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CLASSIFICATION OF INTEGRABLE CASES IN THE DYNAMICS OF A FOUR-DIMENSIONAL RIGID BODY IN A NONCONSERVATIVE FIELD IN THE PRESENCE OF A TRACKING FORCE

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Abstract. This paper is a survey of integrable cases in the dynamics of a four-dimensional rigid body under the action of a nonconservative force field. We review both new results and results obtained earlier. The problems examined are described by dynamical systems with so-called variable dissipation with zero mean.

The problem of a search for complete sets of transcendental first integrals of systems with dissipation is quite current; a large number of works are devoted to it. We introduce a new class of dynamical systems that have a periodic coordinate. Due to the existence of a nontrivial symmetry group of such systems, we can prove that these systems possess variable dissipation with zero mean, which means that on the average for a period with respect to the periodic coordinate, the dissipation in the system is equal to zero, although in various domains of the phase space, either energy pumping or dissipation can occur. Based on the results obtained, we analyze dynamical systems that appear in the dynamics of a four-dimensional rigid body and obtain a series of new cases of complete integrability of the equations of motion in transcendental functions that can be expressed through a finite combination of elementary functions.

CONTENTS

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Introduction

This paper is a survey of integrable cases in the dynamics of a four-dimensional rigid body under the action of a nonconservative force field. We review both new results and results obtained earlier. The problems examined are described by dynamical systems with so-called variable dissipation with zero mean.

We study nonconservative systems for which the usual methods of studying Hamiltonian systems are not applicable. Thus, for such systems, we must "directly" integrate the main equation of dynamics (see [45, 47–52, 58, 60, 64, 65, 70, 79, 127].

We generalize previously known cases and obtain new cases of complete integrability in transcendental functions of the equation of dynamics of a four-dimensional rigid body in a nonconservative force field.

Of course, in the general case, the construction of a theory of integration of nonconservative systems (even of low dimension) is a quite difficult task. In a number of cases where the systems considered have additional symmetries, we succeed in finding first integrals expressed through finite combinations of elementary functions (see [5, 6, 12, 17, 18, 21, 28, 29, 32–35, 37, 42, 91, 92, 146, 148, 151].

We obtain a series of complete integrable nonconservative dynamical systems with nontrivial symmetries. Moreover, in almost all cases, all first integrals are expressed through finite combinations of elementary functions; these first integrals are transcendental functions of their variables. In this case, transcendence is understood in the sense of complex analysis, when the analytic continuation of a function into the complex plane has essentially singular points. This fact is caused by the existence of attracting and repelling limit sets in the system (for example, attracting and repelling focuses).

We discover new integrable cases of the motion of a rigid body, including the classical problem of the motion of a multi-dimensional spherical pendulum in a running flow of a medium.

Chapter 1 is devoted to general aspects of the integrability of dynamical systems with variable dissipation. First, we propose a descriptive characteristic of such systems. The term "variable dissipation" refers to the possibility of alternation of sign rather than to the value of the dissipation coefficient (therefore, it is more reasonable to use the term "sign-alternating").

Later, we define systems with variable dissipation with zero (nonzero) mean based on the divergence of the vector field of the system, which characterizes the change of the phase volume in the phase space of the system considered (see [91, 95, 103, 107, 109, 110, 112, 118, 120]).

We introduce a class of autonomous dynamical systems with one periodic phase coordinate possessing certain symmetries that are typical for pendulum-type systems. We show that this class of systems can be naturally embedded in the class of systems with variable dissipation with zero mean, i.e., on the average for the period with respect to the periodic coordinate, the dissipation in the system is equal to zero, although in various domains of the phase space, either energy pumping or dissipation can occur, but they balance each other in a certain sense. We present some examples of pendulum-type systems on lower-dimension manifolds from the dynamics of a rigid body in a nonconservative field.

For multi-parametric third-order systems, we present sufficient conditions of existence of first integrals that are expressed through finite combinations of elementary functions.

We deal with three properties that seem, at first glance, independent:

(1) a class of systems with symmetries specified above;
(2) the fact that this class consists of systems with variable dissipation with zero mean (with respect to the existing periodic variable), which allows us to consider them as "almost" conservative systems;
(3) in certain (although lower-dimensional) cases, these systems have a complete set of first integrals, which, in general, are transcendental (in the sense of complex analysis).

In Chap. 2, we recall general aspects of the dynamics of a free multi-dimensional rigid body: the notion of the tensor of angular velocity of the body, the joint dynamical equations of motion on the direct product $\mathbb{R}^n \times \mathrm{so}(n)$, and the Euler and Rivals formulas in the multi-dimensional case.

We also consider the tensor of inertia of a four-dimensional (4D) rigid body. In this work, we study two possible cases in which there exist *two* relations between the principal moments of inertia:

(i) there are *three* equal principal moments of inertia ($I_2 = I_3 = I_4$);
(ii) there are *two pairs* of equal principal moments of inertia ($I_1 = I_2$ and $I_3 = I_4$).

In Chaps. 2 and 3, we systematize results on the study of equations of motion of a four-dimensional (4D) rigid body in a nonconservative force field for the case (i). The form of these equations is taken from the dynamics of realistic rigid bodies of lesser dimension that interact with a resisting medium by laws of jet flow when the body is influenced by a nonconservative tracing force. Under the action of this force, the following two cases are possible. In the first case, the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo constraint (see Chap. 2). In the second case, the body is subjected to a nonconservative tracing force such that throughout the motion the center of mass of the body moves rectilinearly and uniformly; this means that in the system there exists a nonconservative couple of forces (see Chap. 3). See also [19–21, 27, 44, 54, 55, 57, 59, 61–63, 66–68, 72–78, 80, 82–87, 90–94, 96–107, 111–117, 119–125, 128–138, 140–144, 149].

Moreover, in Chap. 2, besides the four existing analytic invariant relations (a nonintegrable connection and three integrals that show that the components of the tensor of angular speed vanish), we obtain four additional transcendental first integrals expressed in terms of finite combinations of elementary functions. In Chap. 3, we find additional transcendental first integrals besides the four known analytic first integrals (the squared velocity of the center of mass and the three integrals that show that the components of the tensor of angular speed vanish).

The results relate to the case where all interaction of the medium with the body part is concentrated on a part of the surface of the body that has the form of a three-dimensional disk, while the action of the force is concentrated in the direction perpendicular to this disk. These results are systematized and are preserved in invariant form. Moreover, we introduce an extra dependence of the moment of the nonconservative force on the angular velocity. This dependence can be further extended to cases of motion in spaces of higher dimension.

Many results of this paper were regularly presented at scientific seminars, including the Trofimov seminar "Current problems of geometry and mechanics" (see [22]) under the supervision of D. V. Georgievskii and M. V. Shamolin [1, 2, 23–26].

Chapter 1

INTEGRABILITY IN ELEMENTARY FUNCTIONS OF SOME CLASSES OF NONCONSERVATIVE SYSTEMS

We study nonconservative systems for which the usual methods of studying Hamiltonian systems are not applicable. Thus, for such systems, we must "directly" integrate the main equation of dynamics. We recall known facts in a more universal form and also present some new cases of complete integrability in transcendental functions in the dynamics of a 4D-rigid body in a nonconservative field.

The results of the present paper develop previous studies, including an applied problem from the dynamics of a rigid body (see [42, 44, 46–48, 149]), for which complete lists of transcendental first integrals that can be expressed through finite combinations of elementary functions were obtained.

Later, this allows us to perform a complete analysis of all phase trajectories and to specify their rough properties that are preserved for systems of a more general form. The complete integrability of such systems is related to hidden symmetries.

As is well known, the notion of integrability is, generally speaking, quite vague.

We must always take into account in what sense this notion is understood (what criterion allows one to judge whether trajectories of a dynamical system are simple in one sense or another), in what functional class the first integrals are searched, etc. (see [6, 33, 34, 44, 54, 57]).

In this paper, we consider first integrals that belong to the functional class consisting of transcendental elementary functions. Here the term "transcendental" is meant in the sense of complex analysis, i.e., a transcendental function is a function that possesses essential singularities after an analytic continuation in the complex plane (see [12, 44]).

1. Preliminaries

The construction of a theory of integration of nonconservative systems (even lower-dimensional) is a difficult problem. However, in some cases where the systems being studied possess additional symmetries, one can find first integrals in the form of finite combinations of elementary functions (see [91]).

The present paper is a development of the planar problem on the motion of a rigid body in a resisting medium in which the domain of the contact between the body and the medium is a planar part of the exterior surface of the body. The force field in this problem is constructed by accounting for the action of the medium on the body in the quasi-stationary jet or separated flow. It turns out that the study of such motions can be reduced to systems with dissipation of energy ((purely) dissipative systems or systems in dissipative fields) or to systems with energy pumping (so-called systems with antidissipation or systems with accelerating forces). Note that similar problems have earlier appeared in applied aerodynamics (see [14, 15]).

The problems that were considered earlier stimulated the development of qualitative tools that substantially supplement the qualitative theory of nonconservative systems with dissipation of any sign (see [91]).

Nonlinear effects in problems of planar and spatial dynamics of a rigid body were examined by qualitative methods. We justify the need to introduce the notions of relative roughness and relative nonroughness of different orders (see [4, 28, 29, 39, 52, 60, 64, 88, 91, 149].

In the present work, the following results are obtained.

(1) We develop methods of qualitative analysis of dissipative and antidissipative systems, which allows us to obtain bifurcation conditions for the appearance of stable and unstable self-oscillations and conditions of the absence of singular trajectories. We succeed in extending the study of planar topographical Poincaré systems and comparison systems to higher dimensions. We obtain sufficient Poisson-stability conditions (everywhere density near itself) of some classes of nonclosed trajectories of dynamical systems (see [91]);

(2) in 2D- and 3D-dynamics of a rigid body, we obtain complete lists of first integrals of dissipative and antidissipative systems that are transcendental (in the sense of the classification of their singularities) functions, which, in some cases, can be expressed through elementary functions. We introduce the notions of relative roughness and relative nonroughness of different orders for integrated systems (see [4, 28, 29, 39, 52, 60, 64, 88, 91, 149];

(3) we obtain multi-parameter families of topologically nonequivalent phase portraits that appear in purely dissipative systems (i.e., systems with variable dissipation with nonzero (positive) mean). Almost all portraits of such families are rough (see [91]);

(4) we detect new qualitative analogies between the motion of a free body in a resisting medium and the motion of a fixed body in a flow of a running medium.

2. Dynamical Systems with Variable Dissipation

2.1. Descriptive characteristics of dynamical systems with variable dissipation. As the initial modeling of the action of a medium on a rigid body, we used experimental information on the properties of jet flow. Naturally it became necessary to study the class of dynamical systems that possess the property of (relative) roughness (relative structural stability). Therefore, it is natural to introduce these notions for such systems. Many of the systems considered are rough in the sense of Andronov and Pontryagin (see [4, 28, 29, 39, 52, 60, 64, 88, 91, 149].

After some transformations (for example, in 2D-dynamics), the dynamical part of the general system of the equations of plane-parallel motion can be reduced to a pendulum system of second order containing a linear nonconservative (sign-alternating dissipative) force with a coefficient, which can change sign for different values of the periodic phase coordinate of the system.

Thus, in this case, we speak of systems with so-called variable dissipation, where the term "variable" refers not only to the value of the dissipation coefficient but to its sign (and so the term "sign-alternating" is more adequate).

On the average by a period (with respect to the periodic coordinate), dissipation can be positive ("purely" dissipative systems), negative (systems with accelerating forces), or zero (but does not vanish identically). In the last case, we speak of systems with variable dissipation with zero mean (these systems can be associated with "almost" conservative systems).

As was noted above, we obtain important mechanical analogies in comparing the qualitative properties of a free body and the equilibrium of a pendulum in a flow of a medium. Such analogies have a deep sense since they allow one to transfer properties of a nonlinear dynamical system for a pendulum to dynamical systems for a free body. Both systems belong to the class of so-called pendulum dynamical systems with variable dissipation with zero mean.

Under additional conditions, the equivalence described above can be extended to the case of spatial motion, which allows one to speak of a general character of symmetries of systems with variable dissipation with zero mean in plane-parallel and spatial motions (for planar and spatial versions of a pendulum in a flow of a medium, see also [91]).

Subsequently, we present some classes of nonlinear systems of the second, third, and higher orders that are integrable in the class of transcendental (in the sense of the theory of functions of complex variables) elementary functions, for example, five-parameter dynamical systems including the majority of systems examined earlier in the dynamics of a low-dimensional (2D and 3D) rigid body interacting with a medium:

$$\begin{aligned}
\dot\alpha &= a\sin\alpha + b\omega + \gamma_1\sin^5\alpha + \gamma_2\omega\sin^4\alpha + \gamma_3\omega^2\sin^3\alpha + \gamma_4\omega^3\sin^2\alpha + \gamma_5\omega^4\sin\alpha,\\
\dot\omega &= c\sin\alpha\cos\alpha + d\omega\cos\alpha + \gamma_1\omega\sin^4\alpha\cos\alpha + \gamma_2\omega^2\sin^3\alpha\cos\alpha + \gamma_3\omega^3\sin^2\alpha\cos\alpha +\\
&\quad + \gamma_4\omega^4\sin\alpha\cos\alpha + \gamma_5\omega^5\cos\alpha.
\end{aligned}$$

Purely dissipative dynamical systems (and also (purely) antidissipative systems), which, in our case, can belong to the class of systems with variable dissipation with nonzero mean, are, as a rule, structurally stable ((absolutely) rough), whereas systems with variable dissipation with zero mean (which usually possess additional symmetries) are either structurally unstable (nonrough) or only relatively structurally stable (relatively rough). However, the proof of the last assertion in the general case is a difficult problem.

