alpha v\sin\alpha - \Omega v\sin\alpha + \sigma\Omega^2 &= \tfrac{F_x}{m}, \\ \dot v\sin\alpha + \dot\alpha v\cos\alpha + \Omega v\cos\alpha - \sigma\dot\Omega &= 0, \\ I\dot\Omega &= y_N\left(\alpha, \tfrac{\Omega}{v}\right) s(\alpha)v^2, \end{aligned} \tag{10}$$

where $F_x = -S,\ S = s(\alpha)v^2,\ \sigma = CD$, in this case $(0, y_N(\alpha, \Omega/v))$ are the coordinates of the point $N$ of application of the force $\mathbf{S}$ in the coordinate system $Dx_1x_2 = Dxy$ related to the body (Fig. 2).

The first two equations of the system (10) describe the motion of the center of a mass in the two-dimensional Euclidean plane $\mathbf{E}^2$ in the projections on the coordinate system $Dx_1x_2$. In this case, $Dx_1 = Dx$ is the perpendicular to the plate passing through the center of mass $C$ of the symmetric body and $Dx_2 = Dy$ is an axis along the plate. The third equation of the system (10) is obtained from the theorem on the change of the angular moment of a rigid body in the projection on the axis perpendicular to the figure.

Thus, the direct product $\mathbf{R}^1 \times \mathbf{S}^1 \times \mathrm{so}(2)$ of the two-dimensional cylinder and the Lie algebra so(2) is the phase space of third-order system (10) of the dynamical equations. Herewith, since the medium influence force dos not depend on the position of the body in a plane, the system (10) of the dynamical equations *is separated from the system of kinematic equations* and may be studied independently (see also [3, 4]).

### 5.1. Nonintegrable constraint

If I consider *a more general problem* on the motion of a body under the action of a certain tracing force $\mathbf{T}$ passing through the center of mass and providing the fulfillment of the equality

$$v \equiv \mathrm{const}, \tag{11}$$

during the motion, then $F_x$ in system (10) must be replaced by $T - s(\alpha)v^2$.

As a result of an appropriate choice of the magnitude $T$ of the tracing force, I can achieve the fulfillment of Eq. (11) during the motion. Indeed, if I formally express the value $T$ by virtue of system (10), I obtain (for $\cos\alpha \neq 0$):

$$T = T_v(\alpha, \Omega) = m\sigma\Omega^2 + s(\alpha)v^2 \left[1 - \frac{m\sigma}{I} y_N \left(\alpha, \frac{\Omega}{v}\right) \frac{\sin\alpha}{\cos\alpha}\right].$$

This procedure can be viewed from two standpoints. First, a transformation of the system has occurred at the presence of the tracing (control) force in the system which provides the corresponding class of motions (11). Second, I can consider this procedure as a procedure that allows one to reduce the order of the system. Indeed, system (10) generates an independent second-order system of the following form:

$$\begin{aligned} &\dot{\alpha} v \cos\alpha + \Omega v \cos\alpha - \sigma\dot{\Omega} = 0, \\ &I\dot{\Omega} = y_N \left(\alpha, \frac{\Omega}{v}\right) s(\alpha)v^2, \end{aligned} \qquad (12)$$

where the parameter $v$ is supplemented by the constant parameters specified above.

I can see from (12) that the system cannot be solved uniquely with respect to $\dot{\alpha}$ on the manifold

$$O = \left\{(\alpha, \Omega) \in \mathbf{R}^2 : \ \alpha = \frac{\pi}{2} + \pi k, \ k \in \mathbf{Z}\right\} \qquad (13)$$

Thus, formally speaking, the uniqueness theorem is violated on manifold (13).

This implies that system (12) outside of the manifold (13) (and only outside it) is equivalent to the following system:

$$\begin{aligned} &\dot{\alpha} = -\Omega + \frac{\sigma v}{I} \frac{y_N \left(\alpha, \frac{\Omega}{v}\right) s(\alpha)}{\cos\alpha}, \\ &\dot{\Omega} = \frac{1}{I} y_N \left(\alpha, \frac{\Omega}{v}\right) s(\alpha)v^2. \end{aligned} \qquad (14)$$

The uniqueness theorem is violated for system (12) on the manifold (13) in the following sense: regular phase trajectories of system (12) pass through almost all points of the manifold (13) and intersect the manifold (13) at a right angle, and also there exists a phase trajectory that completely coincides with the specified point at all time instants. However, these trajectories are different since they correspond to different values of the tracing force.

