Certain Integrable Cases in Dynamics of a Multi-Dimensional Rigid Body in a Nonconservative Field

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***Abstract*—This paper is a survey of integrable cases in dynamics of a five-dimensional rigid body under the action of a nonconservative force field. We review both new results and results obtained earlier. Problems examined are described by dynamical systems with so-called variable dissipation with zero mean.**

The problem of the search for complete sets of transcendental first integrals of systems with dissipation is quite actual; a large number of works are devoted to it. We introduce a new class of dynamical systems that have a periodic coordinate. Due to the existence of a nontrivial symmetry groups of such systems, we can prove that these systems possess variable dissipation with zero mean, which means that on the average for a period with respect to the periodic coordinate, the dissipation in the system is equal to zero, although in various domains of the phase space, either the energy pumping or dissipation can occur. Based on facts obtained, we analyze dynamical systems that appear in dynamics of a five-dimensional rigid body and obtain a series of new cases of complete integrability of the equations of motion in transcendental functions that can be expresses through a finite combination of elementary functions.

***Index Terms*—Case of integrability, dynamic part of motion equations, multidimensional rigid body.**

I. Introduction

THIS This paper is a survey of integrable cases in dynamics of a f ve-dimensional rigid body under the action of a nonconservative force field We review both new results and results obtained earlier. Problems examined are described by dynamical systems with so-called variable dissipation with zero mean.

We study nonconservative systems for which usual methods of the study of Hamiltonian system is inapplicable. Thus, for such systems, we must directly integrate the main equation of dynamics (see also [1], [2], [3], [4], [5], [6]).

We generalize previously known cases and obtain new cases of the complete integrability in transcendental functions of the equation of dynamics of a f ve-dimensional rigid body in a nonconservative force field

Of course, in the general case, the construction of a theory of integration of nonconservative systems (even of low dimension) is a quite difficul task. In a number of cases, where the systems considered have additional symmetries, we succeed in findin firs integrals through finit combinations of elementary functions [6], [7], [8], [9].

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In basic part we recall general aspects of the dynamics of a free multi-dimensional rigid body: the notion of the tensor of angular velocity of the body, the joint dynamical equations of motion on the direct product $\mathbf{R}^n \times \mathrm{so}(n)$, and the Euler and Rivals formulas in the multi-dimensional case.

We also consider the tensor of inertia of a f ve-dimensional ($5D-$) rigid body. In this work, we study one of two possible cases in which there exists two relations between the principal moments of inertia:

(i) there are four equal principal moments of inertia ($I_2 = I_3 = I_4 = I_5$).

Furthermore, we systematize results on the study of equations of motion of a f ve-dimensional ($5D-$) rigid body in a nonconservative force fiel for the case (i). The form of these equations is taken from the dynamics of realistic rigid bodies of lesser dimension that interact with a resisting medium by laws of jet fl w when the is influence by a nonconservative tracing force. Under the action of this force, the following two cases are possible. In this case, the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo-constraint (see also [10], [11]).

The results relate to the case where all interaction of the medium with the body part is concentrated on a part of the surface of the body, which has the form of a four-dimensional disk, and the action of the force is concentrated in the direction perpendicular to this disk. These results are systematized and are preserved in the invariant form. Moreover, we introduce an extra dependence of the moment of the nonconservative force on the angular velocity. This dependence can be further extended to cases of the motion in spaces of higher dimension.

Many results of this paper were regularly presented on scientifi seminars, including the seminar Actual problems of geometry and mechanics named after Prof. V. V. Trofim v under the supervision of D. V. Georgievskii and M. V. Shamolin.

II. General Discourse

A. Cases of dynamical symmetry of a five-dimensional body

Let a f ve-dimensional rigid body Θ of mass m with smooth four-dimensional boundary $\partial\Theta$ be under the influenc of a nonconservative force field this can be interpreted as a motion of the body in a resisting medium that fill up f ve-dimensional domain of Euclidean space $\mathbf{E}^5$. We assume that the body

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is dynamically symmetric. If the body has two independent principal moments of inertia, then in some coordinate system $Dx_1x_2x_3x_4x_5$ attached to the body, the operator of inertia has the form

$$\text{diag}\{I_1, I_2, I_2, I_2, I_2\}, \tag{1}$$

or the form

$$\text{diag}\{I_1, I_1, I_3, I_3, I_3\}. \tag{2}$$

In the firs case, the body is dynamically symmetric in the hyperplane $Dx_2x_3x_4x_5$.

B. Dynamics on so(5) *and* $\mathbf{R}^5$

The configuratio space of a free, n-dimensional rigid body is the direct product

$$\mathbf{R}^n \times \text{SO}(n) \tag{3}$$

of the space $\mathbf{R}^n$, which define the coordinates of the center of mass of the body, and the rotation group $\text{SO}(n)$, which define rotations of the body about its center of mass and has dimension

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

Therefore, the dynamical part of equations of motion has the same dimension, whereas the dimension of the phase space is equal to

$$n(n+1).$$

In particular, if Ω is the tensor of angular velocity of a f ve-dimensional rigid body (it is a second-rank tensor, see [12], [13], [14], [15], [16]), $\Omega \in \text{so}(5)$, then the part of dynamical equations of motion corresponding to the Lie algebra $\text{so}(5)$ has the following form (see [17], [18]):

$$\dot{\Omega}\Lambda + \Lambda\dot{\Omega} + [\Omega,\ \Omega\Lambda + \Lambda\Omega] = M, \tag{4}$$

where

$$\Lambda = \text{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}, \tag{5}$$

$$\lambda_1 = \frac{-I_1 + I_2 + I_3 + I_4 + I_5}{2},$$
$$\lambda_2 = \frac{I_1 - I_2 + I_3 + I_4 + I_5}{2},$$
$$\lambda_3 = \frac{I_1 + I_2 - I_3 + I_4 + I_5}{2},$$
$$\lambda_4 = \frac{I_1 + I_2 + I_3 - I_4 + I_5}{2},$$
$$\lambda_5 = \frac{I_1 + I_2 + I_3 + I_4 - I_5}{2},$$

$M = M_F$ is the natural projection of the moment of external forces $\mathbf{F}$ acting to the body in $\mathbf{R}^5$ on the natural coordinates of the Lie algebra $\text{so}(5)$, and $[\ ]$ is the commutator in $\text{so}(5)$. The skew-symmetric matrix corresponding to this second-rank tensor $\Omega \in \text{so}(5)$ we represent in the form

$$\begin{pmatrix} 0 & -\omega_{10} & \omega_9 & -\omega_7 & \omega_4 \\ \omega_{10} & 0 & -\omega_8 & \omega_6 & -\omega_3 \\ -\omega_9 & \omega_8 & 0 & -\omega_5 & \omega_2 \\ \omega_7 & -\omega_6 & \omega_5 & 0 & -\omega_1 \\ -\omega_4 & \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}, \tag{6}$$

where $\omega_1,\ \omega_2,\ \ldots,\ \omega_{10}$ are the components of the tensor of angular velocity corresponding to the projections on the coordinates of the Lie algebra $\text{so}(5)$.

Obviously, the following relations hold:

$$\lambda_i - \lambda_j = I_j - I_i \tag{7}$$

for any $i, j = 1, \ldots, 5$.

For the calculation of the moment of an external force acting to the body, we need to construct the mapping

$$\mathbf{R}^5 \times \mathbf{R}^5 \longrightarrow \text{so}(5), \tag{8}$$

than maps a pair of vectors

$$(\mathbf{DN}, \mathbf{F}) \in \mathbf{R}^5 \times \mathbf{R}^5 \tag{9}$$

from $\mathbf{R}^5 \times \mathbf{R}^5$ to an element of the Lie algebra $\text{so}(5)$, where

$$\begin{aligned} \mathbf{DN} &= \{0, x_{2N}, x_{3N}, x_{4N}, x_{5N}\}, \\ \mathbf{F} &= \{F_1, F_2, F_3, F_4, F_5\}, \end{aligned} \tag{10}$$

and $\mathbf{F}$ is an external force acting to the body. For this end, we construct the following auxiliary matrix

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} & x_{4N} & x_{5N} \\ F_1 & F_2 & F_3 & F_4 & F_5 \end{pmatrix}. \tag{11}$$

Then the right-hand side of system (4) takes the form

$$\begin{gathered} M = \{x_{4N}F_5 - x_{5N}F_4, x_{5N}F_3 - x_{3N}F_5, \\ x_{2N}F_5 - x_{5N}F_2, x_{5N}F_1, x_{3N}F_4 - x_{4N}F_3, \\ x_{4N}F_2 - x_{2N}F_4, -x_{4N}F_1, x_{2N}F_3 - x_{3N}F_2, \\ x_{3N}F_1, -x_{2N}F_1\}. \end{gathered} \tag{12}$$

Dynamical systems studied in the following, generally speaking, are not conservative; they are dynamical systems with variable dissipation with zero mean (see [12]). We need to examine by direct methods a part of the main system of dynamical equations, namely, the Newton equation, which plays the role of the equation of motion of the center of mass, i.e., the part of the dynamical equations corresponding to the space $\mathbf{R}^5$:

$$m\mathbf{w}_C = \mathbf{F}, \tag{13}$$

where $\mathbf{w}_C$ is the acceleration of the center of mass C of the body and m is its mass. Moreover, due to the higher-dimensional Rivals formula (it can be obtained by the operator method) we have the following relations:

$$\mathbf{w}_C = \mathbf{w}_D + \Omega^2\mathbf{DC} + E\mathbf{DC},\ \mathbf{w}_D = \dot{\mathbf{v}}_D + \Omega\mathbf{v}_D,\ E = \dot{\Omega}, \tag{14}$$

where $\mathbf{w}_D$ is the acceleration of the point D, $\mathbf{F}$ is the external force acting on the body (in our case, $\mathbf{F} = \mathbf{S}$), and E is the tensor of angular acceleration (second-rank tensor).

So, the system of equations (4) and (13) of fifteent order on the manifold $\mathbf{R}^5 \times \text{so}(5)$ is a closed system of dynamical equations of the motion of a free f ve-dimensional rigid body under the action of an external force $\mathbf{F}$. This system have been separated from the kinematic part of the equations of motion on the manifold (3) and can be examined independently.

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III. General Problem on the Motion Under a Tracing Force

Consider a motion of a homogeneous, dynamically symmetric (case (1)), rigid body with front end face (a four-dimensional disk interacting with a medium that fill the f ve-dimensional space) in the fiel of a resistance force **S** under the quasi-stationarity conditions.

