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TRAJECTORIES THAT HAVE POINTS AT INFINITY AS LIMIT SETS FOR DYNAMICAL SYSTEMS ON THE PLANE

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Abstract. In this paper, we deal with the existence and uniqueness of trajectories of the dynamical systems on the plane that have infinitely remote points as α- and ω-limit sets. Therefore, on the Riemann or Poincaré sphere, the limit set of such trajectories is the north pole. These are key trajectories by definition since an infinitely remote point is always singular.

1. Preliminaries

At the beginning we consider systems of the form (see also [7, 12])

$$\begin{aligned} \alpha' &= \omega + \frac{\sigma}{I}F(\alpha)\cos\alpha + \sigma\omega^2\sin\alpha, \\ \omega' &= -\frac{1}{I}F(\alpha) - \frac{\sigma}{I}\omega F(\alpha)\sin\alpha\sigma\omega^3\cos\alpha,\ \sigma,\ I > 0, \end{aligned} \tag{1.1}$$

where the condition

$$F \in \Phi \tag{1.2}$$

is fulfilled. The class Φ consists of sufficiently smooth odd π-periodic functions that vanish only at the points $0 \mod \pi/2$ and satisfy the following conditions: $F'(0) > 0$ and $F'(\pi/2) < 0$ [9, 11].

Lemma 1.1. *Let us consider system* (1.1) *on the set*

$$\Pi \cap \{(\alpha, \omega) \in \mathbb{R}^2 \colon \omega > 0\}.$$

Then for any sufficiently smooth function F, there exists a single trajectory going to infinity (*and having the point* $(-0, +\infty)$ *as the ω-limit set*).

Proof. Let us endow a phase plane $\mathbb{R}^2\{x, y\}$ with an infinitely remote point to obtain an extended phase plane $\overline{\mathbb{R}^2\{x, y\}}$. Now map the region $\Pi \cap \{(\alpha, \omega) \in \mathbb{R}^2 : \omega > 0\}$ onto a Riemann or Poincaré sphere. In the vicinity of the north pole of the sphere, there exist new coordinates (α, y), $y = 1/\omega$, to which the former coordinates of the considered region of the extended phase plane are sent by a nonsingular transformation [2, 4, 5].

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In the variables (α, y), system (1.1) is equivalent to the equation

$$\frac{d\alpha}{dy} = \frac{y + \frac{\sigma}{I} y^2 F(\alpha)\cos\alpha + \sigma\sin\alpha}{\frac{y^4}{I} F(\alpha) - \sigma y\cos\alpha + \frac{\sigma}{I} y^3 F(\alpha)\sin\alpha}. \tag{1.3}$$

Given this, the trajectories of Eq. (1.3) are parametrized in a different manner than the trajectories of system (1.1).

One can see that Eq. (1.3) has a singular point $(0,0)$ corresponding to the infinitely remote point $(-0,+\infty)$ of system (1.1). One can readily make sure that the point $(0,0)$ of Eq. (1.3) is a hyperbolic saddle, and Lemma 1.1 follows [1, 6, 8]. □

Let us consider systems of the form

$$\begin{aligned} \alpha' &= \omega + \frac{\sigma}{I} F(\alpha)\cos\alpha + \sigma\omega^2\sin\alpha + \frac{s(\alpha)}{m}\sin\alpha, \\ \omega' &= -\frac{1}{I} F(\alpha) - \frac{\sigma}{I}\omega F(\alpha)\sin\alpha + \sigma\omega^2\cos\alpha + \frac{\omega}{m} s(\alpha)\cos\alpha, \ \sigma,\ I > 0, \end{aligned} \tag{1.4}$$

in the strip Π' under condition (1.2) and

$$s \in \Sigma. \tag{1.5}$$

The class Σ consists of sufficiently smooth 2π-periodic even functions that are equal to zero only at the points $\pi/2 \mod \pi$ and satisfy the conditions

$$s(0) > 0, \quad s'\left(\frac{\pi}{2}\right) < 0, \quad s(\alpha + \pi) = -s(\alpha), \quad \forall \alpha \in \mathbb{R}.$$

Lemma 1.2. *Let us consider system* (1.4) *on the set*

$$\Pi \cap \{(\alpha,\omega) \in \mathbb{R}^2 \colon \omega > 0\}.$$

Then for any sufficiently smooth functions F and s, there exists a unique trajectory going to infinity (*and having the point* $(-0,+\infty)$ *as the ω-limit set*).