For example, the dynamical system of the form

$$\begin{aligned} \dot{\alpha} &= \Omega + \beta \sin \alpha, \\ \dot{\Omega} &= -\beta \sin \alpha \cos \alpha \end{aligned} \tag{2.1}$$

is relatively structurally stable (relatively rough) and is topologically equivalent to the system describing a fixed pendulum in a running flow of a medium (see [91]).

One can obtain its first integral, which is a transcendental (in the sense of the theory of functions of a complex variable, as a function whose analytical continuation in the complex plane has essential singularities) function of phase variables that can be expressed through a finite combination of elementary functions (see [91]). The phase cylinder $\mathbb{R}^2\{\alpha, \Omega\}$ of quasi-velocictes of the system considered has an interesting topological structure of a splitting into trajectories.

Although the dynamical system considered is not conservative, in the rotational domain (and only in this domain) of its phase plane $\mathbb{R}^2\{\alpha, \Omega\}$, it admits the preservation of invariant measure with variable density. This property characterizes this system as a system with variable dissipation with zero mean (see [91]).

2.2. A definition of a system with variable dissipation with zero mean. We study systems of ordinary differential equations that have a periodic phase coordinate. Such systems possess symmetries under which their average phase volume with respect to the periodic coordinate is preserved. For example, the following pendulum system with smooth and periodic (of period T) with respect to α right-hand side $\mathbf{V}(\alpha, \omega)$ of the form

$$\begin{aligned} \dot{\alpha} &= -\omega + f(\alpha), \ \ f(\alpha + T) = f(\alpha), \\ \dot{\omega} &= g(\alpha), \ \ g(\alpha + T) = g(\alpha), \end{aligned} \tag{2.2}$$

preserves its phase area on the phase cylinder within the period T:

$$\int_0^T \operatorname{div} \mathbf{V}(\alpha, \omega) d\alpha = \int_0^T \left(\frac{\partial}{\partial \alpha}(-\omega + f(\alpha)) + \frac{\partial}{\partial \omega} g(\alpha) \right) d\alpha = \int_0^T f'(\alpha) d\alpha = 0. \tag{2.3}$$

This system is equivalent to the equation of a pendulum

$$\ddot{\alpha} - f'(\alpha)\dot{\alpha} + g(\alpha) = 0, \tag{2.4}$$

in which the integral of the coefficient $f'(\alpha)$ of the dissipative term $\dot{\alpha}$ over the period is equal to zero.

We see that this system has symmetries under which it becomes a system with *variable dissipation with zero mean* in the sense of the following definition (see [91]).

Definition 2.1. Consider a smooth autonomous system of order $(n+1)$ in the normal form defined on the cylinder $\mathbb{R}^n\{x\} \times \mathbf{S}^1\{\alpha \mod 2\pi\}$, where α is a periodic coordinate of period $T > 0$. The divergence of the right-hand side $\mathbf{V}(x, \alpha)$ (which, in general, is a function of all phase variables and does not vanish identically) of this system is denoted by $\operatorname{div} \mathbf{V}(x, \alpha)$. This system is called a system with variable dissipation with zero (respectively, nonzero) mean if the function

$$\int_0^T \operatorname{div} \mathbf{V}(x, \alpha) d\alpha \tag{2.5}$$

vanishes (respectively, does not vanish) identically. In some cases (for example, when at some points of the circle $\mathbf{S}^1\{\alpha \mod 2\pi\}$ singularities appear), this integral is understood in the sense of principal value.

We note that it is quite difficult to give a general definition of a system with variable dissipation with zero (nonzero) mean. The definition presented above is based on the notion of divergence (as is well known, the divergence of the right-hand side of a system in the normal form characterizes the change of the phase volume in the phase space of the given system).

3. Systems with Symmetries and Variable Dissipation with Zero Mean

Consider a system of the following form (the dot denotes the derivative with respect to time):

$$\begin{aligned} \dot{\alpha} &= f_\alpha(\omega, \sin\alpha, \cos\alpha), \\ \dot{\omega_k} &= f_k(\omega, \sin\alpha, \cos\alpha), \quad k = 1, \ldots, n, \end{aligned} \tag{3.1}$$

defined on the set

$$\mathbf{S}^1\{\alpha \mod 2\pi\} \setminus K \times \mathbb{R}^n\{\omega\}, \quad \omega = (\omega_1, \ldots, \omega_n), \tag{3.2}$$

where sufficiently smooth functions $f_\lambda(u_1, u_2, u_3)$, $\lambda = \alpha, 1, \ldots, n$, of three variables u_1, u_2, u_3 are such that

$$\begin{aligned} f_\lambda(-u_1, -u_2, u_3) &= -f_\lambda(u_1, u_2, u_3), \\ f_\alpha(u_1, u_2, -u_3) &= f_\alpha(u_1, u_2, u_3), \\ f_k(u_1, u_2, -u_3) &= -f_k(u_1, u_2, u_3); \end{aligned} \tag{3.3}$$

moreover, the functions $f_k(u_1, u_2, u_3)$ are defined for $u_3 = 0$ for any $k = 1, \ldots, n$.

The set K is either empty or consists of a finite number of points of the circle $\mathbf{S}^1\{\alpha \mod 2\pi\}$.

The last two variables u_2, u_3 in the functions $f_\lambda(u_1, u_2, u_3)$ depend on the same parameter α, but we assume that these variables belong to different groups for the following reason. First, they cannot be uniquely expressed through one another on their entire domain and, second, u_2 is an odd function of α whereas u_3 is an even function, which affects the symmetries of system (3.1).

We establish a correspondence between system (3.1) and the following nonautonomous system:

$$\frac{d\omega_k}{d\alpha} = \frac{f_k(\omega, \sin\alpha, \cos\alpha)}{f_\alpha(\omega, \sin\alpha, \cos\alpha)}, \quad k = 1, \ldots, n. \tag{3.4}$$

By the substitution $\tau = \sin\alpha$, it can be reduced to the form

$$\begin{aligned} \frac{d\omega_k}{d\tau} &= \frac{f_k(\omega, \tau, \varphi_k(\tau))}{f_\alpha(\omega, \tau, \varphi_\alpha(\tau))}, \quad k = 1, \ldots, n, \\ \varphi_\lambda(-\tau) &= \varphi_\lambda(\tau), \quad \lambda = \alpha, 1, \ldots, n. \end{aligned} \tag{3.5}$$

The last system, in particular, can have an algebraic right-hand side (i.e., it can be the ratio of two polynomials), which simplifies the search for its first integrals in explicit form.

The following theorem states that the class of systems (3.1) is a subclass of the class of dynamical systems with variable dissipation with zero mean. Note that, in general, the converse is invalid.

Theorem 3.1. *Systems of the form* (3.1) *are dynamical systems with variable dissipation with zero mean.*

Proof. The proof of this theorem is based on certain symmetries (3.3) of system (3.1) listed above and the periodicity of the right-hand side of the system with respect to α.

Indeed, the divergence of the vector field of system (3.1) equals

$$\frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial u_2} \cos\alpha - \frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial u_3} \sin\alpha + \sum_{k=1}^{n} \frac{\partial f_k(\omega, \sin\alpha, \cos\alpha)}{\partial u_1}. \tag{3.6}$$

The following integral of the first two terms in (3.6) vanishes:

$$\int_0^{2\pi} \left\{ \frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial u_2} d\sin\alpha + \frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial u_3} d\cos\alpha \right\}$$

$$= \int_0^{2\pi} \frac{\partial f_\alpha(\omega, \sin\alpha, \cos\alpha)}{\partial \alpha} d\alpha = h_\alpha(\omega) \equiv 0, \quad (3.7)$$

since the function $f_\alpha(\omega, \sin\alpha, \cos\alpha)$ is periodic with respect to α.

Further, by the third equation in (3.3), for any $k = 1, \ldots, n$ we have

$$\frac{\partial f_k(\omega, \sin\alpha, \cos\alpha)}{\partial u_1} = \cos\alpha \cdot \frac{\partial g_k(\omega, \sin\alpha)}{\partial u_1}, \quad (3.8)$$

where the function $g_k(u_1, u_2)$ is sufficiently smooth for any $k = 1, \ldots, n$.

Then the integral over the period 2π of the right-hand side of Eq. (3.8) equals

$$\int_0^{2\pi} \frac{\partial g_k(\omega, \sin\alpha)}{\partial u_1} d\sin\alpha = h_k(\omega) \equiv 0 \quad (3.9)$$

for any $k = 1, \ldots, n$. From Eqs. (3.7) and (3.9) we obtain Theorem 3.1. □

The converse assertion is invalid: there exist dynamical systems on the two-dimensional cylinder that are systems with variable dissipation with zero mean, but do not possess the symmetries listed above.

In this paper, we basically consider the case where the functions $f_\lambda(\omega, \tau, \varphi_k(\tau))$ $(\lambda = \alpha, 1, \ldots, n)$ are polynomials of ω and τ.

Example 3.1. We consider pendulum systems on the two-dimensional cylinder $\mathbf{S}^1\{\alpha \mod 2\pi\} \times \mathbb{R}^1\{\omega\}$ with parameter $b > 0$, which appear in the dynamics of a rigid body (see [91]):

$$\begin{aligned} \dot\alpha &= -\omega + b\sin\alpha, \\ \dot\omega &= \sin\alpha\cos\alpha, \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} \dot\alpha &= -\omega + b\sin\alpha\cos^2\alpha + b\omega^2\sin\alpha, \\ \dot\omega &= \sin\alpha\cos\alpha - b\omega\sin^2\alpha\cos\alpha + b\omega^3\cos\alpha. \end{aligned} \quad (3.11)$$

We establish a correspondence between these systems in the variables (ω, τ) and the equations with algebraic right-hand sides

$$\frac{d\omega}{d\tau} = \frac{\tau}{-\omega + b\tau}, \quad (3.12)$$

and

$$\frac{d\omega}{d\tau} = \frac{\tau + b\omega\left[\omega^2 - \tau^2\right]}{-\omega + b\tau + b\tau\left[\omega^2 - \tau^2\right]} \quad (3.13)$$

of the form (3.5), respectively. These systems are dynamical systems with variable dissipation with zero mean, which can be easily verified.

Indeed, the divergences of their right-hand sides are equal to $b\cos\alpha$ and

$$b\cos\alpha\left[4\omega^2 + \cos^2\alpha - 3\sin^2\alpha\right],$$

respectively; they belong to the class of systems (3.1).

Moreover, each of them possesses a first integral, which is a transcendental (in the sense of the theory of functions of complex variables) function that can be expressed through a finite combination of elementary functions.

We present another important example of a higher-order system that possesses the properties listed above.

Example 3.2. Consider the following system with a parameter b, which is defined in the three-dimensional domain

$$\mathbf{S}^1\{\alpha \mod 2\pi\} \setminus \{\alpha = 0, \alpha = \pi\} \times \mathbb{R}^2\{z_1, z_2\} \tag{3.14}$$

(this system is separated from a system on the tangent bundle $T_*\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$):

$$\begin{aligned} \dot{\alpha} &= -z_2 + b\sin\alpha, \\ \dot{z_2} &= \sin\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha}, \\ \dot{z_1} &= z_1 z_2 \frac{\cos\alpha}{\sin\alpha}. \end{aligned} \tag{3.15}$$

This system describes the motion of a rigid body in a resistive medium. We establish a correspondence between this system and the following nonautonomous system with algebraic right-hand side ($\tau = \sin\alpha$):

$$\begin{aligned} \frac{dz_2}{d\tau} &= \frac{\tau - z_1^2/\tau}{-z_2 + b\tau}, \\ \frac{dz_1}{d\tau} &= \frac{z_1 z_2/\tau}{-z_2 + b\tau}. \end{aligned} \tag{3.16}$$

We see that system (3.15) is a system with variable dissipation with zero mean. To obtain the full correspondence with the definition, we introduce the new phase variable

$$z_1^* = \ln|z_1|. \tag{3.17}$$

The divergence of the right-hand side of system (3.15) in the Cartesian coordinates α, z_1^*, z_2 is equal to $b\cos\alpha$. Taking into account (3.14), we have (in the sense of principal value)

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi-\varepsilon} b\cos\alpha + \lim_{\varepsilon\to 0}\int_{\pi+\varepsilon}^{2\pi-\varepsilon} b\cos\alpha = 0. \tag{3.18}$$

Moreover, this system possesses two first integrals (i.e., a complete set) that are transcendental functions, which can be expressed through a finite combination of elementary functions. This becomes possible after establishing a correspondence between it and the system (nonautonomous, generally speaking) of equations with an algebraic (polynomial) right-hand side (3.16).

Systems (3.10), (3.11), and (3.15) belong to the class of systems (3.1), possess variable dissipation with zero mean, and have a complete set of transcendental first integrals that can be expressed through a finite combination of elementary functions.

So, to find the first integrals of the systems considered, it is convenient to reduce systems of the form (3.1) to systems with polynomial right-hand sides (3.5), which allow one to perform integration in terms of elementary functions of the initial system. Thus, we find sufficient conditions for the integrability in elementary functions of systems with polynomial right-hand sides and examine systems of the most general form.

4. Systems in the Plane and on a Two-Dimensional Cylinder

Earlier, the author proved a series of assertions regarding many-parameter systems of ordinary differential equations with algebraic right-hand side (see, e.g., [91]). We recall some of these assertions.