Let $(v, \alpha, \beta_1, \beta_2, \beta_3)$ be the (generalized) spherical coordinates of the velocity vector of the center of the four-dimensional disk lying on the axis of symmetry of the body,

$$\Omega = \begin{pmatrix} 0 & -\omega_{10} & \omega_9 & -\omega_7 & \omega_4 \\ \omega_{10} & 0 & -\omega_8 & \omega_6 & -\omega_3 \\ -\omega_9 & \omega_8 & 0 & -\omega_5 & \omega_2 \\ \omega_7 & -\omega_6 & \omega_5 & 0 & -\omega_1 \\ -\omega_4 & \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

be the tensor of angular velocity of the body, $Dx_1x_2x_3x_4x_5$ be the coordinate system attached to the body such that the axis of symmetry CD coincides with the axis Dx_1 (recall that C is the center of mass), and the axes Dx_2, Dx_3, Dx_4, Dx_5 lie in the hyperplane of the disk, and $I_1,\ I_2,\ I_3 = I_2,\ I_4 = I_2,\ I_5 = I_2,\ m$ are characteristics of inertia and mass.

We adopt the following expansions in the projections to the axes of the coordinate system $Dx_1x_2x_3x_4x_5$:

$$\mathbf{DC} = \{-\sigma, 0, 0, 0, 0\},$$

$$\mathbf{v}_D = \{v\cos\alpha, v\sin\alpha\cos\beta_1, v\sin\alpha\sin\beta_1\cos\beta_2,$$
$$v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3, v\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3\}. \quad (15)$$

In the case (1) we additionally have the expansion for the function of the influenc of the medium on the f ve-dimensional body:

$$\mathbf{S} = \{-S, 0, 0, 0, 0\}, \quad (16)$$

i.e., in this case $\mathbf{F} = \mathbf{S}$.

Then the part of the dynamical equations of motion (including the analytic Chaplygin functions; see below) that describes the motion of the center of mass and corresponds to the space $\mathbf{R}^5$, in which tangent forces of the influenc of the medium on the four-dimensional disk vanish, takes the form

$$\dot v\cos\alpha - \dot\alpha v\sin\alpha - \omega_{10}v\sin\alpha\cos\beta_1 + \omega_9 v\sin\alpha\sin\beta_1\cos\beta_2 -$$
$$-\omega_7 v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3 + \omega_4 v\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3 +$$
$$+\sigma(\omega_{10}^2 + \omega_9^2 + \omega_7^2 + \omega_4^2) = -\frac{S}{m}, \quad (17)$$

$$\dot v\sin\alpha\cos\beta_1 + \dot\alpha v\cos\alpha\cos\beta_1 - \dot\beta_1 v\sin\alpha\sin\beta_1 +$$
$$+\omega_{10}v\cos\alpha - \omega_8 v\sin\alpha\sin\beta_1\cos\beta_2 +$$
$$+\omega_6 v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3 -$$
$$-\omega_3 v\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3 -$$
$$-\sigma(\omega_9\omega_8 + \omega_6\omega_7 + \omega_3\omega_4) - \sigma\dot\omega_{10} = 0, \quad (18)$$

$$\dot v\sin\alpha\sin\beta_1\cos\beta_2 + \dot\alpha v\cos\alpha\sin\beta_1\cos\beta_2 +$$
$$+\dot\beta_1 v\sin\alpha\cos\beta_1\cos\beta_2 -$$
$$-\dot\beta_2 v\sin\alpha\sin\beta_1\sin\beta_2 - \omega_9 v\cos\alpha + \omega_8 v\sin\alpha\cos\beta_1 -$$
$$-\omega_5 v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3 +$$
$$+\omega_2 v\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3 -$$
$$-\sigma(\omega_8\omega_{10} - \omega_5\omega_7 - \omega_2\omega_4) + \sigma\dot\omega_9 = 0, \quad (19)$$

$$\dot v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3 + \dot\alpha v\cos\alpha\sin\beta_1\sin\beta_2\cos\beta_3 +$$
$$+\dot\beta_1 v\sin\alpha\cos\beta_1\sin\beta_2\cos\beta_3 +$$
$$+\dot\beta_2 v\sin\alpha\sin\beta_1\cos\beta_2\cos\beta_3 -$$
$$-\dot\beta_3 v\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3 + \omega_7 v\cos\alpha - \omega_6 v\sin\alpha\cos\beta_1 +$$
$$+\omega_5 v\sin\alpha\sin\beta_1\cos\beta_2 - \omega_1 v\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3 +$$
$$+\sigma(\omega_6\omega_{10} + \omega_5\omega_9 - \omega_1\omega_4) - \sigma\dot\omega_7 = 0, \quad (20)$$

$$\dot v\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3 + \dot\alpha v\cos\alpha\sin\beta_1\sin\beta_2\sin\beta_3 +$$
$$+\dot\beta_1 v\sin\alpha\cos\beta_1\sin\beta_2\sin\beta_3 +$$
$$+\dot\beta_2 v\sin\alpha\sin\beta_1\cos\beta_2\sin\beta_3 +$$
$$+\dot\beta_3 v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3 - \omega_4 v\cos\alpha + \omega_3 v\sin\alpha\cos\beta_1 -$$
$$-\omega_2 v\sin\alpha\sin\beta_1\cos\beta_2 + \omega_1 v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3 -$$
$$-\sigma(\omega_3\omega_{10} + \omega_2\omega_9 + \omega_1\omega_7) + \sigma\dot\omega_4 = 0, \quad (21)$$

where

$$S = s(\alpha)v^2,\ \sigma = CD,\ v > 0. \quad (22)$$

Further, the auxiliary matrix (11) for the calculation of the moment of the resistance force has the form

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} & x_{4N} & x_{5N} \\ -S & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (23)$$

then the part of the dynamical equations of motion that describes the motion of the body about the center of mass and corresponds to the Lie algebra so(5), becomes

$$(\lambda_4 + \lambda_5)\dot\omega_1 + (\lambda_4 - \lambda_5)(\omega_4\omega_7 + \omega_3\omega_6 + \omega_2\omega_5) = 0, \quad (24)$$

$$(\lambda_3 + \lambda_5)\dot\omega_2 + (\lambda_5 - \lambda_3)(\omega_1\omega_5 - \omega_3\omega_8 - \omega_4\omega_9) = 0, \quad (25)$$

$$(\lambda_2 + \lambda_5)\dot\omega_3 + (\lambda_2 - \lambda_5)(\omega_4\omega_{10} - \omega_2\omega_8 - \omega_1\omega_6) = 0, \quad (26)$$

$$(\lambda_1 + \lambda_5)\dot\omega_4 + (\lambda_5 - \lambda_1)(\omega_3\omega_{10} + \omega_2\omega_9 + \omega_1\omega_7) =$$
$$= -x_{5N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)s(\alpha)v^2, \quad (27)$$

$$(\lambda_3 + \lambda_4)\dot\omega_5 + (\lambda_3 - \lambda_4)(\omega_7\omega_9 + \omega_6\omega_8 + \omega_1\omega_2) = 0, \quad (28)$$

$$(\lambda_2 + \lambda_4)\dot\omega_6 + (\lambda_4 - \lambda_2)(\omega_5\omega_8 - \omega_7\omega_{10} - \omega_1\omega_3) = 0, \quad (29)$$

$$(\lambda_1 + \lambda_4)\dot\omega_7 + (\lambda_1 - \lambda_4)(\omega_1\omega_4 - \omega_6\omega_{10} - \omega_5\omega_9) =$$
$$= x_{4N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)s(\alpha)v^2, \quad (30)$$

$$(\lambda_2 + \lambda_3)\dot\omega_8 + (\lambda_2 - \lambda_3)(\omega_9\omega_{10} + \omega_5\omega_6 + \omega_2\omega_3) = 0, \quad (31)$$

$$(\lambda_1 + \lambda_3)\dot\omega_9 + (\lambda_3 - \lambda_1)(\omega_8\omega_{10} - \omega_5\omega_7 - \omega_2\omega_4) =$$
$$= -x_{3N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)s(\alpha)v^2, \quad (32)$$

$$(\lambda_1 + \lambda_2)\dot\omega_{10} + (\lambda_1 - \lambda_2)(\omega_8\omega_9 + \omega_6\omega_7 + \omega_3\omega_4) =$$

$$= x_{2N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)s(\alpha)v^2. \quad (33)$$

Thus, the phase space of system (17)–(21), (24)–(33) of fifteent order is the direct product of the f ve-dimensional manifold and the Lie algebra so(5):

$$\mathbf{R}^1 \times \mathbf{S}^4 \times \mathrm{so}(5). \quad (34)$$

We note that system (17)–(21), (24)–(33), due to the existing dynamical symmetry

$$I_2 = I_3 = I_4 = I_5, \quad (35)$$

possesses cyclic firs integrals

$$\omega_1 \equiv \omega_1^0,\ \omega_2 \equiv \omega_2^0,\ \omega_3 \equiv \omega_3^0,\ \omega_5 \equiv \omega_5^0,\ \omega_6 \equiv \omega_6^0,\ \omega_8 \equiv \omega_8^0. \quad (36)$$

In the sequel, we consider the dynamics of the system on zero levels:

$$\omega_1^0 = \omega_2^0 = \omega_3^0 = \omega_5^0 = \omega_6^0 = \omega_8^0 = 0. \quad (37)$$

If one considers a more general problem on the motion of a body under a tracing force $\mathbf{T}$ that lies on the straight line $CD = Dx_1$ and provides the fulfillmen of the relation

$$v \equiv \mathrm{const} \quad (38)$$

throughout the motion, then instead of F_1 system (17)–(21), (24)–(33) contains

$$T - s(\alpha)v^2,\ \sigma = DC. \quad (39)$$

Choosing the value T of the tracing force appropriately, one can achieve the equality (38) throughout the motion. Indeed, expressing T due to system (17)–(21), (24)–(33), we obtain for $\cos\alpha \neq 0$ the relation

$$T = T_v(\alpha,\beta_1,\beta_2,\beta_3,\Omega) = m\sigma(\omega_4^2+\omega_7^2+\omega_9^2+\omega_{10}^2)+$$
$$+s(\alpha)v^2\left[1-\frac{m\sigma}{3I_2}\frac{\sin\alpha}{\cos\alpha}\Gamma_v\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\right], \quad (40)$$

where

$$\Gamma_v\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right) =$$
$$= x_{5N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\sin\beta_1\sin\beta_2\sin\beta_3+$$
$$+x_{4N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\sin\beta_1\sin\beta_2\cos\beta_3+$$
$$+x_{3N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\sin\beta_1\cos\beta_2+$$
$$+x_{2N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\cos\beta_1; \quad (41)$$

here we used conditions (36)–(38).