Proof. Following the arguments used in the proof of Lemma 1.1, mapping the extended phase plane to a sphere and making a similar change of coordinates, we obtain the equation [3, 10]

$$\frac{dy}{d\alpha} = \frac{\frac{y^4}{I} F(\alpha) - \sigma y\cos\alpha + \frac{\sigma}{I} y^3 F(\alpha)\sin\alpha - y^3 \frac{s(\alpha)}{m}\cos\alpha}{y + \frac{\sigma}{I} y^2 F(\alpha)\cos\alpha + \sigma\sin\alpha + y^2 \frac{s(\alpha)}{m}\sin\alpha}. \tag{1.6}$$

The trajectories of Eq. (1.6) are parametrized in a different way than the trajectories of system (1.4).

One can see that Eq. (1.6) has a singular point $(0,0)$ corresponding to the infinitely remote point $(-0,+\infty)$ of system (1.4). One can readily make sure that this singular point has the topological type of a hyperbolic saddle, which implies Lemma. □

2. Existence and uniqueness of trajectories going to infinity

Theorem 2.1. (1) *If, after the change of phase variables*

$$(x_1, x_2) \Rightarrow (x_1, y),$$

where $y = 1/x_2$, the equation defined on the sphere acquires the singular point $(x_1^0, 0)$, the system under consideration has a trajectory tending to the straight line

$$\{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 = x_1^0\}$$

and having an infinitely remote point as the limit set.

(2) *If, after the change of phase variables*

$$(x_1, x_2) \Rightarrow (y, x_2),$$

where $y = 1/x_1$, the equation defined on the sphere acquires the singular point $(0, x_2^0)$, the system under consideration has a trajectory tending to the straight line

$$\{(x_1, x_2) \in \mathbb{R}^2 \colon x_2 = x_2^0\}$$

and having an infinitely remote point as the limit set.

Proof. In line with Lemmas 1.1 and 1.2, we endow the phase plane with an infinitely remote point to obtain $\overline{\mathbb{R}^2\{\alpha, \omega\}}$. Then we map the extended plane to a Riemann or Poincaré sphere. In the neighborhood of the north pole of the sphere, one can introduce coordinates mapping this neighborhood to a certain neighborhood of zero of the coordinate plane, such that in case (1) they are equal to (x_1, y), $y = 1/x_2$, and in case (2), to (y, x_2), $y = 1/x_1$. We investigate the infinitely remote points along the x_2 axis in the first case and along the x_1 axis in the second case. Our further arguments are similar to those used in the proofs of Lemmas 1.1 and 1.2. □

Remark 2.1. The number of trajectories going to infinity is determined from the topological type of the infinitely remote singular point. In particular, in systems (1.1) and (1.4) there exists a single trajectory going to infinity since the infinitely remote point is a saddle (if it is not the plane but the phase cylinder that is mapped).

Remark 2.2. There may exist phase trajectories going to infinity on the phase plane along which both phase variables infinitely increase. In this case, changing the variables $x_1 = 1/y_1$, $x_2 = 1/y_2$ and examining the topological type of the north pole of the sphere, which is always a singular point, one can try to prove the existence and uniqueness of trajectories approaching straight lines of the form

$$A_1x_1 + A_2x_2 + A_3 = 0,$$

where $A_1A_2 \neq 0$.

Indeed, to the north pole of the sphere the trajectory in this case tends at a certain angle which corresponds to a trajectory on a plane tending to a straight line with a nonzero and finite slope.

3. Elements of the theory of monotonic vector fields

Let us consider a family of sufficiently smooth vector fields ϑ_ϵ in the region D of a two-dimensional oriented Riemann surface. In the tangent space T_qD of each point $q \in D$, one can measure angles made by the vectors of the family under study.

Definition 3.1. A one-parameter family of fields ϑ_ϵ ($\epsilon \in E$) exhibits a monotonicity in D if for any points $q \in D$, $\epsilon_1 \in E$, and $\epsilon_2 \in E$ in the tangent space T_qD, the angle made by the vectors $\vartheta_{\epsilon_1}, \vartheta_{\epsilon_2} \in T_qD$ is a monotonic function of the difference $\epsilon_2 - \epsilon_1$; the orientation of the angle variation remains unchanged. If the monotonic dependence is strict, we say that ϑ_ϵ possesses a strict monotonicity property.