Proposition 4.1. *A seven-parameter family of systems of equations in the plane* $\mathbb{R}^2\{x, y\}$

$$\begin{aligned} \dot{x} &= a_1 x + b_1 y + \beta_1 x^3 + \beta_2 x^2 y + \beta_3 x y^2, \\ \dot{y} &= c_1 x + d_1 y + \beta_1 x^2 y + \beta_2 x y^2 + \beta_3 y^3, \end{aligned} \tag{4.1}$$

possesses a first integral (*in general, transcendental*) *that can be expressed through elementary functions.*

Corollary 4.1. *For any parameters* a_1, b_1, c_1, d_1, β_1, β_2, *and* β_3, *the system*

$$\begin{aligned} \dot{\alpha} &= a_1 \sin\alpha + b_1\omega + \beta_1 \sin^3\alpha + \beta_2\omega\sin^2\alpha + \beta_3\omega^2\sin\alpha, \\ \dot{\omega} &= c_1 \sin\alpha\cos\alpha + d_1\omega\cos\alpha + \beta_1\omega\sin^2\alpha\cos\alpha + \beta_2\omega^2\sin\alpha\cos\alpha + \beta_3\omega^3\cos\alpha \end{aligned} \tag{4.2}$$

on the two-dimensional cylinder $\{(\alpha, \omega) \in \mathbb{R}^2 : \alpha \mod 2\pi\}$ *possesses a first integral* (*in general, transcendental*) *that can be expressed through elementary functions.*

In particular, systems (3.10) and (3.11) can be obtained from this system if

$$a_1 = b, \quad b_1 = -1, \quad c_1 = 1, \quad d_1 = \beta_1 = \beta_2 = \beta_3 = 0$$

and

$$a_1 = b, \quad b_1 = -1, \quad c_1 = 1, \quad d_1 = -b, \quad \beta_1 = -b, \quad \beta_2 = 0, \quad \beta_3 = b,$$

respectively.

The above arguments can be easily generalized. We consider the possibility of complete integration (in elementary functions) of systems of a more general form: the nonlinearity is characterized by an arbitrary homogeneous form of odd degree $2n - 1$.

In this case, we have the following assertion, which is more general than Proposition 4.1.

Proposition 4.2. *The* $(2n + 3)$*-parameter family of systems of equations*

$$\begin{aligned} \dot{x} &= a_1 x + b_1 y + \delta_1 x^{2n-1} + \delta_2 x^{2n-2} y + \cdots + \delta_{2n-2} x^2 y^{2n-3} + \delta_{2n-1} x y^{2n-2}, \\ \dot{y} &= c_1 x + d_1 y + \delta_1 x^{2n-2} y + \delta_2 x^{2n-3} y^2 + \cdots + \delta_{2n-2} x y^{2n-2} + \delta_{2n-1} y^{2n-1} \end{aligned} \tag{4.3}$$

in the plane $\mathbb{R}^2\{x, y\}$ *possesses a first integral* (*in general, transcendental*), *which can be expressed through elementary functions.*

Indeed, the family of Eqs. (4.3) depends on $2n - 1 + 4$ independent parameters since the total nonlinearity of an odd degree is characterized by $4n$ parameters subject to $2n + 1$ conditions (the other four parameters are contained in the linear part).

Corollary 4.2. *For any parameters* a_1, b_1, c_1, d_1, *and* $\delta_1, \ldots, \delta_{2n-1}$, *the systems*

$$\begin{aligned} \dot{\alpha} &= a\sin\alpha + b\omega + \delta_1 \sin^{2n-1}\alpha + \delta_2\omega\sin^{2n-2}\alpha + \cdots + \delta_{2n-1}\omega^{2n-2}\sin\alpha, \\ \dot{\omega} &= c\sin\alpha\cos\alpha + d\omega\cos\alpha + \delta_1\omega\sin^{2n-2}\alpha\cos\alpha + \delta_2\omega^2\sin^{2n-3}\alpha\cos\alpha + \cdots + \delta_{2n-1}\omega^{2n-1}\cos\alpha \end{aligned} \tag{4.4}$$

on the two-dimensional cylinder $\{(\alpha, \omega) \in \mathbb{R}^2 : \alpha \mod 2\pi\}$ *possesses a transcendental first integral, which can be expressed through elementary functions.*

Systems (3.10), (3.11), and (3.15) are relatively rough (see [91]), but if we violate the symmetries (3.3) introduced for systems of general form (3.1) (for example, by introducing additional terms in their right-hand sides), then the number of topologically distinct phase portraits can substantially change.

In [91], we obtained a multi-parametric family of phase portraits of a system with variable dissipation with nonzero mean (whose typical portraits are (absolutely) rough), which is a perturbation of a dynamical system with variable dissipation with zero mean of the form (3.11). This family (as families obtained earlier, see [91]) contains an infinite number of topologically nonequivalent phase portraits on a two-dimensional phase cylinder.

5. Systems of the Tangent Bundle of the Two-Dimensional Sphere

On the tangent bundle $T_*\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2\{\theta, \psi\}$, we consider the following dynamical system:

$$\begin{gathered}\ddot{\theta} + b\dot{\theta}\cos\theta + \sin\theta\cos\theta - \dot{\psi}^2\frac{\sin\theta}{\cos\theta} = 0,\\ \ddot{\psi} + b\dot{\psi}\cos\theta + \dot{\theta}\dot{\psi}\left[\frac{1+\cos^2\theta}{\sin\theta\cos\theta}\right] = 0.\end{gathered} \tag{5.1}$$

This system describes a spherical pendulum in a flow of a running medium (see [91]). Moreover, the system possesses the conservative moment

$$\sin\theta\cos\theta \tag{5.2}$$

and the force moment, which linearly depends of the velocity with a variable coefficient:

$$b\begin{pmatrix}\dot{\theta}\\ \dot{\psi}\end{pmatrix}\cos\theta. \tag{5.3}$$

Other coefficients in the equations are the connection coefficients, namely,

$$\Gamma^{\theta}_{\psi\psi} = -\frac{\sin\theta}{\cos\theta}, \quad \Gamma^{\psi}_{\theta\psi} = \frac{1+\cos^2\theta}{\sin\theta\cos\theta}. \tag{5.4}$$

In fact, system (5.1) has order 3 since the variable ψ is cyclic and the system contains only the variable $\dot{\psi}$.

Proposition 5.1. *The equation*

$$\dot{\psi} = 0 \tag{5.5}$$

defines a family of integral planes for system (5.1)*.*

Moreover, Eq. (5.5) *reduces system* (5.1) *to an equation that describes a cylindrical pendulum in a flow of a running medium* (*see* [91])*.*

Proposition 5.2. *System* (5.1) *is equivalent to the following system:*

$$\begin{aligned}\dot{\theta} &= -z_2 + b\sin\theta,\\ \dot{z}_2 &= \sin\theta\cos\theta - z_1^2\frac{\cos\theta}{\sin\theta},\\ \dot{z}_1 &= z_1 z_2\frac{\cos\theta}{\sin\theta},\\ \dot{\psi} &= z_1\frac{\cos\theta}{\sin\theta}\end{aligned} \tag{5.6}$$

on the tangent bundle $T_*\mathbf{S}^2\{z_1, z_2, \theta, \psi\}$ of the two-dimensional sphere $\mathbf{S}^2\{\theta, \psi\}$. Moreover, the first three equations of system (5.6) *form a closed system of third order and coincide with system* (3.15) (*if we set $\alpha = \theta$*)*. The fourth equation of system* (5.6) *has been separated due to the cyclicity of the variable ψ.*

Example 5.1. We examine a system of the form (3.15), which can be reduced to (3.16), and the following system, which appears in the spatial (3D) dynamics of a rigid body interacting with a medium:

$$\begin{aligned}
\dot{\alpha} &= -z_2 + b\left(z_1^2 + z_2^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha,\\
\dot{z}_2 &= \sin\alpha\cos\alpha + bz_2\left(z_1^2 + z_2^2\right)\cos\alpha - bz_2\sin^2\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha},\\
\dot{z}_1 &= bz_1\left(z_1^2 + z_2^2\right)\cos\alpha - bz_1\sin^2\alpha\cos\alpha + z_1z_2\frac{\cos\alpha}{\sin\alpha},
\end{aligned} \tag{5.7}$$

which corresponds to the following system with algebraic right-hand side:

$$\begin{aligned}
\frac{dz_2}{d\tau} &= \frac{\tau + bz_2\left(z_1^2 + z_2^2\right) - bz_2\tau^2 - z_1^2/\tau}{-z_2 + b\tau\left(z_1^2 + z_2^2\right) + b\tau\left(1 - \tau^2\right)},\\
\frac{dz_1}{d\tau} &= \frac{bz_1\left(z_1^2 + z_2^2\right) - bz_1\tau^2 + z_1z_2/\tau}{-z_2 + b\tau\left(z_1^2 + z_2^2\right) + b\tau(1 - \tau^2)}.
\end{aligned} \tag{5.8}$$

Thus, we consider two systems: the initial system (5.7) and the corresponding algebraic system (5.8).

Similarly, we can pass to homogeneous coordinates u_k, $k = 1, 2$, by the formulas

$$z_k = u_k\tau. \tag{5.9}$$

By this change of variables, system (3.16) (see above) can be transformed to the form

$$\begin{aligned}
\tau\frac{du_2}{d\tau} + u_2 &= \frac{\tau - u_1^2\tau}{-u_2\tau + b\tau},\\
\tau\frac{du_1}{d\tau} + u_1 &= \frac{u_1u_2\tau}{-u_2\tau + b\tau},
\end{aligned} \tag{5.10}$$

which, in turn, corresponds to the equation

$$\frac{du_2}{du_1} = \frac{1 - bu_2 + u_2^2 - u_1^2}{2u_1u_2 - bu_1}. \tag{5.11}$$

Since the identity

$$d\left(\frac{1 - \beta u_2 + u_2^2}{u_1}\right) + du_1 = 0 \tag{5.12}$$

is integrable, this equation can be integrated in elementary functions and in the coordinates (τ, z_1, z_2) has a first integral of the form

$$\frac{z_1^2 + z_2^2 - \beta z_2\tau + \tau^2}{z_1\tau} = \text{const}\,.$$

System (5.7) after reduction corresponds to the system

$$\begin{aligned}
\tau\frac{du_2}{d\tau} + u_2 &= \frac{\tau + bu_2\tau^3\left(u_1^2 + u_2^2\right) - bu_2\tau^3 - u_1^2\tau}{-u_2\tau + b\tau^3\left(u_1^2 + u_2^2\right) + b\tau\left(1 - \tau^2\right)},\\
\tau\frac{du_1}{d\tau} + u_1 &= \frac{bu_1\tau^3\left(u_1^2 + u_2^2\right) - bu_1\tau^3 + u_1u_2\tau}{-u_2\tau + b\tau^3\left(u_1^2 + u_2^2\right) + b\tau\left(1 - \tau^2\right)},
\end{aligned} \tag{5.13}$$

which can also be reduced to (5.11).

6. Some Generalizations

The following question arises: Can the system

$$\begin{aligned}\frac{dz}{dx} &= \frac{ax+by+cz+c_1z^2/x+c_2zy/x+c_3y^2/x}{d_1x+ey+fz},\\ \frac{dy}{dx} &= \frac{gx+hy+iz+i_1z^2/x+i_2zy/x+i_3y^2/x}{d_1x+ey+fz},\end{aligned} \tag{6.1}$$

possessing a singularity of type $1/x$, be integrated in elementary functions? This system is a generalization of systems (3.16) and (5.8) in three-dimensional phase domains.

A series of results concerning this question has already been obtained (see [91]); here we present a brief review of these results.

As above, we introduce the substitutions

$$y = ux, \quad z = vx \tag{6.2}$$

and reduce system (6.1) to the following form:

$$x\frac{dv}{dx} + v = \frac{ax+bux+cvx+c_1v^2x+c_2vux+c_3u^2x}{d_1x+eux+fvx}, \tag{6.3}$$

$$x\frac{du}{dx} + u = \frac{gx+hux+ivx+i_1v^2x+i_2vux+i_3u^2x}{d_1x+eux+fvx}, \tag{6.4}$$

which is equivalent to

$$x\frac{dv}{dx} = \frac{ax+bux+(c-d_1)vx+(c_1-f)v^2x+(c_2-e)vux+c_3u^2x}{d_1x+eux+fvx}, \tag{6.5}$$

$$x\frac{du}{dx} = \frac{gx+(h-d_1)ux+ivx+i_1v^2x+(i_2-f)vux+(i_3-e)u^2x}{d_1x+eux+fvx}. \tag{6.6}$$

We establish a correspondence between this system and the following nonautonomous equation with algebraic right-hand side:

$$\frac{dv}{du} = \frac{a+bu+cv+c_1v^2+c_2vu+c_3u^2-v[d_1+eu+fv]}{g+hu+iv+i_1v^2+i_2vu+i_3u^2-u[d_1+eu+fv]}. \tag{6.7}$$

Integration of this equation reduces to integration of the equation in complete differentials

$$\begin{aligned}&\left[g+hu+iv+i_1v^2+i_2vu+i_3u^2-d_1u-eu^2-fuv\right]dv\\ &\qquad = \left[a+bu+cv+c_1v^2+c_2vu+c_3u^2-d_1v-euv-fv^2\right]du.\end{aligned} \tag{6.8}$$

Generally speaking, we have a 15-parameter family of equations of the form (6.8). To integrate the last identity in elementary functions as a homogeneous equation, it suffices to impose the following six restrictions:

$$g = 0, \quad i = 0, \quad i_1 = 0, \quad e = c_2, \quad h = c, \quad i_2 = 2c_1 - f. \tag{6.9}$$

We introduce nine parameters $\beta_1, \dots, \beta_9$ and consider them as independent:

$$\beta_1 = a, \quad \beta_2 = b, \quad \beta_3 = c, \quad \beta_4 = c_1, \quad \beta_5 = c_2, \quad \beta_6 = c_3, \quad \beta_7 = d_1, \quad \beta_8 = f, \quad \beta_9 = i_3. \tag{6.10}$$

Thus, Eq. (6.8) under conditions (6.9) and (6.10) is reduced to the form

$$\frac{dv}{du} = \frac{\beta_1+\beta_2u+(\beta_3-\beta_7)v+(\beta_4-\beta_8)v^2+\beta_6u^2}{(\beta_3-\beta_7)u+2(\beta_4-\beta_8)vu+(\beta_9-\beta_5)u^2}, \tag{6.11}$$

whereas system (6.5) (6.6) is reduced to the form

$$x\frac{dv}{dx} = \frac{\beta_1 + \beta_2 u + (\beta_3 - \beta_7)v + (\beta_4 - \beta_8)v^2 + \beta_6 u^2}{\beta_7 + \beta_5 u + \beta_8 v}, \tag{6.12}$$

$$x\frac{du}{dx} = \frac{(\beta_3 - \beta_7)u + 2(\beta_4 - \beta_8)vu + (\beta_9 - \beta_5)u^2}{\beta_7 + \beta_5 u + \beta_8 v}, \tag{6.13}$$

after which Eq. (6.11) can be integrated by a finite combination of elementary functions.