This procedure can be interpreted in two ways. First, we have transformed the system using the tracing force (control) that provides the consideration of the class (38) of motions interesting for us. Second, we can treat this as an order-reduction procedure. Indeed, system (17)–(21), (24)–(33) generates the following independent system of eighth order:

$$\dot\alpha v\cos\alpha\cos\beta_1 - \dot\beta_1 v\sin\alpha\sin\beta_1+$$
$$+\omega_{10}v\cos\alpha - \sigma\dot\omega_{10} = 0, \quad (42)$$
$$\dot\alpha v\cos\alpha\sin\beta_1\cos\beta_2 + \dot\beta_1 v\sin\alpha\cos\beta_1\cos\beta_2-$$
$$-\dot\beta_2 v\sin\alpha\sin\beta_1\sin\beta_2 - \omega_9 v\cos\alpha + \sigma\dot\omega_9 = 0, \quad (43)$$
$$\dot\alpha v\cos\alpha\sin\beta_1\sin\beta_2\cos\beta_3 + \dot\beta_1 v\sin\alpha\cos\beta_1\sin\beta_2\cos\beta_3+$$
$$+\dot\beta_2 v\sin\alpha\sin\beta_1\cos\beta_2\cos\beta_3-$$
$$-\dot\beta_3 v\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3 + \omega_7 v\cos\alpha - \sigma\dot\omega_7 = 0, \quad (44)$$
$$\dot\alpha v\cos\alpha\sin\beta_1\sin\beta_2\sin\beta_3 + \dot\beta_1 v\sin\alpha\cos\beta_1\sin\beta_2\sin\beta_3+$$
$$+\dot\beta_2 v\sin\alpha\sin\beta_1\cos\beta_2\sin\beta_3+$$
$$+\dot\beta_3 v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3 - \omega_4 v\cos\alpha + \sigma\dot\omega_4 = 0, \quad (45)$$
$$3I_2\dot\omega_4 = -x_{5N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)s(\alpha)v^2, \quad (46)$$
$$3I_2\dot\omega_7 = x_{4N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)s(\alpha)v^2, \quad (47)$$
$$3I_2\dot\omega_9 = -x_{3N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)s(\alpha)v^2, \quad (48)$$
$$3I_2\dot\omega_{10} = x_{2N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)s(\alpha)v^2, \quad (49)$$

which, in addition to the permanent parameters specifie above, contains the parameter v.

System (42)–(49) is equivalent to the system

$$\dot\alpha v\cos\alpha + v\cos\alpha\{\omega_{10}\cos\beta_1+$$
$$+[(\omega_7\cos\beta_3 - \omega_4\sin\beta_3)\sin\beta_2 - \omega_9\cos\beta_2]\sin\beta_1\}+$$
$$+\sigma\{-\dot\omega_{10}\cos\beta_1 + [\dot\omega_9\cos\beta_2-$$
$$-(\dot\omega_7\cos\beta_3 - \dot\omega_4\sin\beta_3)\sin\beta_2]\sin\beta_1\} = 0, \quad (50)$$
$$\dot\beta_1 v\sin\alpha + v\cos\alpha\{[(\omega_7\cos\beta_3-$$
$$-\omega_4\sin\beta_3)\sin\beta_2 - \omega_9\cos\beta_2]\cos\beta_1 - \omega_{10}\sin\beta_1\}+$$
$$+\sigma\{[\dot\omega_9\cos\beta_2 - (\dot\omega_7\cos\beta_3-$$
$$-\dot\omega_4\sin\beta_3)\sin\beta_2]\cos\beta_1 + \dot\omega_{10}\sin\beta_1\} = 0, \quad (51)$$
$$\dot\beta_2 v\sin\alpha\sin\beta_1 + v\cos\alpha\{[\omega_7\cos\beta_3-$$
$$-\omega_4\sin\beta_3]\cos\beta_2 + \omega_9\sin\beta_2\}+$$
$$+\sigma\{-[\dot\omega_7\cos\beta_3 - \dot\omega_4\sin\beta_3]\cos\beta_2 - \dot\omega_9\sin\beta_2\} = 0, \quad (52)$$
$$\dot\beta_3 v\sin\alpha\sin\beta_1\sin\beta_2 + v\cos\alpha\{-\omega_4\cos\beta_3 - \omega_7\sin\beta_3\}+$$
$$+\sigma\{\dot\omega_4\cos\beta_3 + \dot\omega_7\sin\beta_3\} = 0, \quad (53)$$
$$\dot\omega_4 = -\frac{v^2}{3I_2}x_{5N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)s(\alpha), \quad (54)$$
$$\dot\omega_7 = \frac{v^2}{3I_2}x_{4N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)s(\alpha), \quad (55)$$
$$\dot\omega_9 = -\frac{v^2}{3I_2}x_{3N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)s(\alpha), \quad (56)$$
$$\dot\omega_{10} = \frac{v^2}{3I_2}x_{2N}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)s(\alpha). \quad (57)$$

Introduce the new quasi-velocities. For this, we transform $\omega_4, \omega_7, \omega_9, \omega_{10}$ by three rotations:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} =$$

$$= T_{3,4}(-\beta_1) \circ T_{2,3}(-\beta_2) \circ T_{1,2}(-\beta_3) \begin{pmatrix} \omega_4 \\ \omega_7 \\ \omega_9 \\ \omega_{10} \end{pmatrix}, \quad (58)$$

where

$$T_{3,4}(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{pmatrix},$$

$$T_{2,3}(\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta & 0 \\ 0 & \sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_{1,2}(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta & 0 & 0 \\ \sin\beta & \cos\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, the following relations hold:

$$\begin{aligned} &z_1 = \omega_4 \cos\beta_3 + \omega_7 \sin\beta_3, \\ &z_2 = (\omega_7 \cos\beta_3 - \omega_4 \sin\beta_3)\cos\beta_2 + \omega_9 \sin\beta_2, \\ &z_3 = [(-\omega_7 \cos\beta_3 + \omega_4 \sin\beta_3)\sin\beta_2 + \\ &+ \omega_9 \cos\beta_2]\cos\beta_1 + \omega_{10}\sin\beta_1, \\ &z_4 = [(\omega_7 \cos\beta_3 - \omega_4 \sin\beta_3)\sin\beta_2 - \\ &- \omega_9 \cos\beta_2]\sin\beta_1 + \omega_{10}\cos\beta_1. \end{aligned} \quad (59)$$

As we see from (50)–(57), we cannot solve the system with respect to $\dot\alpha,\ \dot\beta_1,\ \dot\beta_2,\ \dot\beta_3$ on the manifold

$$O_1 = \{(\alpha, \beta_1, \beta_2, \beta_3, \omega_4, \omega_7, \omega_9, \omega_{10}) \in \mathbf{R}^8 :$$

$$\alpha = \frac{\pi}{2}k,\ \beta_1 = \pi l_1,\ \beta_2 = \pi l_2,\ k, l_1, l_2 \in \mathbf{Z}\}. \quad (60)$$

Therefore, on the manifold (60) the uniqueness theorem formally is violated. Moreover, for even k and any l_1, l_2, an indeterminate form appears due to the degeneration of the spherical coordinates $(v, \alpha, \beta_1, \beta_2, \beta_3)$. For odd k, the uniqueness theorem is obviously violated since the firs equation (50) degenerates.

This implies that system (50)–(57) outside (and only outside) the manifold (60) is equivalent to the system

$$\dot\alpha = -z_4 + \frac{\sigma v}{3I_2}\frac{s(\alpha)}{\cos\alpha}\Gamma_v\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right), \quad (61)$$

$$\begin{aligned} \dot z_4 = &\frac{v^2}{3I_2}s(\alpha)\Gamma_v\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) - \\ &-(z_1^2 + z_2^2 + z_3^2)\frac{\cos\alpha}{\sin\alpha} + \\ &+\frac{\sigma v}{3I_2}\frac{s(\alpha)}{\sin\alpha}\{-z_3\Delta_{v,1}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) + \\ &+z_2\Delta_{v,2}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) - \\ &-z_1\Delta_{v,3}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)\}, \end{aligned} \quad (62)$$

$$\begin{aligned} \dot z_3 = &z_3 z_4\frac{\cos\alpha}{\sin\alpha} + (z_1^2 + z_2^2)\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + \\ &+\frac{\sigma v}{3I_2}\frac{s(\alpha)}{\sin\alpha}\{z_4\Delta_{v,1}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) - \\ &-z_2\Delta_{v,2}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)\frac{\cos\beta_1}{\sin\beta_1} + \\ &+z_1\Delta_{v,3}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)\frac{\cos\beta_1}{\sin\beta_1}\} - \\ &-\frac{v^2}{3I_2}s(\alpha)\Delta_{v,1}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right), \end{aligned} \quad (63)$$

$$\begin{aligned} \dot z_2 = &z_2 z_4\frac{\cos\alpha}{\sin\alpha} - z_2 z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - \\ &-z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2} + \\ &+\frac{\sigma v}{3I_2}\frac{s(\alpha)}{\sin\alpha}\Delta_{v,2}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)\left\{-z_4 + z_3\frac{\cos\beta_1}{\sin\beta_1}\right\} + \\ &+\frac{\sigma v}{3I_2}\frac{s(\alpha)}{\sin\alpha}\Delta_{v,3}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right)\left\{-z_1\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}\right\} + \\ &+\frac{v^2}{3I_2}s(\alpha)\Delta_{v,2}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right), \end{aligned} \quad (64)$$

$$\begin{aligned} \dot z_1 = &z_1 z_4\frac{\cos\alpha}{\sin\alpha} - z_1 z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + \\ &+z_1 z_2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2} + \\ &+\frac{\sigma v}{3I_2}\frac{s(\alpha)}{\sin\alpha}\Delta_{v,3}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) \times \\ &\times\left\{z_4 - z_3\frac{\cos\beta_1}{\sin\beta_1} + z_2\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}\right\} - \\ &-\frac{v^2}{3I_2}s(\alpha)\Delta_{v,3}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right), \end{aligned} \quad (65)$$

$$\dot\beta_1 = z_3\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma v}{3I_2}\frac{s(\alpha)}{\sin\alpha}\Delta_{v,1}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right), \quad (66)$$

$$\begin{aligned} \dot\beta_2 = &-z_2\frac{\cos\alpha}{\sin\alpha\sin\beta_1} + \\ &+\frac{\sigma v}{3I_2}\frac{s(\alpha)}{\sin\alpha\sin\beta_1}\Delta_{v,2}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right), \end{aligned} \quad (67)$$

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$$\dot{\beta}_3 = z_1 \frac{\cos\alpha}{\sin\alpha \sin\beta_1 \sin\beta_2} +$$
$$+ \frac{\sigma v}{3I_2} \frac{s(\alpha)}{\sin\alpha \sin\beta_1 \sin\beta_2} \Delta_{v,3}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right), \quad (68)$$

where

$$\Delta_{v,1}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) =$$
$$= -x_{2N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) \sin\beta_1 +$$
$$+ x_{3N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) \cos\beta_1 \cos\beta_2 +$$
$$+ x_{4N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) \cos\beta_1 \sin\beta_2 \cos\beta_3 +$$
$$+ x_{5N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) \cos\beta_1 \sin\beta_2 \sin\beta_3,$$
$$\Delta_{v,2}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) =$$
$$= -x_{3N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) \sin\beta_2 + \quad (69)$$
$$+ x_{4N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) \cos\beta_2 \cos\beta_3 +$$
$$+ x_{5N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) \cos\beta_2 \sin\beta_3,$$
$$\Delta_{v,3}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) =$$
$$= -x_{4N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) \sin\beta_3 +$$
$$+ x_{5N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) \cos\beta_3,$$

and the function $\Gamma_v(\alpha, \beta_1, \beta_2, \beta_3, \Omega/v)$ can be represented in the form (41).