### 5.2. Constant velocity of the center of mass

If I consider *a more general problem* on the motion of a body under the action of a certain tracing force $\mathbf{T}$ passing through the center of mass and providing the fulfillment of the equality

$$\mathbf{V}_C \equiv \mathbf{const} \tag{15}$$

($\mathbf{V}_C$ is the velocity of the center of mass), then $F_x$ in system (10) must be replaced by zero since the nonconservative couple of the forces acts on the body: $T - s(\alpha)v^2 \equiv 0$.

Obviously, I must choose the value of the tracing force $T$ as follows:

$$T = T_v(\alpha, \Omega) = s(\alpha)v^2, \ \mathbf{T} \equiv -\mathbf{S}. \tag{16}$$

The choice (16) of the magnitude of the tracing force $T$ is a particular case of the possibility of separation of an independent second-order subsystem after a certain transformation of the third-order system (10). Indeed, let the following condition hold for $T$:

$$T = T_v(\alpha, \Omega) = \tau_1\left(\alpha, \frac{\Omega}{v}\right)v^2 + \tau_2\left(\alpha, \frac{\Omega}{v}\right)\Omega v + \tau_3\left(\alpha, \frac{\Omega}{v}\right)\Omega^2 = T_1\left(\alpha, \frac{\Omega}{v}\right)v^2.$$

I can rewrite system (10) as follows:

$$\dot v + \sigma\Omega^2\cos\alpha - \sigma\sin\alpha\left[\frac{v^2}{I}y_N\left(\alpha, \frac{\Omega}{v}\right)s(\alpha)\right] = \frac{T_1\left(\alpha, \frac{\Omega}{v}\right)v^2 - s(\alpha)v^2}{m}\cos\alpha,$$

$$\dot\alpha v + \Omega v - \sigma\cos\alpha\left[\frac{v^2}{I}y_N\left(\alpha, \frac{\Omega}{v}\right)s(\alpha)\right] - \sigma\Omega^2\sin\alpha = \frac{s(\alpha)v^2 - T_1\left(\alpha, \frac{\Omega}{v}\right)v^2}{m}\sin\alpha, \tag{17}$$

$$\dot\Omega = \frac{v^2}{I}y_N\left(\alpha, \frac{\Omega}{v}\right)s(\alpha).$$

If I introduce the new dimensionless phase variable and the differentiation by the formulas $\Omega = n_1 v\omega$, $< \cdot >= n_1 v <'>$, $n_1 > 0$, $n_1 = \mathrm{const}$, then system (17) is reduced to the following form:

$$v' = v\Psi(\alpha, \omega), \tag{18}$$

$$\begin{aligned}
\alpha' &= -\omega + \sigma n_1\omega^2\sin\alpha + \left[\frac{\sigma}{In_1}y_N(\alpha, n_1\omega)s(\alpha)\right]\cos\alpha - \\
&- \frac{T_1(\alpha, n_1\omega) - s(\alpha)}{mn_1}\sin\alpha, \\
\omega' &= \frac{1}{In_1^2}y_N(\alpha, n_1\omega)s(\alpha) - \omega\left[\frac{\sigma}{In_1}y_N(\alpha, n_1\omega)s(\alpha)\right]\sin\alpha + \\
&+ \sigma n_1\omega^3\cos\alpha - \omega\frac{T_1(\alpha, n_1\omega) - s(\alpha)}{mn_1}\cos\alpha,
\end{aligned} \tag{19}$$

$$\Psi(\alpha, \omega) = -\sigma n_1\omega^2\cos\alpha + \left[\frac{\sigma}{In_1}y_N(\alpha, n_1\omega)s(\alpha)\right]\sin\alpha +$$

$$+\frac{T_1(\alpha, n_1\omega) - s(\alpha)}{mn_1}\cos\alpha.$$

I see that the independent second-order subsystem (19) can be substituted into the third-order system (18), (19) and can be considered separately on its own two-dimensional phase cylinder.

I take the function $\mathbf{r}_N$ as follows (the plate $AB$ is given by the equation $x_{1N} \equiv 0$):

$$\mathbf{r}_N = \begin{pmatrix} 0 \\ x_{2N} \end{pmatrix} = R(\alpha)\mathbf{i}_N, \tag{20}$$

where $\mathbf{i}_N = \mathbf{i}_v(\pi/2)$ (see (5)). In our case

$$\mathbf{i}_N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, the equality $x_{2N} = R(\alpha)$ holds and shows that for the considered system, the moment of the nonconservative forces is independent of the angular velocity (it depends only on the angle $\alpha$). For the construction of the force field, I use the pair of dynamical functions $R(\alpha), s(\alpha)$; the information about them is of a qualitative nature. Similarly to the choice of the Chaplygin analytical functions (see [5]), I take the dynamical functions $s$ and $R$ as follows:

$$R(\alpha) = A\sin\alpha,\ s(\alpha) = B\cos\alpha,\ A, B > 0. \tag{21}$$

### 5.3. Reduced systems

**Theorem 1.** The simultaneous equations (3), (6), (9) under conditions (20), (21) can be reduced to the dynamical system on the tangent bundle (2) of the one-dimensional sphere (1).