Indeed, integrating identity (6.8), we obtain

$$d\left[\frac{(\beta_3 - \beta_7)v}{u}\right] + d\left[\frac{(\beta_4 - \beta_8)v^2}{u}\right] + d[(\beta_9 - \beta_5)v] + d\left[\frac{\beta_1}{u}\right] - d[\beta_2 \ln|u|] - d[\beta_6 u] = 0, \tag{6.14}$$

which implies the following invariant relation:

$$\frac{(\beta_3 - \beta_7)v}{u} + \frac{(\beta_4 - \beta_8)v^2}{u} + (\beta_9 - \beta_5)v + \frac{\beta_1}{u} - \beta_2 \ln|u| - \beta_6 u = C_1 = \text{const}, \tag{6.15}$$

and then in the coordinates (x, y, z), the first integral in the form

$$\frac{(\beta_4 - \beta_8)z^2 - \beta_6 y^2 + (\beta_3 - \beta_7)zx + (\beta_9 - \beta_5)zy + \beta_1 x^2}{yx} - \beta_2 \ln\left|\frac{y}{x}\right| = \text{const}\,. \tag{6.16}$$

Therefore, we can conclude that the following, generally speaking nonconservative, system of third order depending on nine parameters is integrable in elementary functions:

$$\begin{aligned} \frac{dz}{dx} &= \frac{\beta_1 x + \beta_2 y + \beta_3 z + \beta_4 z^2/x + \beta_5 zy/x + \beta_6 y^2/x}{\beta_7 x + \beta_5 y + \beta_8 z}, \\ \frac{dy}{dx} &= \frac{\beta_3 y + (2\beta_4 - \beta_8)zy/x + \beta_9 y^2/x}{\beta_7 x + \beta_5 y + \beta_8 z}. \end{aligned} \tag{6.17}$$

Corollary 6.1. *On the set*

$$\mathbf{S}^1\{\alpha \mod 2\pi\} \setminus \{\alpha = 0,\ \alpha = \pi\} \times \mathbb{R}^2\{z_1, z_2\}, \tag{6.18}$$

the third-order system

$$\begin{aligned} \dot\alpha &= \beta_7 \sin\alpha + \beta_5 z_1 + \beta_8 z_2, \\ \dot z_2 &= \beta_1 \sin\alpha\cos\alpha + \beta_2 z_1 \cos\alpha + \beta_3 z_2 \cos\alpha + \beta_4 z_2^2 \frac{\cos\alpha}{\sin\alpha} + \beta_5 z_1 z_2 \frac{\cos\alpha}{\sin\alpha} + \beta_6 z_1^2 \frac{\cos\alpha}{\sin\alpha}, \\ \dot z_1 &= \beta_3 z_1 \cos\alpha + (2\beta_4 - \beta_8) z_1 z_2 \frac{\cos\alpha}{\sin\alpha} + \beta_9 z_1^2 \frac{\cos\alpha}{\sin\alpha}, \end{aligned} \tag{6.19}$$

which depend on nine parameters and possesses, generally speaking, a transcendental first integral, which can be expressed through elementary functions:

$$\frac{(\beta_4 - \beta_8)z_2^2 - \beta_6 z_1^2 + (\beta_3 - \beta_7)z_2 \sin\alpha + (\beta_9 - \beta_5)z_2 z_1 + \beta_1 \sin^2\alpha^2}{z_1 \sin\alpha} - \beta_2 \ln\left|\frac{z_1}{\sin\alpha}\right| = \text{const}\,. \tag{6.20}$$

In particular, system (6.19) for $\beta_1 = 1$, $\beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_9 = 0$, $\beta_6 = \beta_8 = -1$, and $\beta_7 = b$ coincides with system (3.15).

To find an additional first integral of the nonautonomous system (6.1), we can use the first integral (6.16), which is expressed through a finite combination of elementary functions.

First, we transform relation (6.15) as follows:

$$(\beta_4 - \beta_8)v^2 + [(\beta_9 - \beta_5)u + (\beta_3 - \beta_7)]\, v + f_1(u) = 0, \tag{6.21}$$

where

$$f_1(u) = \beta_1 - \beta_6 u^2 - \beta_2 u \ln|u| - C_1 u.$$

Formally, v can be found from the relation

$$v_{1,2}(u) = \frac{1}{2(\beta_4 - \beta_8)} \left\{(\beta_5 - \beta_9)u + (\beta_7 - \beta_3) \pm \sqrt{f_2(u)}\right\}, \tag{6.22}$$

where

$$f_2(u) = A_1 + A_2 u + A_3 u^2 + A_4 u \ln|u|,$$
$$A_1 = (\beta_3 - \beta_7)^2 - 4\beta_1(\beta_4 - \beta_8), \quad A_2 = 2(\beta_9 - \beta_5)(\beta_3 - \beta_7) + 4C_1(\beta_4 - \beta_8),$$
$$A_3 = (\beta_9 - \beta_5)^2 + 4\beta_6(\beta_4 - \beta_8), \quad A_4 = 4\beta_2(\beta_4 - \beta_8).$$

Then the required quadrature for the additional (in general, transcendental) first integral (for example, of system (6.12), (6.13) or (6.5), (6.6), where Eq. (6.13) is used) becomes

$$\int \frac{dx}{x} = \int \frac{[\beta_7 + \beta_5 u + \beta_8 v_{1,2}(u)]du}{(\beta_3 - \beta_7)u + (\beta_9 - \beta_5)u^2 + 2(\beta_4 - \beta_8)uv_{1,2}(u)} = \int \frac{\left[B_1 + B_2 u + B_3\sqrt{f_2(u)}\right] du}{B_4 u\sqrt{f_2(u)}},$$
$$B_k = \text{const}, \quad k = 1, \dots, 4. \tag{6.23}$$

The required quadrature for the search for an additional (in general, transcendental) first integral (for system (6.12), (6.13) or (6.5), (6.6), where Eq. (6.12) is used) becomes

$$\int \frac{dx}{x} = \int \frac{[\beta_7 + \beta_5 u(v) + \beta_8 v]dv}{\beta_1 + \beta_2 u(v) + (\beta_3 - \beta_7)v + (\beta_4 - \beta_8)v^2 + \beta_6 u^2(v)}; \tag{6.24}$$

in this case, the function $u(v)$ must be obtained by solving the implicit equation (6.15) with respect to u (which, in the general case, is not evident).

Sufficient conditions of expressability of integrals in (6.24) through finite combinations of elementary functions are stated by the following lemma.

Lemma 6.1. *For $A_4 = 0$, i.e., for*

$$\beta_2 = 0 \tag{6.25}$$

or for

$$\beta_4 = \beta_8, \tag{6.26}$$

the indefinite integral in (6.24) *can be expressed through a finite combinations of elementary functions.*

Theorem 6.1. *Under the sufficient conditions of Lemma* 6.1 (*in this case, property* (6.25) *holds*), *system* (6.19) *possesses a complete set of first integrals that can be expressed through a finite combination of elementary functions.*

The dynamical systems considered in the present paper are systems with variable dissipation with zero mean with respect to the periodic coordinate. In many cases, such systems possess a complete set of first integrals that can be expressed through elementary functions.

We have presented several cases of complete integrability in the dynamics of the spatial (3D) motion of a body in a nonconservative field. Moreover, we are dealing with three properties that, at first glance, seem to be independent:

(1) the class of systems (3.1) with marked symmetries specified above;
(2) this class of systems possesses variable dissipation with zero mean (with respect to the variable α); this allows one to consider them as "almost" conservative systems;
(3) in some (sufficiently low-dimensional) cases, these systems possess a complete set of (generally speaking, transcendental from the standpoint of complex analysis) first integrals.

The method of reduction of initial systems whose right-hand sides contain polynomials of trigonometric functions to systems with polynomial right-hand sides allows one to find (or to prove the absence) of first integrals for systems of a more general form that perhaps do not possess the symmetries mentioned above (see [91]).

Chapter 2

CASES OF INTEGRABILITY CORRESPONDING TO THE MOTION OF A RIGID BODY IN FOUR-DIMENSIONAL SPACE, I

In this chapter, we systematize some earlier and newer results on the study of the equations of motion of axisymmetric four-dimensional (4D) rigid bodies in nonconservative force fields. The form of these equations is taken from the dynamics of real lower-dimensional rigid bodies interacting with a resisting medium by the laws of jet flows, where a body is influenced by a nonconservative tracing force; under the action of this force, the velocity of some characteristic point of the body remains constant, which means that the system possesses a nonintegrable servo constraint (see [5, 31, 36, 46, 53, 71, 77, 81, 88, 139, 152]).

Earlier (see [42, 81]), the present author proved the complete integrability of the equations of plane-parallel motion of a body in a resisting medium under jet flow conditions when the system of dynamical equations possesses a first integral which is a transcendental (in the sense of the theory of functions of a complex variable) function of quasi-velocities having essential singularities. It was assumed that the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a (one-dimensional) plate.

Subsequently (see [76, 77, 95]), the planar problem was generalized to the spatial (three-dimensional) case, where the system of dynamical equations possesses a complete set of transcendental first integrals. In this case, it was assumed that the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a planar (two-dimensional) disk.

In this chapter, we discuss results, both new results and results obtained earlier, concerning the case where the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a three-dimensional disk and the force acts in the direction perpendicular to the disk. We systematize these results and formulate them in invariant form. We also introduce an additional dependence of the moment of a nonconservative force on the angular velocity; this dependence can be generalized to the motion in higher-dimensional spaces.

7. General Discussion

7.1. Two cases of dynamical symmetry of a four-dimensional body. Assume that a four-dimensional rigid body Θ of mass m with smooth three-dimensional boundary $\partial\Theta$ is under the influence of a nonconservative force field; this can be interpreted as motion of the body in a resisting medium that fills up the four-dimensional domain of Euclidean space $\mathbf{E}^4$. We assume that the body is dynamically symmetric. If the body has two independent principal moments of inertia, then in some coordinate system $Dx_1x_2x_3x_4$ attached to the body the operator of inertia has the form

$$\operatorname{diag}\{I_1, I_2, I_2, I_2\} \tag{7.1}$$

or the form

$$\operatorname{diag}\{I_1, I_1, I_3, I_3\}. \tag{7.2}$$

In the first case, the body is dynamically symmetric in the hyperplane $Dx_2x_3x_4$ while in the second case the two-dimensional planes Dx_1x_2 and Dx_3x_4 are planes of dynamical symmetry of the body.

7.2. Dynamics on so(4) **and** $\mathbb{R}^4$**.** The configuration space of a free, n-dimensional rigid body is the direct product

$$\mathbb{R}^n \times \mathrm{SO}(n) \tag{7.3}$$

of the space $\mathbb{R}^n$, which defines the coordinates of the center of mass of the body, and the rotation group $\mathrm{SO}(n)$, which defines rotations of the body about its center of mass and has dimension

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

Therefore, the dynamical part of the equations of motion has the same dimension, whereas the dimension of the phase space is equal to $n(n+1)$.

In particular, if Ω is the tensor of the angular velocity of a four-dimensional rigid body (it is a second-rank tensor, see [18–22, 25–27, 40, 41, 66]), $\Omega \in \mathrm{so}(4)$, then the part of the dynamical equations of motion corresponding to the Lie algebra so(4) has the following form (see [7, 9, 10, 13, 66, 147–149]):

$$\dot{\Omega}\Lambda + \Lambda\dot{\Omega} + [\Omega, \Omega\Lambda + \Lambda\Omega] = M, \tag{7.4}$$

where

$$\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}, \tag{7.5}$$

$$\lambda_1 = \frac{-I_1 + I_2 + I_3 + I_4}{2}, \quad \lambda_2 = \frac{I_1 - I_2 + I_3 + I_4}{2},$$

$$\lambda_3 = \frac{I_1 + I_2 - I_3 + I_4}{2}, \quad \lambda_4 = \frac{I_1 + I_2 + I_3 - I_4}{2},$$

$M = M_F$ is the natural projection of the moment of external forces $\mathbf{F}$ acting on the body in $\mathbb{R}^4$ on the natural coordinates of the Lie algebra so(4) and [ ] is the commutator in so(4). The skew-symmetric matrix corresponding to this second-rank tensor $\Omega \in \mathrm{so}(4)$ is represented in the form

$$\begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}, \tag{7.6}$$

where ω_1, ω_2, ω_3, ω_4, ω_5, and ω_6 are the components of the tensor of angular velocity corresponding to the projections on the coordinates of the Lie algebra so(4).