Here and in the sequel, the dependence on the group of variables $(\alpha, \beta_1, \beta_2, \beta_3, \Omega/v)$ is meant as the composite dependence on $(\alpha, \beta_1, \beta_2, \beta_3, z_1/v, z_2/v, z_3/v, z_4/v)$ due to (59).

The uniqueness theorem for system (50)–(57) on the manifold (60) for odd k violates in the following sense: for odd k through almost all points of the manifold (60), passes a nonsingular phase trajectory of system (50)–(57) intersecting the manifold (60) at right angle and there exists a phase trajectory that at any time instants completely coincides with the point specified However, physically these trajectories are different since they correspond to different values of the tracing force. Prove this.

As was shown above, to maintain the constraint of the form (38), we must take a value of T for $\cos\alpha \neq 0$ according to (40).

Let

$$\lim_{\alpha\to\pi/2} \frac{s(\alpha)}{\cos\alpha} \Gamma_v\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) =$$
$$= L\left(\beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right). \quad (70)$$

Note that $|L| < +\infty$ if and only if

$$\lim_{\alpha\to\pi/2} \left| \frac{\partial}{\partial\alpha} \left(\Gamma_v\left(\alpha, \beta_1, \beta_2, \beta_3 \frac{\Omega}{v}\right) s(\alpha) \right) \right| < +\infty. \quad (71)$$

For $\alpha = \pi/2$, the required value of the tracing force is define by the equation

$$T = T_v\left(\frac{\pi}{2}, \beta_1, \beta_2, \beta_3, \Omega\right) =$$
$$= m\sigma(\omega_4^2 + \omega_7^2 + \omega_9^2 + \omega_{10}^2) - \frac{m\sigma L v^2}{2I_2}. \quad (72)$$

where $\omega_4, \omega_7, \omega_9, \omega_{10}$ are arbitrary.

On the other hand, maintaining the rotation about some point W by the tracing force, we must choose this force according to the relation

$$T = T_v\left(\frac{\pi}{2}, \beta_1, \beta_2, \beta_3, \Omega\right) = \frac{mv^2}{R_0}, \quad (73)$$

where R_0 is the distance CW.

Relations (72) and (73) define in general, different values of the tracing force T for almost all points of the manifold (60), which proves our assertion.

IV. Case Where the Moment of a Nonconservative Force Is Independent of the Angular Velocity

A. Reduced system

Similarly to the choice of Chaplygin analytic functions, we take the dynamical functions s, x_{2N}, x_{3N}, x_{4N}, and x_{5N} in the following form:

$$s(\alpha) = B\cos\alpha,$$
$$x_{2N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) =$$
$$= x_{2N0}(\alpha, \beta_1, \beta_2, \beta_3) = A\sin\alpha\cos\beta_1,$$
$$x_{3N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) =$$
$$= x_{3N0}(\alpha, \beta_1, \beta_2, \beta_3) = A\sin\alpha\sin\beta_1\cos\beta_2, \quad (74)$$
$$x_{4N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) =$$
$$= x_{4N0}(\alpha, \beta_1, \beta_2, \beta_3) = A\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3,$$
$$x_{5N}\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) =$$
$$= x_{5N0}(\alpha, \beta_1, \beta_2, \beta_3) =$$
$$= A\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3, \ A, B > 0, \ v \neq 0.$$

We see that in the system considered, the moment of nonconservative forces in independent of the angular velocity (but depends on the angles $\alpha, \beta_1, \beta_2, \beta_3$).

Herewith, the functions $\Gamma_v(\alpha, \beta_1, \beta_2, \beta_3, \Omega/v)$, $\Delta_{v,s}(\alpha, \beta_1, \beta_2, \beta_3, \Omega/v)$, $s = 1, 2, 3$, in system (61)–(68), take the following form:

$$\Gamma_v\left(\alpha, \beta_1, \beta_2, \beta_3, \frac{\Omega}{v}\right) = A\sin\alpha,$$

$$\Delta_{v,s}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)\equiv 0,\ s=1,2,3. \tag{75}$$

Then, due to the nonintegrable constraint (38), outside the manifold (60), the dynamical part of the equations of motion (system (61)–(68)) has the form of the following analytic system:

$$\alpha' = -z_4 + \frac{\sigma ABv}{3I_2}\sin\alpha, \tag{76}$$

$$z_4' = \frac{ABv^2}{3I_2}\sin\alpha\cos\alpha - (z_1^2+z_2^2+z_3^2)\frac{\cos\alpha}{\sin\alpha}, \tag{77}$$

$$z_3' = z_3z_4\frac{\cos\alpha}{\sin\alpha} + (z_1^2+z_2^2)\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{78}$$

$$z_2' = z_2z_4\frac{\cos\alpha}{\sin\alpha} - z_2z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - -z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}, \tag{79}$$

$$z_1' = z_1z_4\frac{\cos\alpha}{\sin\alpha} - z_1z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + +z_1z_2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}, \tag{80}$$

$$\beta_1' = z_3\frac{\cos\alpha}{\sin\alpha}, \tag{81}$$

$$\beta_2' = -z_2\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \tag{82}$$

$$\beta_3' = z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2}. \tag{83}$$

Further, introducing the dimensionless variables, parameters, and the differentiation as follows:

$$z_k \mapsto n_0vz_k,\ k=1,2,3,4,\ n_0^2=\frac{AB}{3I_2},$$
$$b=\sigma n_0,\quad <\cdot>=n_0v<'>, \tag{84}$$

we reduce system (76)–(83) to the form

$$\alpha' = -z_4 + b\sin\alpha, \tag{85}$$

$$z_4' = \sin\alpha\cos\alpha - (z_1^2+z_2^2+z_3^2)\frac{\cos\alpha}{\sin\alpha}, \tag{86}$$

$$z_3' = z_3z_4\frac{\cos\alpha}{\sin\alpha} + (z_1^2+z_2^2)\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{87}$$

$$z_2' = z_2z_4\frac{\cos\alpha}{\sin\alpha} - z_2z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - -z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}, \tag{88}$$

$$z_1' = z_1z_4\frac{\cos\alpha}{\sin\alpha} - z_1z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + +z_1z_2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}, \tag{89}$$

$$\beta_1' = z_3\frac{\cos\alpha}{\sin\alpha}, \tag{90}$$

$$\beta_2' = -z_2\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \tag{91}$$

$$\beta_3' = z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2}. \tag{92}$$

We see that the eighth-order system (85)–(92) (which can be considered as a system on the tangent bundle $T\mathbf{S}^4$ of the four-dimensional sphere $\mathbf{S}^4$, see below) contains the independent seventh-order system (85)–(91) on its own seven-dimensional manifold.

For the complete integration of system (85)–(92), in general, we need seven independent firs integrals. However, after the change of variables

$$\begin{pmatrix} z_4 \\ z_3 \\ z_2 \\ z_1 \end{pmatrix} \to \begin{pmatrix} w_4 \\ w_3 \\ w_2 \\ w_1 \end{pmatrix},$$
$$w_4 = z_4,\ w_3 = \sqrt{z_1^2+z_2^2+z_3^2},$$
$$w_2 = \frac{z_2}{z_1},\ w_1 = \frac{z_3}{\sqrt{z_1^2+z_2^2}}, \tag{93}$$

system (85)–(92) splits as follows:

$$\alpha' = -w_4 + b\sin\alpha, \tag{94}$$

$$w_4' = \sin\alpha\cos\alpha - w_3^2\frac{\cos\alpha}{\sin\alpha}, \tag{95}$$

$$w_3' = w_3w_4\frac{\cos\alpha}{\sin\alpha}, \tag{96}$$

$$w_2' = d_2(w_4,w_3,w_2,w_1;\alpha,\beta_1,\beta_2,\beta_3)\frac{1+w_2^2}{w_2}\frac{\cos\beta_2}{\sin\beta_2},\quad \beta_2' = d_2(w_4,w_3,w_2,w_1;\alpha,\beta_1,\beta_2,\beta_3), \tag{97}$$

$$w_1' = d_1(w_4,w_3,w_2,w_1;\alpha,\beta_1,\beta_2,\beta_3)\frac{1+w_1^2}{w_1}\frac{\cos\beta_1}{\sin\beta_1},\quad \beta_1' = d_1(w_4,w_3,w_2,w_1;\alpha,\beta_1,\beta_2,\beta_3), \tag{98}$$

$$\beta_3' = d_3(w_4,w_3,w_2,w_1;\alpha,\beta_1,\beta_2,\beta_3), \tag{99}$$

where

$$\begin{aligned}
&d_1(w_4,w_3,w_2,w_1;\alpha,\beta_1,\beta_2,\beta_3) = \\
&= Z_3(w_4,w_3,w_2,w_1)\frac{\cos\alpha}{\sin\alpha}, \\
&d_2(w_4,w_3,w_2,w_1;\alpha,\beta_1,\beta_2,\beta_3) = \\
&= -Z_2(w_4,w_3,w_2,w_1)\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \\
&d_3(w_4,w_3,w_2,w_1;\alpha,\beta_1,\beta_2,\beta_3) = \\
&= Z_1(w_4,w_3,w_2,w_1)\frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2},
\end{aligned} \tag{100}$$

herewith

$$z_k = Z_k(w_4,w_3,w_2,w_1),\ k=1,2,3, \tag{101}$$

are the functions due to the change of variables (93).

We see that the eighth-order system splits into independent subsystems of lower order: system (94)–(96) has order three and systems (97), (98) (after the change of the independent variable) have order two. Thus, for the complete integration of system (94)–(99) it suffice to specify two independent firs integrals of system (94)–(96), one firs integral of each system (97), (98), and an additional firs integral that attaches Eq. (99).

Note that system (94)–(96) can be considered on the tangent bundle $T\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$.

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B. Complete list of invariant relations

System (94)–(96) has the form of a system that appears in the dynamics of a three-dimensional ($3D-$) rigid body in a fiel of nonconservative forces.