Theorem 3.1. *Let a field ϑ_ϵ possess the monotonicity property in the region D of a plane $\mathbb{R}^2$. Let x_0 be a nonsingular initial condition for the phase trajectory of the field ϑ_ϵ for all $\epsilon \in E$.*

Then if for any $\epsilon \in E$, the limit set of trajectories beginning at x_0 is a set γ_0, $\{A, B\} = \partial\gamma_0$, A is the limit set of the trajectory of the field ϑ_{ϵ_1}, and B is the limit set of the trajectory of the field ϑ_{ϵ_2}, $\epsilon_1 < \epsilon_2$, then we have $\epsilon \in (\epsilon_1, \epsilon_2)$ if and only if there exists a set C that is the limit set of the trajectory of the field ϑ_ϵ, which, when increasing, shifts monotonically from A to B. (*We speak here simultaneously either of α- or ω-limit sets of the family of trajectories.*) *The required phase trajectory is unique if the monotonicity property is strict.*

The scheme of the proof. For any ϵ, the set γ_0 can be assumed to consist of ω-limit sets. According to the theorem on the continuous dependence of solutions on initial conditions and right-hand sides of equations, for a small change parameter ϵ, the limit set will remain in a small neighborhood of the initial one (in the case where the set γ_0 is simply connected). If the set γ_0 is multiply connected, we successively look over each of the connected components. Within the framework of the theory of comparison systems, in view of the monotonicity property, a nonmonotonic dependence of the trajectory on the parameter ϵ is excluded.

Suppose a system possesses a strict monotonicity property. On the contrary, for a point $M \in \gamma_0$, let there exist at least two parameters ϵ^1 and ϵ^2 for which the trajectories of the fields ϑ_{ϵ^1} and ϑ_{ϵ^2} tend to the point M. Then the trajectories the fields $\vartheta_{\bar\epsilon}$, $\bar\epsilon \in [\epsilon^1, \epsilon^2]$, tend to the point M (by virtue of the monotonicity property). Since the monotonicity property is strict, for any $\delta > 0$, the system with the vector field $\vartheta_{\epsilon+\delta}$ ($\epsilon+\delta \in E$) is the comparison system for ϑ_ϵ. It can be easily understood that the trajectory of the field $\vartheta_{\epsilon+\delta}$ starting from a nonsingular initial condition never intersects the corresponding trajectory of the field ϑ_ϵ starting from the same initial condition. In view of this, the trajectories of the fields ϑ_{k_1} and ϑ_{k_2} have different limit sets and $\epsilon^1 < k_1 < k_2 < \epsilon^2$, which is a contradiction. This completely proves the theorem.

This may also be the scheme of the proof of a qualitatively different proposition that holds for any smooth two-dimensional oriented manifold.

Lemma 3.1. *Let us consider a family of fields ϑ_ϵ ($\epsilon \in E$) in a region of the sphere S^2 of the following form*: *the south* (S) *and the north* (N) *poles of the sphere are saddles. Let this family of fields possess a strict monotonicity property so that for a certain ϵ_1, the ω-limit set of the trajectory emanating from the south pole is the south pole, and for a certain $\epsilon_2 > \epsilon_1$, the ω-limit set of the trajectory emanating from the north pole is the north pole. Both these situations are homoclinic on the sphere when there exists only one rest point* (*in addition to N and S*) *located in the region bounded by the indicated separatrices. The sphere contains no other nontrivial limit sets.*

Then there exists only one value of the parameter $\epsilon = \epsilon_0 \in (\epsilon_1, \epsilon_2)$ *such that the trajectory emanating from the south* (*north*) *pole enters the north* (*south*) *pole* (*this is a heteroclinic situation on the sphere*).

Proof. Uniqueness. On the contrary, let two parameters $\overline{\epsilon}$ and $\overline{\overline{\epsilon}}$ possess the indicated property. Then by virtue of the monotonicity property, all the parameters from the interval $(\overline{\epsilon}, \overline{\overline{\epsilon}})$ possess this property. With arguments similar to those used in Theorem 3.1, we obtain a contradiction to the monotonicity property.

Existence. Therefore, there exists a unique value of the parameter $\epsilon = \epsilon_0$ such that for $\epsilon < \epsilon_0$ and $\epsilon > \epsilon_0$, different homoclinic situations are realized on the sphere. On the contrary, for $\epsilon = \epsilon_0$, let there be one of the homoclinic situations. Then there exists a neighborhood of the value $\epsilon = \epsilon_0$

$$U = U^{\delta}_{\epsilon_0} = \{\epsilon : |\epsilon - \epsilon_0| < \delta\}$$

such that for any $\epsilon \in U$, the same homoclinic situation holds, which is a contradiction. This completely proves the proposition. □

Remark 3.1. We have obtained another method of the proof of Lemmas 1.1 and 1.2. Indeed, the unknown fields satisfy the conditions of Lemma 3.1, since the infinitely remote point is projected into the north pole of the Riemann (or Poincaré) sphere and the point $(-\pi/2, 0)$ is projected into the south pole.

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