Obviously, the following relations hold:

$$\lambda_i - \lambda_j = I_j - I_i, \quad i, j = 1, \ldots, 4. \tag{7.7}$$

To calculate the moment of an external force acting to the body, we need to construct the mapping

$$\mathbb{R}^4 \times \mathbb{R}^4 \to \mathrm{so}(4) \tag{7.8}$$

that maps a pair of vectors

$$(\mathbf{DN}, \mathbf{F}) \in \mathbb{R}^4 \times \mathbb{R}^4 \tag{7.9}$$

into an element of the Lie algebra so(4), where

$$\mathbf{DN} = \{0, x_{2N}, x_{3N}, x_{4N}\}, \quad \mathbf{F} = \{F_1, F_2, F_3, F_4\}, \tag{7.10}$$

and $\mathbf{F}$ is an external force acting on the body. For this purpose, we construct the following auxiliary matrix

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} & x_{4N} \\ F_1 & F_2 & F_3 & F_4 \end{pmatrix}. \tag{7.11}$$

Then the right-hand side of system (7.4) takes the form

$$M = \Big\{ x_{3N}F_4 - x_{4N}F_3,\ x_{4N}F_2 - x_{2N}F_4,\ -x_{4N}F_1,\ x_{2N}F_3 - x_{3N}F_2,\ x_{3N}F_1,\ -x_{2N}F_1 \Big\}. \tag{7.12}$$

The dynamical systems studied in the following chapters, generally speaking, are not conservative and in fact are dynamical systems with variable dissipation with zero mean (see [91]). We need to examine by direct methods one part of the main system of dynamical equations, namely, the Newton equation, which serves as the equation of motion of the center of mass, i.e., the part of the dynamical equations of motion corresponding to the space $\mathbb{R}^4$:

$$m\mathbf{w}_C = \mathbf{F}, \tag{7.13}$$

where $\mathbf{w}_C$ is the acceleration of the center of mass C of the body and m is its mass. Moreover, due to the higher-dimensional Rivals formula (it can be obtained by the operator method) we have the following relations:

$$\mathbf{w}_C = \mathbf{w}_D + \Omega^2\mathbf{DC} + E\mathbf{DC}, \quad \mathbf{w}_D = \dot{\mathbf{v}}_D + \Omega\mathbf{v}_D, \quad E = \dot{\Omega}, \tag{7.14}$$

where $\mathbf{w}_D$ is the acceleration of the point D, $\mathbf{F}$ is the external force acting on the body (in our case, $\mathbf{F} = \mathbf{S}$), and E is the tensor of angular acceleration (second-rank tensor).

Thus, the system of equations (7.4) and (7.13) of tenth order on the manifold $\mathbb{R}^4 \times \mathrm{so}(4)$ is a *closed* system of dynamical equations of the motion of a free four-dimensional rigid body under the action of an external force $\mathbf{F}$. This system is separated from the kinematic part of the equations of motion on the manifold (7.3) and can be examined independently.

8. General Problem of Motion Under a Tracing Force

Consider a motion of a homogeneous, dynamically symmetric (case (7.1)), rigid body with "front end face" (a three-dimensional disk interacting with a medium that fills four-dimensional space) in the field of a resistance force $\mathbf{S}$ under quasi-stationarity conditions (see [16, 17, 30, 35, 36, 42, 43, 89, 108, 126, 145, 152].

Let $(v, \alpha, \beta_1, \beta_2)$ be the (generalized) spherical coordinates of the velocity vector of the center of the three-dimensional disk lying on the axis of symmetry of the body, let

$$\Omega = \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

be the tensor of angular velocity of the body, and let $Dx_1x_2x_3x_4$ be the coordinate system attached to the body such that the axis of symmetry CD coicides with the axis Dx_1 (recall that C is the center of mass) and the axes Dx_2, Dx_3, and Dx_4 lie in the hyperplane of the disk, while I_1, I_2, $I_3 = I_2$, $I_4 = I_2$, and m are the characteristics of inertia and mass.

We adopt the following expansions in projections onto the axes of the coordinate system $Dx_1x_2x_3x_4$:

$$\begin{aligned} &\mathbf{DC} = \{-\sigma, 0, 0, 0\}, \\ &\mathbf{v}_D = \Big\{ v\cos\alpha,\ v\sin\alpha\cos\beta_1,\ v\sin\alpha\sin\beta_1\cos\beta_2,\ v\sin\alpha\sin\beta_1\sin\beta_2 \Big\}. \end{aligned} \tag{8.1}$$

In the case (7.1) we additionally have an expansion for the function of the influence of the medium on the four-dimensional body:

$$\mathbf{S} = \{-S, 0, 0, 0\}, \tag{8.2}$$

i.e., in this case $\mathbf{F} = \mathbf{S}$.

Then the part of the dynamical equations of motion (including the analytic Chaplygin functions [16, 17]; see below) that describes the motion of the center of mass and corresponds to the space $\mathbb{R}^4$, in which tangent forces of the influence of the medium on the three-dimensional disk vanish, takes the form

$$\dot v\cos\alpha-\dot\alpha v\sin\alpha-\omega_6 v\sin\alpha\cos\beta_1+\omega_5 v\sin\alpha\sin\beta_1\cos\beta_2-\omega_3 v\sin\alpha\sin\beta_1\sin\beta_2 \\ +\sigma\left(\omega_6^2+\omega_5^2+\omega_3^2\right)=-\frac{S}{m}, \tag{8.3}$$

$$\dot v\sin\alpha\cos\beta_1+\dot\alpha v\cos\alpha\cos\beta_1-\dot\beta_1 v\sin\alpha\sin\beta_1+\omega_6 v\cos\alpha-\omega_4 v\sin\alpha\sin\beta_1\cos\beta_2 \\ +\omega_2 v\sin\alpha\sin\beta_1\sin\beta_2-\sigma(\omega_4\omega_5+\omega_2\omega_3)-\sigma\dot\omega_6=0, \tag{8.4}$$

$$\dot v\sin\alpha\sin\beta_1\cos\beta_2+\dot\alpha v\cos\alpha\sin\beta_1\cos\beta_2+\dot\beta_1 v\sin\alpha\cos\beta_1\cos\beta_2-\dot\beta_2 v\sin\alpha\sin\beta_1\sin\beta_2 \\ -\omega_5 v\cos\alpha+\omega_4 v\sin\alpha\cos\beta_1-\omega_1 v\sin\alpha\sin\beta_1\sin\beta_2-\sigma(-\omega_1\omega_2+\omega_4\omega_6)+\sigma\dot\omega_5=0, \tag{8.5}$$

$$\dot v\sin\alpha\sin\beta_1\sin\beta_2+\dot\alpha v\cos\alpha\sin\beta_1\sin\beta_2+\dot\beta_1 v\sin\alpha\cos\beta_1\sin\beta_2+\dot\beta_2 v\sin\alpha\sin\beta_1\cos\beta_2 \\ +\omega_3 v\cos\alpha-\omega_2 v\sin\alpha\cos\beta_1+\omega_1 v\sin\alpha\sin\beta_1\cos\beta_2+\sigma(\omega_2\omega_6+\omega_1\omega_5)-\sigma\dot\omega_3=0, \tag{8.6}$$

where

$$S=s(\alpha)v^2,\quad \sigma=CD,\quad v>0. \tag{8.7}$$

Further, the auxiliary matrix (7.11) for calculating the moment of the resistance force has the form

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} & x_{4N} \\ -S & 0 & 0 & 0 \end{pmatrix}; \tag{8.8}$$

then the part of the dynamical equations of motion that describes the motion of the body about the center of mass and corresponds to the Lie algebra so(4), becomes

$$(\lambda_4+\lambda_3)\dot\omega_1+(\lambda_3-\lambda_4)(\omega_3\omega_5+\omega_2\omega_4)=0, \tag{8.9}$$

$$(\lambda_2+\lambda_4)\dot\omega_2+(\lambda_2-\lambda_4)(\omega_3\omega_6-\omega_1\omega_4)=0, \tag{8.10}$$

$$(\lambda_4+\lambda_1)\dot\omega_3+(\lambda_4-\lambda_1)(\omega_2\omega_6+\omega_1\omega_5)=x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2, \tag{8.11}$$

$$(\lambda_3+\lambda_2)\dot\omega_4+(\lambda_2-\lambda_3)(\omega_5\omega_6+\omega_1\omega_2)=0, \tag{8.12}$$

$$(\lambda_1+\lambda_3)\dot\omega_5+(\lambda_3-\lambda_1)(\omega_4\omega_6-\omega_1\omega_3)=-x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2, \tag{8.13}$$

$$(\lambda_1+\lambda_2)\dot\omega_6+(\lambda_1-\lambda_2)(\omega_4\omega_5+\omega_2\omega_3)=x_{2N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)v^2. \tag{8.14}$$

Thus, the phase space of system (8.3)–(8.6), (8.9)–(8.14) of tenth order is the direct product of the four-dimensional manifold and the Lie algebra so(4):

$$\mathbb{R}^1\times\mathbf{S}^3\times\mathrm{so}(4). \tag{8.15}$$

We note that system (8.3)–(8.6), (8.9)–(8.14), due to the existing dynamical symmetry

$$I_2=I_3=I_4, \tag{8.16}$$

possesses cyclic first integrals

$$\omega_1\equiv\omega_1^0=\mathrm{const},\quad \omega_2\equiv\omega_2^0=\mathrm{const},\quad \omega_4\equiv\omega_4^0=\mathrm{const}\,. \tag{8.17}$$

We will henceforth consider the dynamics of the system on zero levels:

$$\omega_1^0=\omega_2^0=\omega_4^0=0. \tag{8.18}$$

If we consider a more general problem on the motion of a body under a tracing force $\mathbf{T}$ that lies on the straight line $CD = Dx_1$ and assume that the relation

$$v \equiv \text{const} \tag{8.19}$$

is satisfied throughout the motion (see [91]), then instead of F_1 system (8.3)–(8.6), (8.9)–(8.14) contains

$$T - s(\alpha)v^2, \quad \sigma = DC. \tag{8.20}$$

Choosing the value T of the tracing force appropriately, we can assume Eq. (8.19) throughout the motion. Indeed, expressing T on the basis of system (8.3)–(8.6), (8.9)–(8.14), we obtain for $\cos\alpha \neq 0$ the relation

$$T = T_v(\alpha, \beta_1, \beta_2, \Omega) = m\sigma\left(\omega_3^2 + \omega_5^2 + \omega_6^2\right) + s(\alpha)v^2\left[1 - \frac{m\sigma}{2I_2}\frac{\sin\alpha}{\cos\alpha}\Gamma_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\right], \tag{8.21}$$

where

$$\Gamma_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\sin\beta_1\sin\beta_2 + x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\sin\beta_1\cos\beta_2 + x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\cos\beta_1; \tag{8.22}$$

here we used conditions (8.17)–(8.19).

This procedure can be interpreted in two ways. First, we have transformed the system using the tracing force (control) that enables us to consider the class (8.19) of motions of interest. Second, we can treat this as an order-reduction procedure. Indeed, system (8.3)–(8.6), (8.9)–(8.14) generates the following independent system of sixth order:

$$\dot\alpha v\cos\alpha\cos\beta_1 - \dot\beta_1 v\sin\alpha\sin\beta_1 + \omega_6 v\cos\alpha - \sigma\dot\omega_6 = 0, \tag{8.23}$$

$$\dot\alpha v\cos\alpha\sin\beta_1\cos\beta_2 + \dot\beta_1 v\sin\alpha\cos\beta_1\cos\beta_2 - \dot\beta_2 v\sin\alpha\sin\beta_1\sin\beta_2 - \omega_5 v\cos\alpha + \sigma\dot\omega_5 = 0, \tag{8.24}$$

$$\dot\alpha v\cos\alpha\sin\beta_1\sin\beta_2 + \dot\beta_1 v\sin\alpha\cos\beta_1\sin\beta_2 + \dot\beta_2 v\sin\alpha\sin\beta_1\cos\beta_2 + \omega_3 v\cos\alpha - \sigma\dot\omega_3 = 0, \tag{8.25}$$

$$2I_2\dot\omega_3 = x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2, \tag{8.26}$$

$$2I_2\dot\omega_5 = -x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2, \tag{8.27}$$

$$2I_2\dot\omega_6 = x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2, \tag{8.28}$$

which, in addition to the permanent parameters specified above, contains the parameter v.