First, to the third-order system (94)–(96), we put in correspondence the nonautonomous second-order system

$$\begin{aligned}\frac{dw_4}{d\alpha} &= \frac{\sin\alpha\cos\alpha - w_3^2\cos\alpha/\sin\alpha}{-w_4 + b\sin\alpha},\\ \frac{dw_3}{d\alpha} &= \frac{w_3 w_4\cos\alpha/\sin\alpha}{-w_4 + b\sin\alpha}.\end{aligned} \tag{102}$$

Applying the substitution $\tau = \sin\alpha$, we rewrite system (102) in the algebraic form

$$\begin{aligned}\frac{dw_4}{d\tau} &= \frac{\tau - w_3^2/\tau}{-w_4 + b\tau},\\ \frac{dw_3}{d\tau} &= \frac{w_3 w_4/\tau}{-w_4 + b\tau}.\end{aligned} \tag{103}$$

Later on, introducing the homogeneous variables by the formulas

$$w_3 = u_1\tau,\ w_4 = u_2\tau, \tag{104}$$

we reduce system (103) to the following form:

$$\begin{aligned}\tau\frac{du_2}{d\tau} + u_2 &= \frac{1 - u_1^2}{-u_2 + b},\\ \tau\frac{du_1}{d\tau} + u_1 &= \frac{u_1 u_2}{-u_2 + b},\end{aligned} \tag{105}$$

which is equivalent to the system

$$\begin{aligned}\tau\frac{du_2}{d\tau} &= \frac{1 - u_1^2 + u_2^2 - bu_2}{-u_2 + b},\\ \tau\frac{du_1}{d\tau} &= \frac{2u_1u_2 - bu_1}{-u_2 + b}.\end{aligned} \tag{106}$$

To the second-order system (106), we put in correspondence the nonautonomous first-orde equation

$$\frac{du_2}{du_1} = \frac{1 - u_1^2 + u_2^2 - bu_2}{2u_1u_2 - bu_1}, \tag{107}$$

which can be easily reduced to the exact-differential form:

$$d\left(\frac{u_2^2 + u_1^2 - bu_2 + 1}{u_1}\right) = 0. \tag{108}$$

Thus, Eq. (107) has the following firs integral:

$$\frac{u_2^2 + u_1^2 - bu_2 + 1}{u_1} = C_1 = \text{const}, \tag{109}$$

which in the previous variables has the form

$$\frac{w_4^2 + w_3^2 - bw_4\sin\alpha + \sin^2\alpha}{w_3\sin\alpha} = C_1 = \text{const}. \tag{110}$$

Remark 1. Consider system (94)–(96) with variable dissipation with zero mean, that becomes conservative for $b = 0$:

$$\begin{aligned}\alpha' &= -w_4,\\ w_4' &= \sin\alpha\cos\alpha - w_3^2\frac{\cos\alpha}{\sin\alpha},\\ w_3' &= w_3w_4\frac{\cos\alpha}{\sin\alpha}.\end{aligned} \tag{111}$$

It possesses two analytic firs integrals of the form

$$w_4^2 + w_3^2 + \sin^2\alpha = C_1^* = \text{const}, \tag{112}$$

$$w_3\sin\alpha = C_2^* = \text{const}. \tag{113}$$

Obviously, the ratio of two firs integrals (112), (113) is also a firs integral of system (111). But for $b \neq 0$, each of the functions

$$w_4^2 + w_3^2 - bw_4\sin\alpha + \sin^2\alpha \tag{114}$$

and (113) is not a firs integral of system (94)–(96). However, but their ratio is a firs integral for any b.

Further, we fin the explicit form of the additional firs integral of the third-order system (94)–(96). For this, we transform the invariant relation (109) for $u_1 \neq 0$ as follows:

$$\left(u_2 - \frac{b}{2}\right)^2 + \left(u_1 - \frac{C_1}{2}\right)^2 = \frac{b^2 + C_1^2}{4} - 1. \tag{115}$$

We see that the parameters of this invariant relation satisfy the condition

$$b^2 + C_1^2 - 4 \geq 0, \tag{116}$$

and the phase space of system (94)–(96) is stratifie into the family of surfaces define by Eq. (115).

Thus, by relation (109), the firs equation of system (106) has the form

$$\tau\frac{du_2}{d\tau} = \frac{2(1 - bu_2 + u_2^2) - C_1U_1(C_1, u_2)}{-u_2 + b}, \tag{117}$$

where

$$U_1(C_1, u_2) = \frac{1}{2}\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + 1)}\}; \tag{118}$$

the integration constant C_1 is define by condition (116).

Therefore, the quadrature for the search for the additional firs integral of system (94)–(96) becomes

$$\begin{gathered}\int\frac{d\tau}{\tau} = \\ = \int\frac{(b - u_2)du_2}{2A^0 - C_1\{C_1 \pm \sqrt{C_1^2 - 4A^0}\}/2},\\ A^0 = 1 - bu_2 + u_2^2.\end{gathered} \tag{119}$$

Obviously, the left-hand side (up to an additive constant) equals

$$\ln|\sin\alpha|. \tag{120}$$

If

$$u_2 - \frac{b}{2} = r_1,\ b_1^2 = b^2 + C_1^2 - 4, \tag{121}$$

then the right-hand side of Eq. (119) has the form

$$\begin{gathered}-\frac{1}{4}\int\frac{d(b_1^2 - 4r_1^2)}{(b_1^2 - 4r_1^2) \pm C_1\sqrt{b_1^2 - 4r_1^2}} - \\ -b\int\frac{dr_1}{(b_1^2 - 4r_1^2) \pm C_1\sqrt{b_1^2 - 4r_1^2}} = \\ = -\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2 - 4r_1^2}}{C_1} \pm 1\right| \pm \frac{b}{2}I_1,\end{gathered} \tag{122}$$

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where

$$I_1 = \int \frac{dr_3}{\sqrt{b_1^2 - r_3^2}(r_3 \pm C_1)}, \; r_3 = \sqrt{b_1^2 - 4r_1^2}. \quad (123)$$

In the calculation of integral (123), the following three cases are possible.

I. $b > 2$.

$$I_1 = -\frac{1}{2\sqrt{b^2-4}} \ln \left| \frac{\sqrt{b^2-4} + \sqrt{b_1^2 - r_3^2}}{r_3 \pm C_1} \pm \frac{C_1}{\sqrt{b^2-4}} \right| + $$
$$+ \frac{1}{2\sqrt{b^2-4}} \ln \left| \frac{\sqrt{b^2-4} - \sqrt{b_1^2 - r_3^2}}{r_3 \pm C_1} \mp \frac{C_1}{\sqrt{b^2-4}} \right| + $$
$$+\text{const.} \quad (124)$$

II. $b < 2$.

$$I_1 = \frac{1}{\sqrt{4-b^2}} \arcsin \frac{\pm C_1 r_3 + b_1^2}{b_1(r_3 \pm C_1)} + \text{const.} \quad (125)$$

III. $b = 2$.

$$I_1 = \mp \frac{\sqrt{b_1^2 - r_3^2}}{C_1(r_3 \pm C_1)} + \text{const.} \quad (126)$$

Returning to the variable

$$r_1 = \frac{w_4}{\sin\alpha} - \frac{b}{2}, \quad (127)$$

we obtain the fina expression for I_1:

I. $b > 2$.

$$I_1 = -\frac{1}{2\sqrt{b^2-4}} \ln \left| \frac{\sqrt{b^2-4} \pm 2r_1}{\sqrt{b_1^2 - 4r_1^2} \pm C_1} \pm \frac{C_1}{\sqrt{b^2-4}} \right| + $$
$$+ \frac{1}{2\sqrt{b^2-4}} \ln \left| \frac{\sqrt{b^2-4} \mp 2r_1}{\sqrt{b_1^2 - 4r_1^2} \pm C_1} \mp \frac{C_1}{\sqrt{b^2-4}} \right| + \text{const.} \quad (128)$$

II. $b < 2$.

$$I_1 = \frac{1}{\sqrt{4-b^2}} \arcsin \frac{\pm C_1 \sqrt{b_1^2 - 4r_1^2} + b_1^2}{b_1(\sqrt{b_1^2 - 4r_1^2} \pm C_1)} + \text{const.} \quad (129)$$

III. $b = 2$.

$$I_1 = \mp \frac{2r_1}{C_1(\sqrt{b_1^2 - 4r_1^2} \pm C_1)} + \text{const.} \quad (130)$$

Thus, we have found an additional firs integral for the third-order system (94)–(96) and we have the complete set of firs integrals that are transcendental functions of their phase variables.

Remark 2. We must substitute the left-hand side of the firs integral (109) in the expression of this firs integral instead C_1.

Then the additional firs integral obtained has the following structure (similar to the transcendental firs integral in planar dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha, \frac{w_4}{\sin\alpha}, \frac{w_3}{\sin\alpha}\right) = C_2 = \text{const.} \quad (131)$$

Thus, for the integration of the eighth-order system (94)–(99), we have found two independent firs integrals. For the complete integration, as was mentioned above, it suffice to fin one firs integral for each (potentially separated) system (97), (98), and an additional firs integral that attaches Eq. (99).

To fin a firs integral for each (potentially separated) system (97), (98), we put in correspondence the following nonautonomous first-orde equation:

$$\frac{dw_s}{d\beta_s} = \frac{1 + w_s^2}{w_s} \frac{\cos\beta_s}{\sin\beta_s}, \; s = 1, 2. \quad (132)$$

After integration, this leads to the invariant relation

$$\frac{\sqrt{1 + w_s^2}}{\sin\beta_s} = C_{s+2} = \text{const}, \; s = 1, 2. \quad (133)$$

Further, for the search for an additional firs integral that attaches Eq. (99), to Eqs. (99) and (97) we put in correspondence the following nonautonomous equation:

$$\frac{dw_2}{d\beta_3} = -(1 + w_2^2)\cos\beta_2. \quad (134)$$

Since, by (133),

$$C_4 \cos\beta_2 = \pm\sqrt{C_4^2 - 1 - w_2^2}, \quad (135)$$

we have

$$\frac{dw_2}{d\beta_3} = \mp \frac{1}{C_4}(1 + w_2^2)\sqrt{C_4^2 - 1 - w_2^2}. \quad (136)$$

Integrating the last relation, we arrive at the following quadrature:

$$\mp(\beta_3 + C_5) = \int \frac{C_4 dw_2}{(1 + w_2^2)\sqrt{C_4^2 - 1 - w_2^2}}, \; C_5 = \text{const.} \quad (137)$$

Integrating this relation we obtain

$$\mp \text{tg}(\beta_3 + C_5) = \frac{C_4 w_2}{\sqrt{C_4^2 - 1 - w_2^2}}, \; C_5 = \text{const.} \quad (138)$$

Finally, we have the following form of the additional firs integral that attaches Eq. (99):

$$\text{arctg} \frac{C_4 w_2}{\sqrt{C_4^2 - 1 - w_2^2}} \pm \beta_3 = C_5, \; C_5 = \text{const.} \quad (139)$$

Thus, in the case considered, the system of dynamical equations (17)–(21), (24)–(33) under condition (74) has twelve invariant relations: the nonintegrable analytic constraint of the form (38), the cyclic firs integrals of the form (36), (37), the firs integral of the form (110), the firs integral expressed by relations (124)–(131), which is a transcendental function of the phase variables (in the sense of complex analysis) expressed through a finit combination of elementary functions, and, finall , the transcendental firs integrals of the form (133) and (139).