Indeed, if I introduce the dimensionless parameter and the differentiation by the formulas

$$b_* = ln_0,\ n_0^2 = \frac{AB}{I},\ \ <\cdot>= n_0 v_\infty <'>, \tag{22}$$

then the obtained equation has the following form:

$$\xi'' + b_*\xi'\cos\xi + \sin\xi\cos\xi = 0. \tag{23}$$

After the transition from the variables $z$ (about the variables $z$ see (8)) to the variables $w$ $w_1 = -1/n_0 v_\infty z_1 - b_*\sin\xi$, Eq. (23) is equivalent to the system

$$\begin{aligned} \xi' &= -w_1 - b_*\sin\xi, \\ w_1' &= \sin\xi\cos\xi, \end{aligned} \tag{24}$$

on the tangent bundle $T_*\mathbf{S}^1\{(w_1;\xi) \in \mathbf{R}^2 : \ \xi \bmod 2\pi\}$ of the one-dimensional sphere $\mathbf{S}^1\{\xi \in \mathbf{R}^1 : \ \xi \bmod 2\pi\}$.

The phase pattern of the system (24) ($\alpha \leftrightarrow \xi - \pi, \ \omega \leftrightarrow w_1$) is shown in the Fig. 3.

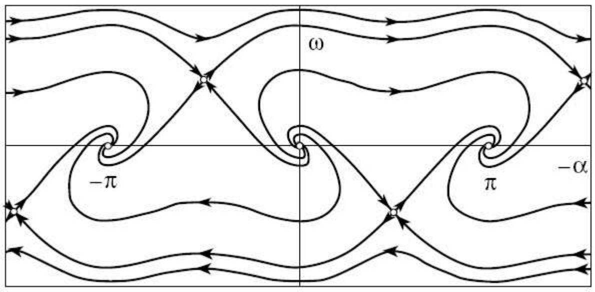


**Figure 3.** Variable dissipation dynamical system

### 5.4. Transcendental first integral

I turn now to the integration of the desired second-order system (24). In the variables $(\xi, w_1)$ the found first integrals have the following forms:

**I.** $b_*^2 - 4 < 0$.

$$[\sin^2\xi + b_* w_1 \sin\xi + w_1^2] \times \exp\left\{\frac{2b_*}{\sqrt{4-b_*^2}}\operatorname{arctg}\frac{2w_1 + b_*\sin\xi}{\sqrt{4-b_*^2}\sin\xi}\right\} = \text{const.} \tag{25}$$

**II.** $b_*^2 - 4 > 0$.

$$[\sin^2\xi + b_* w_1 \sin\xi + w_1^2] \times \left|\frac{2w_1 + b_*\sin\xi + \sqrt{b_*^2-4}\sin\xi}{2w_1 + b_*\sin\xi - \sqrt{b_*^2-4}\sin\xi}\right|^{-b_*/\sqrt{b_*^2-4}} = \text{const.} \tag{26}$$

**III.** $b_*^2 - 4 = 0$.

$$(w_1 - \sin\xi)\exp\left\{\frac{\sin\xi}{w_1 - \sin\xi}\right\} = \text{const.} \tag{27}$$

Therefore, in the considered case the system of dynamical equations (24) has the first integral expressed by relations (25)–(27), which is a transcendental function of its phase variables (in the sense of complex analysis) and is expressed as a finite combination of elementary functions.

**Theorem 2.** Three sets of relations (3), (6), (9) under conditions (20), (21) possess the first integral (the complete set), which is a transcendental function (in the sense of complex analysis) and is expressed as a finite combination of elementary functions.

### 5.5. Topological analogies

Now I present two groups of analogies related to the system (10), which describes the motion of a free body in the presence of a tracking force.

*The first group of analogies* deals with the case of the presence the nonintegrable constraint (11) in the system. In this case the dynamical part of the motion equations under certain conditions is reduced to a system (14).