System (8.23)–(8.28) is equivalent to the system

$$\begin{aligned}\dot\alpha v\cos\alpha + v\cos\alpha\left[\omega_6\cos\beta_1 - \omega_5\sin\beta_1\cos\beta_2 + \omega_3\sin\beta_1\sin\beta_2\right]\\ +\sigma\left[-\dot\omega_6\cos\beta_1 + \dot\omega_5\sin\beta_1\cos\beta_2 - \dot\omega_3\sin\beta_1\sin\beta_2\right] = 0,\end{aligned} \tag{8.29}$$

$$\begin{aligned}\dot\beta_1 v\sin\alpha - v\cos\alpha\left[\omega_5\cos\beta_1\cos\beta_2 + \omega_6\sin\beta_1 - \omega_3\cos\beta_1\sin\beta_2\right]\\ +\sigma\left[\dot\omega_5\cos\beta_1\cos\beta_2\dot\omega_6\sin\beta_1 - \dot\omega_3\cos\beta_1\sin\beta_2\right] = 0,\end{aligned} \tag{8.30}$$

$$\dot\beta_2 v\sin\alpha\sin\beta_1 + v\cos\alpha\left[\omega_3\cos\beta_2 + \omega_5\sin\beta_2\right] + \sigma\left[-\dot\omega_3\cos\beta_2 - \dot\omega_5\sin\beta_2\right] = 0, \tag{8.31}$$

$$\dot\omega_3 = \frac{v^2}{2I_2}x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha), \tag{8.32}$$

$$\dot{\omega}_5 = -\frac{v^2}{2I_2} x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha), \tag{8.33}$$

$$\dot{\omega}_6 = \frac{v^2}{2I_2} x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha). \tag{8.34}$$

We introduce new quasi-velocities. For this purpose, we transform ω_3, ω_5, and ω_6 by means of two rotations:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = T_1(-\beta_1) \circ T_3(-\beta_2) \begin{pmatrix} \omega_3 \\ \omega_5 \\ \omega_6 \end{pmatrix}, \tag{8.35}$$

where

$$T_1(\beta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta_1 & -\sin\beta_1 \\ 0 & \sin\beta_1 & \cos\beta_1 \end{pmatrix}, \quad T_3(\beta_2) = \begin{pmatrix} \cos\beta_2 & -\sin\beta_2 & 0 \\ \sin\beta_2 & \cos\beta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8.36}$$

Therefore, the following relations hold:

$$\begin{aligned} z_1 &= \omega_3 \cos\beta_2 + \omega_5 \sin\beta_2, \\ z_2 &= -\omega_3 \cos\beta_1 \sin\beta_2 + \omega_5 \cos\beta_1 \cos\beta_2 + \omega_6 \sin\beta_1, \\ z_3 &= \omega_3 \sin\beta_1 \sin\beta_2 - \omega_5 \sin\beta_1 \cos\beta_2 + \omega_6 \cos\beta_1. \end{aligned} \tag{8.37}$$

As we see from (8.29)–(8.34), we cannot solve the system with respect to $\dot{\alpha}$, $\dot{\beta}_1$, and $\dot{\beta}_2$ on the manifold

$$O_1 = \left\{(\alpha, \beta_1, \beta_2, \omega_3, \omega_5, \omega_6) \in \mathbb{R}^6 : \alpha = \frac{\pi}{2}k,\ \beta_1 = \pi l,\ k, l \in \mathbb{Z}\right\}. \tag{8.38}$$

Therefore, on the manifold (8.38) the uniqueness theorem is formally violated. Moreover, for even k and any l, an indeterminate form appears due to the degeneration of the spherical coordinates $(v, \alpha, \beta_1, \beta_2)$. For odd k, the uniqueness theorem is obviously violated since the first equation (8.29) is degenerate.

This implies that system (8.29)–(8.34) outside (and only outside) the manifold (8.38) is equivalent to the system

$$\dot{\alpha} = -z_3 + \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\cos\alpha} \Gamma_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right), \tag{8.39}$$

$$\begin{aligned} \dot{z_3} = {} & \frac{v^2}{2I_2} s(\alpha) \Gamma_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) - \left(z_1^2 + z_2^2\right) \frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} z_2 \Delta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) \\ & + \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} z_1 \Theta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right), \end{aligned} \tag{8.40}$$

$$\begin{aligned} \dot{z_2} = {} & -\frac{v^2}{2I_2} s(\alpha) \Delta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) + z_2 z_3 \frac{\cos\alpha}{\sin\alpha} + z_1^2 \frac{\cos\alpha}{\sin\alpha} \frac{\cos\beta_1}{\sin\beta_1} + \\ & + \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} z_3 \Delta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) - -\frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} \frac{\cos\beta_1}{\sin\beta_1} z_1 \Theta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right), \end{aligned} \tag{8.41}$$

$$\begin{aligned} \dot{z_1} = {} & \frac{v^2}{2I_2} s(\alpha) \Theta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) + z_1 z_3 \frac{\cos\alpha}{\sin\alpha} - z_1 z_2 \frac{\cos\alpha}{\sin\alpha} \frac{\cos\beta_1}{\sin\beta_1} - \\ & - \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} z_3 \Theta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) + \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} \frac{\cos\beta_1}{\sin\beta_1} z_2 \Theta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right), \end{aligned} \tag{8.42}$$

$$\dot{\beta}_1 = z_2 \frac{\cos\alpha}{\sin\alpha} + \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha} \Delta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right), \tag{8.43}$$

$$\dot{\beta}_2 = -z_1 \frac{\cos\alpha}{\sin\alpha \sin\beta_1} + \frac{\sigma v}{2I_2} \frac{s(\alpha)}{\sin\alpha \sin\beta_1} \Theta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right), \tag{8.44}$$

where

$$\begin{aligned} \Delta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) = x_{4N} \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) \cos\beta_1 \sin\beta_2 + x_{3N} \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) \cos\beta_1 \cos\beta_2 \\ - x_{2N} \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) \sin\beta_1, \end{aligned} \tag{8.45}$$

$$\Theta_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) = x_{4N} \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) \cos\beta_2 - x_{3N} \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) \sin\beta_2, \tag{8.46}$$

and the function $\Gamma_v (\alpha, \beta_1, \beta_2, \Omega/v)$ can be represented in the form (8.22).

Here and in the discussion that follows, the dependence on the group of variables $(\alpha, \beta_1, \beta_2, \Omega/v)$ is meant as a composite dependence on $(\alpha, \beta_1, \beta_2, z_1/v, z_2/v, z_3/v)$ due to (8.37).

The uniqueness theorem for system (8.29)–(8.34) on the manifold (8.38) for odd k is violated in the following sense: for odd k, a nonsingular phase trajectory of system (8.29)–(8.34) passes through almost all points of the manifold (8.38), intersecting the manifold (8.38) at right angle, and there exists a phase trajectory that at any moment of time completely coincides with the point specified. However, physically these trajectories are different since they correspond to different values of the tracing force.

We prove this assertion. As was shown above, to maintain a constraint of the form (8.19), we must take a value of T for $\cos\alpha \neq 0$ according to (8.21).

Let

$$\lim_{\alpha \to \pi/2} \frac{s(\alpha)}{\cos\alpha} \Gamma_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) = L \left(\beta_1, \beta_2, \frac{\Omega}{v} \right). \tag{8.47}$$

Note that $|L| < +\infty$ if and only if

$$\lim_{\alpha \to \pi/2} \left| \frac{\partial}{\partial \alpha} \left(\Gamma_v \left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v} \right) s(\alpha) \right) \right| < +\infty. \tag{8.48}$$

For $\alpha = \pi/2$, the required value of the tracing force is defined by the equation

$$T = T_v \left(\frac{\pi}{2}, \beta_1, \beta_2, \Omega \right) = m\sigma \left(\omega_3^2 + \omega_5^2 + \omega_6^2 \right) - \frac{m\sigma L v^2}{2I_2}. \tag{8.49}$$

where ω_3, ω_5, and ω_6 are arbitrary.

On the other hand, maintaining the rotation about some point W by the tracing force, we must choose this force according to the relation

$$T = T_v \left(\frac{\pi}{2}, \beta_1, \beta_2, \Omega \right) = \frac{mv^2}{R_0}, \tag{8.50}$$

where R_0 is the distance between C and W.

Relations (8.21) and (8.50) define, in general, different values of the tracing force T for almost all points of the manifold (8.38), which proves our assertion.

9. Case Where the Moment of a Nonconservative Force Is Independent of the Angular Velocity

9.1. Reduced system. As in the choice of Chaplygin analytic functions (see [16, 17]), we take the dynamical functions s, x_{2N}, x_{3N}, and x_{4N} in the following form:

$$\begin{aligned} s(\alpha) &= B\cos\alpha, \\ x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= x_{2N0}(\alpha, \beta_1, \beta_2) = A\sin\alpha\cos\beta_1, \\ x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= x_{3N0}(\alpha, \beta_1, \beta_2) = A\sin\alpha\sin\beta_1\cos\beta_2, \\ x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= x_{4N0}(\alpha, \beta_1, \beta_2) = A\sin\alpha\sin\beta_1\sin\beta_2, \end{aligned} \tag{9.1}$$

where $A, B > 0$ and $v \neq 0$. We see that in the system considered, the moment of nonconservative forces in independent of the angular velocity (but depends on the angles α, β_1, and β_2). Moreover, the functions $\Gamma_v(\alpha, \beta_1, \beta_2, \Omega/v)$, $\Delta_v(\alpha, \beta_1, \beta_2, \Omega/v)$, $\Theta_v(\alpha, \beta_1, \beta_2, \Omega/v)$ in system (8.39)–(8.44) assume the following form:

$$\Gamma_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = A\sin\alpha, \quad \Delta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) \equiv \Theta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) \equiv 0. \tag{9.2}$$

Then, due to the nonintegrable constraint (8.19) outside the manifold (8.38), the dynamical part of the equations of motion (system (8.39)–(8.44)) has the form of the following analytic system:

$$\dot\alpha = -z_3 + \frac{\sigma ABv}{2I_2}\sin\alpha, \tag{9.3}$$

$$\dot z_3 = \frac{ABv^2}{2I_2}\sin\alpha\cos\alpha - \left(z_1^2 + z_2^2\right)\frac{\cos\alpha}{\sin\alpha}, \tag{9.4}$$

$$\dot z_2 = z_2 z_3\frac{\cos\alpha}{\sin\alpha} + z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{9.5}$$

$$\dot z_1 = z_1 z_3\frac{\cos\alpha}{\sin\alpha} - z_1 z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{9.6}$$

$$\dot\beta_1 = z_2\frac{\cos\alpha}{\sin\alpha}, \tag{9.7}$$

$$\dot\beta_2 = -z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{9.8}$$

Further, introducing dimensionless variables and parameters and a new differentiation as follows:

$$z_k \mapsto n_0 v z_k,\ k = 1, 2, 3, \quad n_0^2 = \frac{AB}{2I_2}, \quad b = \sigma n_0, \quad \langle\dot{\ }\rangle = n_0 v\langle\ '\rangle, \tag{9.9}$$

we reduce system (9.3)–(9.8) to the form

$$\alpha' = -z_3 + b\sin\alpha, \tag{9.10}$$

$$z_3' = \sin\alpha\cos\alpha - \left(z_1^2 + z_2^2\right)\frac{\cos\alpha}{\sin\alpha}, \tag{9.11}$$

$$z_2' = z_2 z_3\frac{\cos\alpha}{\sin\alpha} + z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{9.12}$$

$$z_1' = z_1 z_3\frac{\cos\alpha}{\sin\alpha} - z_1 z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{9.13}$$

$$\beta_1' = z_2\frac{\cos\alpha}{\sin\alpha}, \tag{9.14}$$

$$\beta_2' = -z_1 \frac{\cos\alpha}{\sin\alpha \sin\beta_1}. \tag{9.15}$$

We see that the sixth-order system (9.10)–(9.15) (which can be considered as a system on the tangent bundle $T\mathbf{S}^3$ of the three-dimensional sphere $\mathbf{S}^3$, see below) contains the independent fifth-order system (9.10)–(9.14) on its own five-dimensional manifold.

For the complete integration of system (9.10)–(9.15), in general, we need five independent first integrals. However, after the change of variables

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} z \\ z_* \end{pmatrix}, \quad z = \sqrt{z_1^2 + z_2^2}, \quad z_* = z_2/z_1, \tag{9.16}$$

system (9.10)–(9.15) splits as follows:

$$\alpha' = -z_3 + b\sin\alpha, \tag{9.17}$$

$$z_3' = \sin\alpha\cos\alpha - z^2 \frac{\cos\alpha}{\sin\alpha}, \tag{9.18}$$

$$z' = zz_3 \frac{\cos\alpha}{\sin\alpha}, \tag{9.19}$$

$$z_*' = (\pm) z\sqrt{1 + z_*^2} \frac{\cos\alpha}{\sin\alpha} \frac{\cos\beta_1}{\sin\beta_1}, \tag{9.20}$$

$$\beta_1' = (\pm) \frac{zz_*}{\sqrt{1 + z_*^2}} \frac{\cos\alpha}{\sin\alpha}, \tag{9.21}$$

$$\beta_2' = (\mp) \frac{z}{\sqrt{1 + z_*^2}} \frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{9.22}$$

We see that the sixth-order system splits into independent subsystems of lower order: system (9.17)–(9.19) has order three and system (9.20), (9.21) (after a change of the independent variable) order two. Thus, for the complete integration of system (9.17)–(9.22) it suffices to specify two independent first integrals of system (9.17)–(9.19), one first integral of system (9.20), (9.21), and an additional first integral that "attaches" Eq. (9.22).

Note that system (9.17)–(9.19) can be considered on the tangent bundle $T\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$.

9.2. Complete list of invariant relations. System (9.17)–(9.19) has the form of a system that appears in the dynamics of a three-dimensional (3D) rigid body in a field of nonconservative forces.