Theorem 1. *System (17)–(21), (24)–(33) under conditions (38), (74), (37) possesses twelve invariant relations (complete set), five of which transcendental functions from the point of view of complex analysis. Herewith, all relations are expressed through finite combinations of elementary functions.*

C. Topological analogies

Consider the following seventh-order system:

$$
\begin{aligned}
&\ddot{\xi} + b_*\dot{\xi}\cos\xi + \sin\xi\cos\xi - \\
&- [\dot{\eta}_1{}^2 + \dot{\eta}_2{}^2\sin^2\eta_1 + \dot{\eta}_3{}^2\sin^2\eta_1\sin^2\eta_2]\frac{\sin\xi}{\cos\xi} = 0, \\
&\ddot{\eta}_1 + b_*\dot{\eta}_1\cos\xi + \dot{\xi}\dot{\eta}_1\frac{1+\cos^2\xi}{\cos\xi\sin\xi} - \\
&- (\dot{\eta}_2{}^2 + \dot{\eta}_3{}^2\sin^2\eta_2)\sin\eta_1\cos\eta_1 = 0, \\
&\ddot{\eta}_2 + b_*\dot{\eta}_2\cos\xi + \dot{\xi}\dot{\eta}_2\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\
&+ 2\dot{\eta}_1\dot{\eta}_2\frac{\cos\eta_1}{\sin\eta_1} - \dot{\eta}_3{}^2\sin\eta_2\cos\eta_2 = 0, \\
&\ddot{\eta}_3 + b_*\dot{\eta}_3\cos\xi + \dot{\xi}\dot{\eta}_3\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\
&+ 2\dot{\eta}_1\dot{\eta}_3\frac{\cos\eta_1}{\sin\eta_1} + 2\dot{\eta}_2\dot{\eta}_3\frac{\cos\eta_2}{\sin\eta_2} = 0, \ b_* > 0,
\end{aligned}
\tag{140}
$$

which describes a fi ed f ve-dimensional pendulum in a fl w of a running medium for which the moment of forces is independent of the angular velocity, i.e., a mechanical system in a nonconservative field In general the order of such a system is equal to 8, but the phase variable η_3 is a cyclic variable, which leads to the stratificatio of the phase space and reduces the order of the system.

The phase space of this system is the tangent bundle

$$T\mathbf{S}^3\{\dot{\xi}, \dot{\eta}_1, \dot{\eta}_2, \dot{\eta}_3, \xi, \eta_1, \eta_2, \eta_3\} \tag{141}$$

of the four-dimensional sphere $\mathbf{S}^4\{\xi, \eta_1, \eta_2, \eta_3\}$. The equation that transforms system (140) the system on the tangent bundle of the three-dimensional sphere

$$\dot{\eta}_3 \equiv 0, \tag{142}$$

and the equations of great circles

$$\dot{\eta}_1 \equiv 0, \ \dot{\eta}_2 \equiv 0, \ \dot{\eta}_3 \equiv 0 \tag{143}$$

defin families of integral manifolds.

It is easy to verify that system (140) is equivalent to the dynamical system with variable dissipation with zero mean on the tangent bundle (141) of the four-dimensional sphere. Moreover, the following theorem holds.

Theorem 2. *System (17)–(21), (24)–(33) under conditions (38), (74), (37) is equivalent to the dynamical system (140).*

Indeed it suffice to set $\alpha = \xi, \ \beta_1 = \eta_1, \ \beta_2 = \eta_2, \ \beta_3 = \eta_3, \ b = -b_*$.

V. Case Where the Moment of a Nonconservative Force Depends on the Angular Velocity

A. Introduction of the dependence on the angular velocity

This chapter is devoted to the dynamics of a f ve-dimensional rigid body in the f ve-dimensional space. Since the present section is devoted to the study of the motion in the case where the moment of forces depends on the tensor of angular velocity, we introduce this dependence in a more general situation. This also allows us to introduce this dependence for multi-dimensional bodies.

Let $x = (x_{1N}, x_{2N}, x_{3N}, x_{4N}, x_{5N})$ be the coordinates of the point N of application of a nonconservative force (influ ence of the medium) acting on the four-dimensional disk and $Q = (Q_1, Q_2, Q_3, Q_4, Q_5)$ be the components independent of the tensor of the angular velocity. We consider only linear dependence of the functions $(x_{1N}, x_{2N}, x_{3N}, x_{4N}, x_{5N})$ on the tensor of angular velocity since this introduction itself is not obvious.

We adopt the following dependence:

$$x = Q + R, \tag{144}$$

where $R = (R_1, R_2, R_3, R_4, R_5)$ is a vector-valued function containing the components of the tensor of angular velocity. The dependence of the function R on the components of the tensor of angular velocity is gyroscopic:

$$
R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{pmatrix} =
$$

$$
= -\frac{1}{v}\begin{pmatrix} 0 & -\omega_{10} & \omega_9 & -\omega_7 & \omega_4 \\ \omega_{10} & 0 & -\omega_8 & \omega_6 & -\omega_3 \\ -\omega_9 & \omega_8 & 0 & -\omega_5 & \omega_2 \\ \omega_7 & -\omega_6 & \omega_5 & 0 & -\omega_1 \\ -\omega_4 & \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \end{pmatrix}, \tag{145}
$$

where $(h_1, h_2, h_3, h_4, h_5)$ are some positive parameters.

Since $x_{1N} \equiv 0$, we have

$$
\begin{aligned}
x_{2N} &= Q_2 - h_1\frac{\omega_{10}}{v}, \\
x_{3N} &= Q_3 + h_1\frac{\omega_9}{v}, \\
x_{4N} &= Q_4 - h_1\frac{\omega_7}{v}, \\
x_{5N} &= Q_5 + h_1\frac{\omega_4}{v}.
\end{aligned}
\tag{146}
$$

B. Reduced system

Similarly to the choice of the Chaplygin analytic functions

$$
\begin{gathered}
Q_2 = A\sin\alpha\cos\beta_1, \ Q_3 = A\sin\alpha\sin\beta_1\cos\beta_2, \\
Q_4 = A\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3, \\
Q_5 = A\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3, \ A > 0,
\end{gathered}
\tag{147}
$$

we take the dynamical functions s, x_{2N}, x_{3N}, x_{4N}, and x_{5N} in the following form:

$$
\begin{gathered}
s(\alpha) = B\cos\alpha, \ B > 0, \\
x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = A\sin\alpha\cos\beta_1 - h\frac{\omega_{10}}{v}, \\
x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = A\sin\alpha\sin\beta_1\cos\beta_2 + h\frac{\omega_9}{v}, \\
x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = A\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_3 - h\frac{\omega_7}{v},
\end{gathered}
\tag{148}
$$

$$x_{5N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)=A\sin\alpha\sin\beta_1\sin\beta_2\sin\beta_3+$$
$$+h\frac{\omega_4}{v},\ h=h_1>0,\ v\neq 0.$$

This shows that in the problem considered, there is an additional damping (but accelerating in certain domains of the phase space) moment of a nonconservative force (i.e., there is a dependence of the moment on the components of the tensor of angular velocity). Moreover, $h_2=h_3=h_4=h_5$ due to the dynamical symmetry of the body.

In this case, the functions $\Gamma_v(\alpha,\beta_1,\beta_2,\beta_3,\Omega/v)$, $\Delta_{v,s}(\alpha,\beta_1,\beta_2,\beta_3,\Omega/v)$, $s=1,2,3$, in system (61)–(68) have the following form:

$$\Gamma_v\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)=A\sin\alpha-\frac{h}{v}z_4,$$
$$\Delta_{v,1}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)=\frac{h}{v}z_3, \tag{149}$$
$$\Delta_{v,2}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)=-\frac{h}{v}z_2,$$
$$\Delta_{v,3}\left(\alpha,\beta_1,\beta_2,\beta_3,\frac{\Omega}{v}\right)=\frac{h}{v}z_1.$$

Then, due to the nonintegrable constraint (38), outside the manifold (60) the dynamical part of the equations of motion (system (61)–(68)) takes the form of the analytic system

$$\dot\alpha=-\left(1+\frac{\sigma Bh}{3I_2}\right)z_4+\frac{\sigma ABv}{3I_2}\sin\alpha, \tag{150}$$

$$\dot z_4=\frac{ABv^2}{3I_2}\sin\alpha\cos\alpha-$$
$$-\left(1+\frac{\sigma Bh}{3I_2}\right)(z_1^2+z_2^2+z_3^2)\frac{\cos\alpha}{\sin\alpha}-\frac{Bhv}{3I_2}z_4\cos\alpha, \tag{151}$$

$$\dot z_3=\left(1+\frac{\sigma Bh}{3I_2}\right)z_3z_4\frac{\cos\alpha}{\sin\alpha}+$$
$$+\left(1+\frac{\sigma Bh}{3I_2}\right)(z_1^2+z_2^2)\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}-\frac{Bhv}{3I_2}z_3\cos\alpha, \tag{152}$$

$$\dot z_2=\left(1+\frac{\sigma Bh}{3I_2}\right)z_2z_4\frac{\cos\alpha}{\sin\alpha}-$$
$$-\left(1+\frac{\sigma Bh}{3I_2}\right)z_2z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}-$$
$$-\left(1+\frac{\sigma Bh}{3I_2}\right)z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}-\frac{Bhv}{3I_2}z_2\cos\alpha, \tag{153}$$

$$\dot z_1=\left(1+\frac{\sigma Bh}{3I_2}\right)z_1z_4\frac{\cos\alpha}{\sin\alpha}-$$
$$-\left(1+\frac{\sigma Bh}{3I_2}\right)z_1z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}+$$
$$+\left(1+\frac{\sigma Bh}{3I_2}\right)z_1z_2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}-$$
$$-\frac{Bhv}{3I_2}z_1\cos\alpha, \tag{154}$$

$$\dot\beta_1=\left(1+\frac{\sigma Bh}{3I_2}\right)z_3\frac{\cos\alpha}{\sin\alpha}, \tag{155}$$
$$\dot\beta_2=-\left(1+\frac{\sigma Bh}{3I_2}\right)z_2\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \tag{156}$$
$$\dot\beta_3=\left(1+\frac{\sigma Bh}{3I_2}\right)z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2}. \tag{157}$$

Introducing the dimensionless variables, parameters, and the differentiation as follows:

$$\begin{gathered}z_k\mapsto n_0vz_k,\ k=1,2,3,4,\ n_0^2=\frac{AB}{3I_2},\\ b=\sigma n_0,\ H_1=\frac{Bh}{3I_2n_0},\ \ <\cdot>=n_0v<'>,\end{gathered} \tag{158}$$

we reduce system (150)–(157) to the form

$$\alpha'=-(1+bH_1)z_4+b\sin\alpha, \tag{159}$$

$$z_4'=\sin\alpha\cos\alpha-$$
$$-(1+bH_1)(z_1^2+z_2^2+z_3^2)\frac{\cos\alpha}{\sin\alpha}-H_1z_4\cos\alpha, \tag{160}$$

$$z_3'=(1+bH_1)z_3z_4\frac{\cos\alpha}{\sin\alpha}+$$
$$+(1+bH_1)(z_1^2+z_2^2)\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}-H_1z_3\cos\alpha, \tag{161}$$

$$z_2'=(1+bH_1)z_2z_4\frac{\cos\alpha}{\sin\alpha}-$$
$$-(1+bH_1)z_2z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}-$$
$$-(1+bH_1)z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}-H_1z_2\cos\alpha, \tag{162}$$

$$z_1'=(1+bH_1)z_1z_4\frac{\cos\alpha}{\sin\alpha}-$$
$$-(1+bH_1)z_1z_3\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}+$$
$$+(1+bH_1)z_1z_2\frac{\cos\alpha}{\sin\alpha}\frac{1}{\sin\beta_1}\frac{\cos\beta_2}{\sin\beta_2}-H_1z_1\cos\alpha, \tag{163}$$

$$\beta_1'=(1+bH_1)z_3\frac{\cos\alpha}{\sin\alpha}, \tag{164}$$
$$\beta_2'=-(1+bH_1)z_2\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \tag{165}$$
$$\beta_3'=(1+bH_1)z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2}. \tag{166}$$

We see that the eighth-order system (159)–(166) (which can be considered on the tangent bundle $T\mathbf{S}^4$ of the four-dimensional sphere $\mathbf{S}^4$), contains an independent seventh-order system (159)–(165) on its own seven-dimensional manifold.