Under onditions (20), (21) the system (14) has the form

$$\begin{aligned} \alpha' &= -\omega + b\sin\alpha, \\ \omega' &= \sin\alpha\cos\alpha, \end{aligned} \tag{28}$$

if I introduce the dimensionless parameter, the variable, and the differentiation analogously to (22):

$$b = \sigma n_0,\ n_0^2 = \frac{AB}{I},\ \Omega = n_0 v\omega,\ \ <\cdot>= n_0 v <'> . \tag{29}$$

**Theorem 3.** System (28) (for the case of a free body) is equivalent to the system (24) (for the case of a fixed pendulum).

Indeed, it is sufficient to substitute

$$\xi = \alpha,\ w_1 = \omega,\ b_* = -b. \tag{30}$$

**Corollary 1.**

1. The phase pattern of the system (28) is shown in the Fig. 3.

2. The angle of attack $\alpha$ for a free body (Fig. 2) is equivalent to the angle of body deviation $\xi$ of a fixed pendulum (Fig. 1).

3. The distance $\sigma = CD$ for a free body corresponds to the length of a holder $l = OD$ of a fixed pendulum.

4. The first integral of a system (28) can be automatically obtained through the Eqs. (25)–(27) after substitutions (30):

   **I.** $b^2 - 4 < 0$.

$$[\sin^2\alpha - b\omega\sin\alpha + \omega^2] \times \exp\left\{-\frac{2b}{\sqrt{4-b^2}}\text{arctg}\frac{2\omega - b\sin\alpha}{\sqrt{4-b^2}\sin\alpha}\right\} = \text{const}. \tag{31}$$

   **II.** $b^2 - 4 > 0$.

$$[\sin^2\alpha - b\omega\sin\alpha + \omega^2] \times \left|\frac{2\omega - b\sin\alpha + \sqrt{b^2-4}\sin\alpha}{2\omega - b\sin\alpha - \sqrt{b^2-4}\sin\alpha}\right|^{b/\sqrt{b^2-4}} = \text{const}. \tag{32}$$

**III.** $b^2 - 4 = 0$.

$$(\omega - \sin\alpha)\exp\left\{\frac{\sin\alpha}{\omega - \sin\alpha}\right\} = \text{const}. \tag{33}$$

*The second group of analogies* deals with the case of a motion with the constant velocity of the center of mass of a body, i.e., when the property (15) holds. In this case the dynamical part of the motion equations under certain conditions is reduced to a system (19).

Then, under conditions (15), (20), (21), and (29), the reduced dynamical part of the motion equations (system (19)) has the form of analytical system

$$\begin{aligned} \alpha' &= -\omega + b\sin\alpha\cos^2\alpha + b\omega^2\sin\alpha, \\ \omega' &= \sin\alpha\cos\alpha - b\omega\sin^2\alpha\cos\alpha + b\omega^3\cos\alpha, \end{aligned} \tag{34}$$

in this case, I choose the constant $n_1$ as follows: $n_1 = n_0$.

If the problem on the first integral of the system (28) is solved using Corollary 1, the same problem for the system (34) can be solved by the following theorem 4.

**Theorem 4.** The first integral of the system (34) is a transcendental function of its own phase variables and is expressed as a finite combination of elementary functions.

Because of cumbersome character of form of the first integral obtained, I represent this form in the case $b = 2$ only:

$$\exp\left\{\frac{\sin\alpha + \omega}{\sin\alpha - \omega}\right\}\frac{1 - 4\omega\sin\alpha + 4\omega^2}{(\omega - \sin\alpha)^2} = C_1 = \text{const}. \tag{35}$$

**Theorem 5.** The first integral of system (28) is constant on the phase trajectories of the system (34).

Thus, I have the following topological and mechanical analogies in the sense explained above.

(1) A motion of a fixed physical pendulum on a cylindrical hinge in a flowing medium (nonconservative force fields).

(2) A plane-parallel free motion of a rigid body in a nonconservative force field under a tracing force (in the presence of a nonintegrable constraint).

(3) A plane-parallel composite motion of a rigid body rotating about its center of mass, which moves rectilinearly and uniformly, in a nonconservative force field.

## 6. Cases of integrability corresponding to the motion of a pendulum in the three-dimensional space

In [4, 6, 7], the planar problem was generalized to the spatial (three-dimensional) case, where the system of dynamical equations has a complete set of transcendental first integrals. It was assumed that the interaction of the homogeneous medium flow with the fixed body (the spherical pendulum) is concentrated on a part of the body surface that has the form of a planar (two-dimensional) disk.

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