First, we establish a correspondence between the third-order system (9.17)–(9.19) and the nonautonomous second-order system

$$\begin{aligned} \frac{dz_3}{d\alpha} &= \frac{\sin\alpha\cos\alpha - z^2\cos\alpha/\sin\alpha}{-z_3 + b\sin\alpha}, \\ \frac{dz}{d\alpha} &= \frac{zz_3\cos\alpha/\sin\alpha}{-z_3 + b\sin\alpha}. \end{aligned} \tag{9.23}$$

Applying the substitution $\tau = \sin\alpha$, we rewrite system (9.23) in algebraic form

$$\begin{aligned} \frac{dz_3}{d\tau} &= \frac{\tau - z^2/\tau}{-z_3 + b\tau}, \\ \frac{dz}{d\tau} &= \frac{zz_3/\tau}{-z_3 + b\tau}. \end{aligned} \tag{9.24}$$

Introducing homogeneous variables by the formulas

$$z = u_1\tau, \quad z_3 = u_2\tau, \tag{9.25}$$

we reduce system (9.24) to the following form:

$$\tau\frac{du_2}{d\tau}+u_2=\frac{1-u_1^2}{-u_2+b},$$
$$\tau\frac{du_1}{d\tau}+u_1=\frac{u_1u_2}{-u_2+b}, \tag{9.26}$$

which is equivalent to the system

$$\tau\frac{du_2}{d\tau}=\frac{1-u_1^2+u_2^2-bu_2}{-u_2+b},$$
$$\tau\frac{du_1}{d\tau}=\frac{2u_1u_2-bu_1}{-u_2+b}. \tag{9.27}$$

We establish a correspondence between the second-order system (9.27) and the nonautonomous first-order equation

$$\frac{du_2}{du_1}=\frac{1-u_1^2+u_2^2-bu_2}{2u_1u_2-bu_1}, \tag{9.28}$$

which can be easily reduced to exact-differential form:

$$d\left(\frac{u_2^2+u_1^2-bu_2+1}{u_1}\right)=0. \tag{9.29}$$

Thus, Eq. (9.28) has the following first integral:

$$\frac{u_2^2+u_1^2-bu_2+1}{u_1}=C_1=\text{const}, \tag{9.30}$$

which expresses in terms of the previous variables has the form

$$\frac{z_3^2+z^2-bz_3\sin\alpha+\sin^2\alpha}{z\sin\alpha}=C_1=\text{const}\,. \tag{9.31}$$

Remark 9.1. Consider system (9.17)–(9.19) with variable dissipation with zero mean (see [91]) which becomes conservative for $b=0$:

$$\alpha'=-z_3,$$
$$z_3'=\sin\alpha\cos\alpha-z^2\frac{\cos\alpha}{\sin\alpha}, \tag{9.32}$$
$$z'=zz_3\frac{\cos\alpha}{\sin\alpha}.$$

It possesses two analytic first integrals of the form

$$z_3^2+z^2+\sin^2\alpha=C_1^*=\text{const}, \tag{9.33}$$

$$z\sin\alpha=C_2^*=\text{const}\,. \tag{9.34}$$

Obviously, the ratio of the two first integrals (9.33) and (9.34) is also a first integral of system (9.32). However, for $b\neq0$, neither of the functions

$$z_3^2+z^2-bz_3\sin\alpha+\sin^2\alpha \tag{9.35}$$

and (9.34) is a first integral of system (9.17)–(9.19) but their ratio is a first integral for any b.

Further, we find the explicit form of the additional first integral of the third-order system (9.17)–(9.19). For this purpose, we transform the invariant relation (9.30) for $u_1\neq0$ as follows:

$$\left(u_2-\frac{b}{2}\right)^2+\left(u_1-\frac{C_1}{2}\right)^2=\frac{b^2+C_1^2}{4}-1. \tag{9.36}$$

We see that the parameters of this invariant relation satisfy the condition

$$b^2 + C_1^2 - 4 \geq 0, \tag{9.37}$$

and the phase space of system (9.17)–(9.19) is stratified into a family of surfaces defined by Eq. (9.36).

Thus, by relation (9.30), the first equation of system (9.27) has the form

$$\tau \frac{du_2}{d\tau} = \frac{2\left(1 - bu_2 + u_2^2\right) - C_1 U_1(C_1, u_2)}{-u_2 + b}, \tag{9.38}$$

where

$$U_1(C_1, u_2) = \frac{1}{2}\left\{C_1 \pm \sqrt{C_1^2 - 4\left(u_2^2 - bu_2 + 1\right)}\right\}; \tag{9.39}$$

the integration constant C_1 is defined by condition (9.37).

Therefore, the quadrature for the search for the additional first integral of system (9.17)–(9.19) becomes

$$\int \frac{d\tau}{\tau} = \int \frac{(b - u_2)du_2}{2\left(1 - bu_2 + u_2^2\right) - C_1\left\{C_1 \pm \sqrt{C_1^2 - 4\left(u_2^2 - bu_2 + 1\right)}\right\}/2}. \tag{9.40}$$

Obviously, the left-hand side (up to an additive constant) equals

$$\ln|\sin\alpha|. \tag{9.41}$$

If

$$u_2 - \frac{b}{2} = w_1, \quad b_1^2 = b^2 + C_1^2 - 4, \tag{9.42}$$

then the right-hand side of Eq. (9.40) has the form

$$-\frac{1}{4}\int \frac{d\left(b_1^2 - 4w_1^2\right)}{\left(b_1^2 - 4w_1^2\right) \pm C_1\sqrt{b_1^2 - 4w_1^2}} - b\int \frac{dw_1}{\left(b_1^2 - 4w_1^2\right) \pm C_1\sqrt{b_1^2 - 4w_1^2}}$$
$$= -\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2 - 4w_1^2}}{C_1} \pm 1\right| \pm \frac{b}{2}I_1, \tag{9.43}$$

where

$$I_1 = \int \frac{dw_3}{\sqrt{b_1^2 - w_3^2}(w_3 \pm C_1)}, \quad w_3 = \sqrt{b_1^2 - 4w_1^2}. \tag{9.44}$$

In the calculation of the integral (9.44), the following three cases are possible.

I. $b > 2$:

$$I_1 = -\frac{1}{2\sqrt{b^2 - 4}}\ln\left|\frac{\sqrt{b^2 - 4} + \sqrt{b_1^2 - w_3^2}}{w_3 \pm C_1} \pm \frac{C_1}{\sqrt{b^2 - 4}}\right| +$$
$$+ \frac{1}{2\sqrt{b^2 - 4}}\ln\left|\frac{\sqrt{b^2 - 4} - \sqrt{b_1^2 - w_3^2}}{w_3 \pm C_1} \mp \frac{C_1}{\sqrt{b^2 - 4}}\right| + \text{const}; \tag{9.45}$$

II. $b < 2$:

$$I_1 = \frac{1}{\sqrt{4 - b^2}}\arcsin\frac{\pm C_1 w_3 + b_1^2}{b_1(w_3 \pm C_1)} + \text{const}; \tag{9.46}$$

III. $b = 2$:

$$I_1 = \mp\frac{\sqrt{b_1^2 - w_3^2}}{C_1(w_3 \pm C_1)} + \text{const}. \tag{9.47}$$

Returning to the variable

$$w_1 = \frac{z_3}{\sin\alpha} - \frac{b}{2}, \tag{9.48}$$

we obtain the final expression for I_1:

I. $b > 2$:

$$I_1 = -\frac{1}{2\sqrt{b^2-4}} \ln\left|\frac{\sqrt{b^2-4} \pm 2w_1}{\sqrt{b_1^2-4w_1^2} \pm C_1} \pm \frac{C_1}{\sqrt{b^2-4}}\right| + \frac{1}{2\sqrt{b^2-4}} \ln\left|\frac{\sqrt{b^2-4} \mp 2w_1}{\sqrt{b_1^2-4w_1^2} \pm C_1} \mp \frac{C_1}{\sqrt{b^2-4}}\right| + \text{const}; \tag{9.49}$$

II. $b < 2$:

$$I_1 = \frac{1}{\sqrt{4-b^2}} \arcsin \frac{\pm C_1\sqrt{b_1^2-4w_1^2} + b_1^2}{b_1\left(\sqrt{b_1^2-4w_1^2} \pm C_1\right)} + \text{const}; \tag{9.50}$$

III. $b = 2$:

$$I_1 = \mp\frac{2w_1}{C_1\left(\sqrt{b_1^2-4w_1^2} \pm C_1\right)} + \text{const}\,. \tag{9.51}$$

Thus, we have found an additional first integral for the third-order system (9.17)–(9.19) and we have the complete set of first integrals that are transcendental functions of their phase variables.

Remark 9.2. We must substitute the left-hand side of the first integral (9.30) into the expression of this first integral instead of C_1. Then the additional first integral obtained has the following structure (similar to the transcendental first integral in planar dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha, \frac{z_3}{\sin\alpha}, \frac{z}{\sin\alpha}\right) = C_2 = \text{const}\,. \tag{9.52}$$

Thus, for the integration of the sixth-order system (9.17)–(9.22), we have found two independent first integrals. As was mentioned above, to integrate it completely, it suffices to find one first integral for (potentially separated) system (9.20), (9.21) and an additional first integral that "attaches" Eq. (9.22).

To find a first integral for (potentially separated) system (9.20), (9.21), we establish a correspondence between it and the following nonautonomous first-order equation:

$$\frac{dz_*}{d\beta_1} = \frac{1+z_*^2}{z_*}\frac{\cos\beta_1}{\sin\beta_1}. \tag{9.53}$$

After integration, this leads to the invariant relation

$$\frac{\sqrt{1+z_*^2}}{\sin\beta_1} = C_3 = \text{const}, \tag{9.54}$$

which in the variables z_1 and z_2 has the form

$$\frac{\sqrt{z_1^2+z_2^2}}{z_1\sin\beta_1} = C_3 = \text{const}\,. \tag{9.55}$$

Further, to find an additional first integral that "attaches" Eq. (9.22), we establish a correspondence between Eqs. (9.22) and (9.20) and the following nonautonomous equation:

$$\frac{dz_*}{d\beta_2} = -\left(1+z_*^2\right)\cos\beta_1. \tag{9.56}$$

Since, by (9.54),

$$C_3\cos\beta_1 = \pm\sqrt{C_3^2-1-z_*^2}, \tag{9.57}$$

we have

$$\frac{dz_*}{d\beta_2} = \mp\frac{1}{C_3}\left(1+z_*^2\right)\sqrt{C_3^2-1-z_*^2}. \tag{9.58}$$

Integrating the last relation, we arrive at the following quadrature:

$$\mp(\beta_2+C_4) = \int \frac{C_3 dz_*}{\left(1+z_*^2\right)\sqrt{C_3^2-1-z_*^2}}, \quad C_4 = \text{const}\,. \tag{9.59}$$

Integrating this relation we obtain

$$\mp\tan(\beta_2+C_4) = \frac{C_3 z_*}{\sqrt{C_3^2-1-z_*^2}}, \quad C_4 = \text{const}\,. \tag{9.60}$$

In the variables z_1 and z_2 the last invariant relation has the form

$$\mp\tan(\beta_2+C_4) = \frac{C_3 z_2}{\sqrt{\left(C_3^2-1\right)z_1^2-z_2^2}}, \quad C_4 = \text{const}\,. \tag{9.61}$$

Finally, we have the following form of the additional first integral that "attaches" Eq. (9.22):

$$\arctan\frac{C_3 z_*}{\sqrt{C_3^2-1-z_*^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const} \tag{9.62}$$

or

$$\arctan\frac{C_3 z_2}{\sqrt{\left(C_3^2-1\right)z_1^2-z_2^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const}\,. \tag{9.63}$$

Thus, in the case considered, the system of dynamical equations (8.3)–(8.6) and (8.9)–(8.14) under condition (9.1) has eight invariant relations: the nonintegrable analytic constraint of the form (8.19), the cyclic first integrals of the form (8.17), (8.18), the first integral of the form (9.31), the first integral expressed by relations (9.45)–(9.52), which is a transcendental function of the phase variables (in the sense of complex analysis) expressed through a finite combination of elementary functions, and, finally, the transcendental first integrals of the form (9.54) (or (9.55)) and (9.62) (or (9.63)).

Theorem 9.1. *System* (8.3)–(8.6), (8.9)–(8.14) *under conditions* (8.19), (9.1), (8.18) *possesses eight invariant relations (complete set), four of which are transcendental functions from the point of view of complex analysis. Moreover, all the relations are expressed through finite combinations of elementary functions.*

9.3. Topological analogies. Consider the following fifth-order system:

$$\begin{gathered}
\ddot{\xi} + b_*\dot{\xi}\cos\xi + \sin\xi\cos\xi - \left[\dot{\eta}_1{}^2 + \dot{\eta}_2{}^2\sin^2\eta_1\right]\frac{\sin\xi}{\cos\xi} = 0,\\
\ddot{\eta}_1 + b_*\dot{\eta}_1\cos\xi + \dot{\xi}\dot{\eta}_1\frac{1+\cos^2\xi}{\cos\xi\sin\xi} - \dot{\eta}_2{}^2\sin\eta_1\cos\eta_1 = 0,\\
\ddot{\eta}_2 + b_*\dot{\eta}_2\cos\xi + \dot{\xi}\dot{\eta}_2\frac{1+\cos^2\xi}{\cos\xi\sin\xi} + 2\dot{\eta}_1\dot{\eta}_2\frac{\cos\eta_1}{\cos\eta_1} = 0, \quad b_* > 0,
\end{gathered} \tag{9.64}$$

which describes a fixed four-dimensional pendulum in a flow of a running medium for which the moment of forces is independent of the angular velocity, i.e., a mechanical system in a nonconservative field (see [14, 15, 150]). In general, the order of such a system is equal to 6, but the phase variable η_2 is a cyclic variable, which leads to the stratification of the phase space and reduces the order of the system.