For the complete integration of system (159)–(166), we need, in general, seven independent firs integrals. However, after the change of variables

$$\begin{pmatrix}z_4\\z_3\\z_2\\z_1\end{pmatrix}\to\begin{pmatrix}w_4\\w_3\\w_2\\w_1\end{pmatrix},$$

$$w_4 = z_4,\ w_3 = \sqrt{z_1^2 + z_2^2 + z_3^2},$$

$$w_2 = \frac{z_2}{z_1},\ w_1 = \frac{z_3}{\sqrt{z_1^2 + z_2^2}}, \tag{167}$$

system (159)–(166) splits as follows:

$$\alpha' = -(1 + bH_1)w_4 + b\sin\alpha, \tag{168}$$

$$w_4' = \sin\alpha\cos\alpha - (1 + bH_1)w_3^2\frac{\cos\alpha}{\sin\alpha} - H_1 w_4\cos\alpha, \tag{169}$$

$$w_3' = (1 + bH_1)w_3 w_4\frac{\cos\alpha}{\sin\alpha} - H_1 w_3\cos\alpha, \tag{170}$$

$$\begin{aligned} w_2' &= d_2(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3)\times \\ &\times \frac{1 + w_2^2}{w_2}\frac{\cos\beta_2}{\sin\beta_2}, \\ \beta_2' &= d_2(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3), \end{aligned} \tag{171}$$

$$\begin{aligned} w_1' &= d_1(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3)\times \\ &\times \frac{1 + w_1^2}{w_1}\frac{\cos\beta_1}{\sin\beta_1}, \\ \beta_1' &= d_1(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3), \end{aligned} \tag{172}$$

$$\beta_3' = d_3(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3), \tag{173}$$

where

$$\begin{aligned} & d_1(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3) = \\ & = (1 + bH_1)Z_3(w_4, w_3, w_2, w_1)\frac{\cos\alpha}{\sin\alpha}, \\ & d_2(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3) = \\ & = -(1 + bH_1)Z_2(w_4, w_3, w_2, w_1)\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \\ & d_3(w_4, w_3, w_2, w_1; \alpha, \beta_1, \beta_2, \beta_3) = \\ & = (1 + bH_1)Z_1(w_4, w_3, w_2, w_1)\frac{\cos\alpha}{\sin\alpha\sin\beta_1\sin\beta_2}, \end{aligned} \tag{174}$$

herewith,

$$z_k = Z_k(w_4, w_3, w_2, w_1),\ k = 1, 2, 3, \tag{175}$$

are the functions, due to the change of variables (167).

We see that the eighth-order system splits into independent subsystems of lower orders: system (168)–(170) of order 3 and each of system (171), (172) (certainly, after a choice of the independent variables) of order 2. Thus, for the complete integration of system (168)–(173), it suffice to fin two independent firs integrals of system (168)–(170), one firs integral of each system (171), (172), and an additional firs integral that attaches Eq. (173).

Note that system (168)–(170) can be considered on the tangent bundle $T\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$.

C. Complete list of invariant relation

System (168)–(170) has the form of a system of equations that appears in the dynamics of a three-dimensional ($3D-$) rigid body in a nonconservative field

First, to the third-order system (168)–(170), we put in correspondence the nonautonomous second-order system

$$\begin{aligned} & \frac{dw_4}{d\alpha} = \\ & = \frac{\sin\alpha\cos\alpha - (1 + bH_1)w_3^2\cos\alpha/\sin\alpha - H_1 w_4\cos\alpha}{-(1 + bH_1)w_4 + b\sin\alpha}, \\ & \frac{dw_3}{d\alpha} = \\ & = \frac{(1 + bH_1)w_3 w_4\cos\alpha/\sin\alpha - H_1 w_3\cos\alpha}{-(1 + bH_1)w_4 + b\sin\alpha}. \end{aligned} \tag{176}$$

Using the substitution $\tau = \sin\alpha$, we rewrite system (176) in the algebraic form:

$$\begin{aligned} \frac{dw_4}{d\tau} &= \frac{\tau - (1 + bH_1)w_3^2/\tau - H_1 w_4}{-(1 + bH_1)w_4 + b\tau}, \\ \frac{dw_3}{d\tau} &= \frac{(1 + bH_1)w_3 w_4/\tau - H_1 w_3}{-(1 + bH_1)w_4 + b\tau}. \end{aligned} \tag{177}$$

Further, introducing the homogeneous variables by the formulas

$$w_3 = u_1\tau,\ w_4 = u_2\tau, \tag{178}$$

we reduce system (177) to the following form:

$$\begin{aligned} \tau\frac{du_2}{d\tau} + u_2 &= \frac{1 - (1 + bH_1)u_1^2 - H_1 u_2}{-(1 + bH_1)u_2 + b}, \\ \tau\frac{du_1}{d\tau} + u_1 &= \frac{(1 + bH_1)u_1 u_2 - H_1 u_1}{-(1 + bH_1)u_2 + b}, \end{aligned} \tag{179}$$

which is equivalent to

$$\begin{aligned} \tau\frac{du_2}{d\tau} &= \frac{(1 + bH_1)(u_2^2 - u_1^2) - (b + H_1)u_2 + 1}{-(1 + bH_1)u_2 + b}, \\ \tau\frac{du_1}{d\tau} &= \frac{2(1 + bH_1)u_1 u_2 - (b + H_1)u_1}{-(1 + bH_1)u_2 + b}. \end{aligned} \tag{180}$$

To the second-order system (180), we put in correspondence the nonautonomous first-orde equation

$$\frac{du_2}{du_1} = \frac{1 - (1 + bH_1)(u_1^2 - u_2^2) - (b + H_1)u_2}{2(1 + bH_1)u_1 u_2 - (b + H_1)u_1}, \tag{181}$$

which can be easily reduce to the exact-differential form:

$$d\left(\frac{(1 + bH_1)(u_2^2 + u_1^2) - (b + H_1)u_2 + 1}{u_1}\right) = 0. \tag{182}$$

Thus, Eq. (181) has the following firs integral:

$$\begin{aligned} & \frac{(1 + bH_1)(u_2^2 + u_1^2) - (b + H_1)u_2 + 1}{u_1} = \\ & = C_1 = \text{const}, \end{aligned} \tag{183}$$

which in the original variables has the form

$$\begin{aligned} & \frac{(1 + bH_1)(w_4^2 + w_3^2) - (b + H_1)w_4\sin\alpha + \sin^2\alpha}{w_3\sin\alpha} = \\ & = C_1 = \text{const}. \end{aligned} \tag{184}$$

Remark 3. Consider system (168)–(170) with variable dissipation with zero mean, which becomes conservative for $b = H_1$:

$$\begin{aligned} &\alpha' = -(1+b^2)w_4 + b\sin\alpha, \\ &w_4' = \sin\alpha\cos\alpha - (1+b^2)w_3^2\frac{\cos\alpha}{\sin\alpha} - bw_4\cos\alpha, \\ &w_3' = (1+b^2)w_3w_4\frac{\cos\alpha}{\sin\alpha} - bw_3\cos\alpha. \end{aligned} \tag{185}$$

It possesses the following two analytic firs integrals:

$$(1+b^2)(w_4^2+w_3^2) - 2bw_4\sin\alpha + \sin^2\alpha = C_1^* = \text{const}, \tag{186}$$

$$w_3\sin\alpha = C_2^* = \text{const}. \tag{187}$$

Obviously, the ratio of the two firs integrals (186), (187) is also a firs integral of system (185). But for $b \neq H_1$ none of the functions

$$(1+bH_1)(w_4^2+w_3^2) - (b+H_1)w_4\sin\alpha + \sin^2\alpha \tag{188}$$

and (187) is a firs integral of system (168)–(170). However, the ratio of the functions (188), (187) is a firs integral of system (168)–(170) for any b, H_1.

We fin the explicit form of the additional firs integral of the third-order system (168)–(170). First, we transform the invariant relation (183) for $u_1 \neq 0$ as follows:

$$\left(u_2 - \frac{b+H_1}{2(1+bH_1)}\right)^2 + \left(u_1 - \frac{C_1}{2(1+bH_1)}\right)^2 = \\ = \frac{(b-H_1)^2 + C_1^2 - 4}{4(1+bH_1)^2}. \tag{189}$$

We see that the parameters of this invariant relation must satisfy the condition

$$(b-H_1)^2 + C_1^2 - 4 \geq 0, \tag{190}$$

and the phase space of system (168)–(170) is stratifie into the family of surfaces define by Eq. (189).

Thus, due to relation (183), the firs equation of system (180) has the form

$$\tau\frac{du_2}{d\tau} = \\ = \frac{2(1+bH_1)u_2^2 - 2(b+H_1)u_2 + 2 - C_1U_1(C_1,u_2)}{b - (1+bH_1)u_2}, \tag{191}$$

where

$$U_1(C_1,u_2) = \frac{1}{2(1+bH_1)}\{C_1 \pm U_2(C_1,u_2)\}, \tag{192}$$

$$U_2(C_1,u_2) = \\ = \sqrt{C_1^2 - 4(1+bH_1)(1-(b+H_1)u_2 + (1+bH_1)u_2^2)},$$

and the integration constant C_1 is define by condition (190).

Therefore, the quadrature for the search for an additional firs integral of system (168)–(170) becomes

$$\int\frac{d\tau}{\tau} = \\ = \int\frac{(b-(1+bH_1)u_2)du_2}{2A^1 - C_1\{C_1 \pm U_2(C_1,u_2)\}/(2(1+bH_1))}, \tag{193}$$

$$A^1 = 1 - (b+H_1)u_2 + (1+bH_1)u_2^2.$$

Obviously, the left-hand side (up to an additive constant) is equal to

$$\ln|\sin\alpha|. \tag{194}$$

If

$$u_2 - \frac{b+H_1}{2(1+bH_1)} = r_1,\ b_1^2 = (b-H_1)^2 + C_1^2 - 4, \tag{195}$$

then the right-hand side of Eq. (193) becomes

$$-\frac{1}{4}\int\frac{d(b_1^2 - 4(1+bH_1)r_1^2)}{(b_1^2 - 4(1+bH_1)r_1^2) \pm C_1\sqrt{b_1^2 - 4(1+bH_1)r_1^2}} - \\ -(b-H_1)(1+bH_1)\times \\ \times\int\frac{dr_1}{(b_1^2 - 4(1+bH_1)r_1^2) \pm C_1\sqrt{b_1^2 - 4(1+bH_1)r_1^2}} = \\ = -\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2 - 4(1+bH_1)r_1^2}}{C_1} \pm 1\right| \pm \frac{b-H_1}{2}I_1, \tag{196}$$

where

$$I_1 = \int\frac{dr_3}{\sqrt{b_1^2 - r_3^2}(r_3 \pm C_1)}, \\ r_3 = \sqrt{b_1^2 - 4(1+bH_1)r_1^2}. \tag{197}$$

In the calculation of integral (197), the following three cases are possible:

I. $|b - H_1| > 2$.

$$I_1 = -\frac{1}{2\sqrt{(b-H_1)^2 - 4}}\times \\ \times\ln\left|\frac{\sqrt{(b-H_1)^2-4} + \sqrt{b_1^2 - r_3^2}}{r_3 \pm C_1} \pm \frac{C_1}{\sqrt{(b-H_1)^2-4}}\right| + \\ +\frac{1}{2\sqrt{(b-H_1)^2-4}}\times \\ \times\ln\left|\frac{\sqrt{(b-H_1)^2-4} - \sqrt{b_1^2 - r_3^2}}{r_3 \pm C_1} \mp \frac{C_1}{\sqrt{(b-H_1)^2-4}}\right| + \\ +\text{const}. \tag{198}$$

II. $|b - H_1| < 2$.

$$I_1 = \frac{1}{\sqrt{4-(b-H_1)^2}}\arcsin\frac{\pm C_1r_3 + b_1^2}{b_1(r_3 \pm C_1)} + \text{const}. \tag{199}$$

III. $|b - H_1| = 2$.

$$I_1 = \mp\frac{\sqrt{b_1^2 - r_3^2}}{C_1(r_3 \pm C_1)} + \text{const}. \tag{200}$$

Returning to the variable

$$r_1 = \frac{w_3}{\sin\alpha} - \frac{b+H_1}{2(1+bH_1)}, \tag{201}$$

we have the following fina form of I_1:

I. $|b - H_1| > 2$.

$$I_1 = -\frac{1}{2\sqrt{(b-H_1)^2-4}}\times$$

$$\times \ln\left|\frac{\sqrt{(b-H_1)^2-4} \pm 2(1+bH_1)r_1}{\sqrt{b_1^2-4(1+bH_1)^2r_1^2} \pm C_1} \pm \frac{C_1}{\sqrt{(b-H_1)^2-4}}\right| +$$
$$+\frac{1}{2\sqrt{(b-H_1)^2-4}}\times$$
$$\times \ln\left|\frac{\sqrt{(b-H_1)^2-4} \mp 2(1+bH_1)r_1}{\sqrt{b_1^2-4(1+bH_1)^2r_1^2} \pm C_1} \mp \frac{C_1}{\sqrt{(b-H_1)^2-4}}\right| +$$
$$+\text{const.} \tag{202}$$

II. $|b-H_1|<2$.

$$I_1 = \frac{1}{\sqrt{4-(b-H_1)^2}}\times$$
$$\times \arcsin\frac{\pm C_1\sqrt{b_1^2-4(1+bH_1)^2r_1^2}+b_1^2}{b_1(\sqrt{b_1^2-4(1+bH_1)^2r_1^2} \pm C_1)} + \text{const.} \tag{203}$$

III. $|b-H_1|=2$.

$$I_1 = \mp\frac{2(1+bH_1)r_1}{C_1(\sqrt{b_1^2-4(1+bH_1)^2r_1^2} \pm C_1)} + \text{const.} \tag{204}$$

Thus, we have found an additional firs integral for the third-order system (168)–(170) and we have the complete set of firs integrals that are transcendental functions of their phase variables.

Remark 4. Formally, in the expression of the found firs integral, we must substitute instead of C_1 the left-hand side of the firs integral (183).

Then the obtained additional firs integral has the following structure (similar to the transcendental firs integral from planar dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha, \frac{w_4}{\sin\alpha}, \frac{w_3}{\sin\alpha}\right) = C_2 = \text{const.} \tag{205}$$

Thus, to integrate the eighth-order system (168)–(173), we have already found two independent firs integrals. For the complete integration, as was mentioned above, it suffice to fin one firs integral for each (potentially separated) system (171), (172), and an additional firs integral that attaches Eq. (173).

To fin a firs integral of each (potentially separated) system (171), (172), we put in correspondence the following nonautonomous first-orde equation:

$$\frac{dw_s}{d\beta_s} = \frac{1+w_s^2}{w_s}\frac{\cos\beta_s}{\sin\beta_s}, s=1,2. \tag{206}$$

After integration we obtain the required invariant relation

$$\frac{\sqrt{1+w_s^2}}{\sin\beta_s} = C_{s+2} = \text{const}, s=1,2. \tag{207}$$

Further, to obtain an additional firs integral that attaches Eq. (173), to Eqs. (173) and (171) we put in correspondence the following nonautonomous equation:

$$\frac{dw_2}{d\beta_3} = -(1+w_2^2)\cos\beta_2. \tag{208}$$

Since

$$C_4\cos\beta_2 = \pm\sqrt{C_4^2-1-w_2^2}, \tag{209}$$

by (207), we have

$$\frac{dw_2}{d\beta_3} = \mp\frac{1}{C_4}(1+w_2^2)\sqrt{C_4^2-1-w_2^2}. \tag{210}$$

Integrating this relation, we arrive at the following quadrature:

$$\mp(\beta_3+C_5) = \int\frac{C_4dw_2}{(1+w_2^2)\sqrt{C_4^2-1-w_2^2}},$$
$$C_5 = \text{const.} \tag{211}$$

Integration leads to the relation

$$\mp\text{tg}(\beta_3+C_5) = \frac{C_4w_2}{\sqrt{C_4^2-1-w_2^2}},\ C_5 = \text{const.} \tag{212}$$

Finally, we have the following additional firs integral that attaches Eq. (173):

$$\text{arctg}\frac{C_4w_2}{\sqrt{C_4^2-1-w_2^2}} \pm \beta_3 = C_5,\ C_5 = \text{const.} \tag{213}$$

Thus, in the case considered, the system of dynamical equations (17)–(21), (24)–(33) under condition (148) has twelve invariant relations: the analytic nonintegrable constraint of the form (38), the cyclic firs integrals of the form (36) and (37), the firs integral of the form (184), the firs integral expressed by relations (198)–(205), which is a transcendental function of the phase variables (in the sense of complex analysis) expressed through a finit combination of functions, and the transcendental firs integrals of the form (207) and (213).

Theorem 3. *System (17)–(21), (24)–(33) under conditions (38), (148), (37) possesses twelve invariant relations (complete set); five of them are transcendental functions from the point of view of complex analysis. All relations are expressed through finite combinations of elementary functions.*

D. Topological analogies

Consider the following seventh-order system:

$$\begin{gathered}\ddot\xi + (b_*-H_{1*})\dot\xi\cos\xi + \sin\xi\cos\xi - \\ -[\dot\eta_1{}^2 + \dot\eta_2{}^2\sin^2\eta_1 + \dot\eta_3{}^2\sin^2\eta_1\sin^2\eta_2]\frac{\sin\xi}{\cos\xi} = 0, \\ \ddot\eta_1 + (b_*-H_{1*})\dot\eta_1\cos\xi + \dot\xi\dot\eta_1\frac{1+\cos^2\xi}{\cos\xi\sin\xi} - \\ -(\dot\eta_2{}^2 + \dot\eta_3{}^2\sin^2\eta_2)\sin\eta_1\cos\eta_1 = 0, \\ \ddot\eta_2 + (b_*-H_{1*})\dot\eta_2\cos\xi + \dot\xi\dot\eta_2\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\ + 2\dot\eta_1\dot\eta_2\frac{\cos\eta_1}{\sin\eta_1} - \dot\eta_3{}^2\sin\eta_2\cos\eta_2 = 0, \\ \ddot\eta_3 + (b_*-H_{1*})\dot\eta_3\cos\xi + \dot\xi\dot\eta_3\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + \\ + 2\dot\eta_1\dot\eta_3\frac{\cos\eta_1}{\sin\eta_1} + 2\dot\eta_2\dot\eta_3\frac{\cos\eta_2}{\sin\eta_2} = 0, \\ b_*>0,\ H_{1*}>0.\end{gathered} \tag{214}$$

This system describes a fi ed f ve-dimensional pendulum in a fl w of a running medium for which the moment of forces depends on the angular velocity, i.e., a mechanical system in a nonconservative field Generally speaking, the order of this system must be equal to 8, but the phase variable η_3 is a cyclic variable, which leads to the stratificatio of the phase space and reduced the order of the system.

The phase space of this system is the tangent bundle

$$TS^3\{\dot{\xi}, \dot{\eta_1}, \dot{\eta_2}, \dot{\eta_3}, \xi, \eta_1, \eta_2, \eta_3\} \tag{215}$$

of the four-dimensional sphere $\mathbf{S}^4\{\xi, \eta_1, \eta_2, \eta_3\}$. The equation that transforms system (140) into the system on the tangent bundle of the three-dimensional sphere

$$\dot{\eta_3} \equiv 0, \tag{216}$$

and the equations of great circles

$$\dot{\eta_1} \equiv 0, \ \dot{\eta_2} \equiv 0, \ \dot{\eta_3} \equiv 0 \tag{217}$$

defin families of integral manifolds.

It is easy to verify that system (214) is equivalent to the dynamical system with variable dissipation with zero mean on the tangent bundle (215) of the four-dimensional sphere. Moreover, the following theorem holds.

Theorem 4. *System (17)–(21), (24)–(33) under conditions (38), (148), (37) is equivalent to the dynamical system (214).*

Indeed, it suffice to set $\alpha = \xi, \ \beta_1 = \eta_1, \ \beta_2 = \eta_2, \ \beta_3 = \eta_3, \ b = -b_*, \ H_1 = -H_{1*}$.

VI. Conclusion

In the previous studies of the author, the problems on the motion of the lower-dimensional solid were already considered in a nonconservative force fiel in the presence of the following force. This study opens a new cycle of works on integration of a multidimensional solid in the nonconservative fiel because previously, as was already specified we considered only such motions of a solid when the fiel of external forces was the potential.

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