The phase space of this system is the tangent bundle

$$T\mathbf{S}^3\left\{\dot{\xi}, \dot{\eta}_1, \dot{\eta}_2, \xi, \eta_1, \eta_2\right\} \tag{9.65}$$

of the three-dimensional sphere $\mathbf{S}^3\{\xi, \eta_1, \eta_2\}$. The equation that transforms system (9.64) into the system on the tangent bundle of the two-dimensional sphere

$$\dot{\eta}_2 \equiv 0, \tag{9.66}$$

and the equations of great circles

$$\dot{\eta}_1 \equiv 0, \quad \dot{\eta}_2 \equiv 0 \tag{9.67}$$

define families of integral manifolds.

It is easy to verify that system (9.64) is equivalent to a dynamical system with variable dissipation with zero mean on the tangent bundle (9.65) of the three-dimensional sphere. Moreover, the following theorem holds.

Theorem 9.2. *System* (8.3)–(8.6), (8.9)–(8.14) *under conditions* (8.19), (9.1), *and* (8.18) *is equivalent to the dynamical system* (9.64).

Proof. Indeed it suffices to set $\alpha = \xi$, $\beta_1 = \eta_1$, $\beta_2 = \eta_2$, and $b = -b_*$. □

For more general topological analogies, see [91].

10. Case Where the Moment of a Nonconservative Force Depends on the Angular Velocity

10.1. Introduction of the dependence on the angular velocity. This chapter is devoted to the dynamics of a four-dimensional rigid body in four-dimensional space. Since the present section is devoted to the study of motion in the case where the moment of forces depends on the tensor of angular velocity, we introduce this dependence in a more general situation. This also allows us to introduce this dependence for multi-dimensional bodies.

Let $x = (x_{1N}, x_{2N}, x_{3N}, x_{4N})$ be the coordinates of the point N of application of a nonconservative force (influence of the medium) acting on the three-dimensional disk and let $Q = (Q_1, Q_2, Q_3, Q_4)$ be the components of the force $\mathbf{S}$ of the influence of the medium independent of the tensor of the angular velocity. We consider only the linear dependence of the functions $(x_{1N}, x_{2N}, x_{3N}, x_{4N})$ on the tensor of angular velocity since this introduction is not itself a priori obvious (see [14, 15]).

We adopt the following dependence:

$$x = Q + R, \tag{10.1}$$

where $R = (R_1, R_2, R_3, R_4)$ is a vector-valued function containing the components of the tensor of angular velocity. The dependence of the function R on the components of the tensor of angular velocity is gyroscopic:

$$R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = -\frac{1}{v} \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}, \tag{10.2}$$

where (h_1, h_2, h_3, h_4) are some positive parameters (cf. [14, 15, 91]).

Since $x_{1N} \equiv 0$, we have

$$x_{2N} = Q_2 - h_1 \frac{\omega_6}{v}, \quad x_{3N} = Q_3 + h_1 \frac{\omega_5}{v}, \quad x_{4N} = Q_4 - h_1 \frac{\omega_3}{v}. \tag{10.3}$$

10.2. Reduced system. As in the choice of the Chaplygin analytic functions (see [16, 17])

$$\begin{aligned} Q_2 &= A\sin\alpha\cos\beta_1,\\ Q_3 &= A\sin\alpha\sin\beta_1\cos\beta_2,\\ Q_4 &= A\sin\alpha\sin\beta_1\sin\beta_2, \quad A>0, \end{aligned} \tag{10.4}$$

we take the dynamical functions s, x_{2N}, x_{3N}, and x_{4N} in the following form:

$$\begin{aligned} s(\alpha) &= B\cos\alpha,\ B>0,\\ x_{2N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) &= A\sin\alpha\cos\beta_1 - h\frac{\omega_6}{v}, \quad h=h_1>0,\ v\neq 0,\\ x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) &= A\sin\alpha\sin\beta_1\cos\beta_2 + h\frac{\omega_5}{v}, \quad h=h_1>0,\ v\neq 0,\\ x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) &= A\sin\alpha\sin\beta_1\sin\beta_2 - h\frac{\omega_3}{v}, \quad h=h_1>0,\ v\neq 0. \end{aligned} \tag{10.5}$$

This shows that in the problem considered, there is an additional damping (but accelerating in certain domains of the phase space) moment of a nonconservative force (i.e., there is a dependence of the moment on the components of the tensor of angular velocity). Moreover, $h_2 = h_3 = h_4$ due to the dynamical symmetry of the body.

In this case, the functions $\Gamma_v(\alpha,\beta_1,\beta_2,\Omega/v)$, $\Delta_v(\alpha,\beta_1,\beta_2,\Omega/v)$, and $\Theta_v(\alpha,\beta_1,\beta_2,\Omega/v)$ in the system (8.39)–(8.44) have the following form:

$$\begin{aligned} \Gamma_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) &= A\sin\alpha - \frac{h}{v}z_3,\\ \Delta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) &= \frac{h}{v}z_2,\\ \Theta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) &= -\frac{h}{v}z_1. \end{aligned} \tag{10.6}$$

Then, due to the nonintegrable constraint (8.19), outside the manifold (8.38) the dynamical part of the equations of motion (system (8.39)–(8.44)) takes the form of the analytic system

$$\dot\alpha = -\left(1+\frac{\sigma Bh}{2I_2}\right)z_3 + \frac{\sigma ABv}{2I_2}\sin\alpha, \tag{10.7}$$

$$\dot z_3 = \frac{ABv^2}{2I_2}\sin\alpha\cos\alpha - \left(1+\frac{\sigma Bh}{2I_2}\right)(z_1^2+z_2^2)\frac{\cos\alpha}{\sin\alpha} - \frac{Bhv}{2I_2}z_3\cos\alpha, \tag{10.8}$$

$$\dot z_2 = \left(1+\frac{\sigma Bh}{2I_2}\right)z_2z_3\frac{\cos\alpha}{\sin\alpha} + \left(1+\frac{\sigma Bh}{2I_2}\right)z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - \frac{Bhv}{2I_2}z_2\cos\alpha, \tag{10.9}$$

$$\dot z_1 = \left(1+\frac{\sigma Bh}{2I_2}\right)z_1z_3\frac{\cos\alpha}{\sin\alpha} - \left(1+\frac{\sigma Bh}{2I_2}\right)z_1z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - \frac{Bhv}{2I_2}z_1\cos\alpha, \tag{10.10}$$

$$\dot\beta_1 = \left(1+\frac{\sigma Bh}{2I_2}\right)z_2\frac{\cos\alpha}{\sin\alpha}, \tag{10.11}$$

$$\dot\beta_2 = -\left(1+\frac{\sigma Bh}{2I_2}\right)z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{10.12}$$

Introducing dimensionless variables and parameters and a new differentiation as follows:

$$z_k \mapsto n_0 v z_k,\ k=1,2,3, \quad n_0^2 = \frac{AB}{2I_2}, \quad b=\sigma n_0, \quad H_1 = \frac{Bh}{2I_2 n_0}, \quad \langle\dot{\ }\rangle = n_0 v\langle\ '\rangle, \tag{10.13}$$

we reduce system (10.7)–(10.12) to the form

$$\dot{\alpha} = -\left(1 + bH_1\right) z_3 + b \sin\alpha, \tag{10.14}$$

$$\dot{z}_3 = \sin\alpha\cos\alpha - \left(1 + bH_1\right)\left(z_1^2 + z_2^2\right)\frac{\cos\alpha}{\sin\alpha} - H_1 z_3 \cos\alpha, \tag{10.15}$$

$$\dot{z}_2 = \left(1 + bH_1\right) z_2 z_3 \frac{\cos\alpha}{\sin\alpha} + \left(1 + bH_1\right) z_1^2 \frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - H_1 z_2 \cos\alpha, \tag{10.16}$$

$$\dot{z}_1 = \left(1 + bH_1\right) z_1 z_3 \frac{\cos\alpha}{\sin\alpha} - \left(1 + bH_1\right) z_1 z_2 \frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} - H_1 z_1 \cos\alpha, \tag{10.17}$$

$$\dot{\beta}_1 = \left(1 + bH_1\right) z_2 \frac{\cos\alpha}{\sin\alpha}, \tag{10.18}$$

$$\dot{\beta}_2 = -\left(1 + bH_1\right) z_1 \frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{10.19}$$

We see that the sixth-order system (10.14)–(10.19) (which can be considered on the tangent bundle $T\mathbf{S}^3$ of the three-dimensional sphere $\mathbf{S}^3$), contains an independent fifth-order system (10.14)–(10.18) on its own five-dimensional manifold.

For complete integration of system (10.14)–(10.19), we need, in general, five independent first integrals. However, after the change of variables

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to \begin{pmatrix} z \\ z_* \end{pmatrix}, \quad z = \sqrt{z_1^2 + z_2^2}, \quad z_* = z_2/z_1, \tag{10.20}$$

system (10.14)–(10.19) splits as follows:

$$\alpha' = -(1 + bH_1) z_3 + b \sin\alpha, \tag{10.21}$$

$$z_3' = \sin\alpha\cos\alpha - (1 + bH_1) z^2 \frac{\cos\alpha}{\sin\alpha} - H_1 z_3 \cos\alpha, \tag{10.22}$$

$$z' = (1 + bH_1) z z_3 \frac{\cos\alpha}{\sin\alpha} - H_1 z \cos\alpha, \tag{10.23}$$

$$z_*' = (\pm)(1 + bH_1) z \sqrt{1 + z_*^2}\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{10.24}$$

$$\beta_1' = (\pm)(1 + bH_1)\frac{z z_*}{\sqrt{1 + z_*^2}}\frac{\cos\alpha}{\sin\alpha}, \tag{10.25}$$

$$\beta_2' = (\mp)(1 + bH_1)\frac{z}{\sqrt{1 + z_*^2}}\frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{10.26}$$

We see that the sixth-order system splits into independent subsystems of lower orders: system (10.21)–(10.23) of order 3 and system (10.24), (10.25) (certainly, after a choice of the independent variables) of order 2. Thus, to integrate system (10.21)–(10.26) completely, it suffices to find two independent first integrals of system (10.21)–(10.23), one first integral of system (10.24), (10.25), and an additional first integral that "attaches" Eq. (10.26).

Note that system (10.21)–(10.23) can be considered on the tangent bundle $T\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$.

10.3. Complete list of invariant relation. System (10.21)–(10.23) has the form of a system of equations that appears in the dynamics of a three-dimensional (3D) rigid body in a nonconservative field.

First, we establish a correspondence between the third-order system (10.21)–(10.23) and the nonautonomous second-order system

$$\begin{aligned} \frac{dz_3}{d\alpha} &= \frac{\sin\alpha\cos\alpha - (1+bH_1)z^2\cos\alpha/\sin\alpha - H_1 z_3\cos\alpha}{-(1+bH_1)z_3 + b\sin\alpha}, \\ \frac{dz}{d\alpha} &= \frac{(1+bH_1)zz_3\cos\alpha/\sin\alpha - H_1 z\cos\alpha}{-(1+bH_1)z_3 + b\sin\alpha}. \end{aligned} \tag{10.27}$$

Using the substitution $\tau = \sin\alpha$, we rewrite system (10.27) in the algebraic form:

$$\begin{aligned} \frac{dz_3}{d\tau} &= \frac{\tau - (1+bH_1)z^2/\tau - H_1 z_3}{-(1+bH_1)z_3 + b\tau}, \\ \frac{dz}{d\tau} &= \frac{(1+bH_1)zz_3/\tau - H_1 z}{-(1+bH_1)z_3 + b\tau}. \end{aligned} \tag{10.28}$$

Further, introducing homogeneous variables by the formulas

$$z = u_1\tau, \quad z_3 = u_2\tau, \tag{10.29}$$

we reduce system (10.28) to the following form:

$$\begin{aligned} \tau\frac{du_2}{d\tau} + u_2 &= \frac{1 - (1+bH_1)u_1^2 - H_1 u_2}{-(1+bH_1)u_2 + b}, \\ \tau\frac{du_1}{d\tau} + u_1 &= \frac{(1+bH_1)u_1u_2 - H_1 u_1}{-(1+bH_1)u_2 + b}, \end{aligned} \tag{10.30}$$

which is equivalent to

$$\begin{aligned} \tau\frac{du_2}{d\tau} &= \frac{(1+bH_1)\left(u_2^2 - u_1^2\right) - (b+H_1)u_2 + 1}{-(1+bH_1)u_2 + b}, \\ \tau\frac{du_1}{d\tau} &= \frac{2(1+bH_1)u_1u_2 - (b+H_1)u_1}{-(1+bH_1)u_2 + b}. \end{aligned} \tag{10.31}$$

We establish a correspondence between the second-order system (10.31) and the nonautonomous first-order equation

$$\frac{du_2}{du_1} = \frac{1 - (1+bH_1)\left(u_1^2 - u_2^2\right) - (b+H_1)u_2}{2(1+bH_1)u_1u_2 - (b+H_1)u_1}, \tag{10.32}$$

which can be easily reduced to exact-differential form:

$$d\left(\frac{(1+bH_1)\left(u_2^2 + u_1^2\right) - (b+H_1)u_2 + 1}{u_1}\right) = 0. \tag{10.33}$$

Thus, Eq. (10.32) has the following first integral:

$$\frac{(1+bH_1)\left(u_2^2 + u_1^2\right) - (b+H_1)u_2 + 1}{u_1} = C_1 = \text{const}, \tag{10.34}$$

which in the original variables has the form

$$\frac{(1+bH_1)\left(z_3^2 + z^2\right) - (b+H_1)z_3\sin\alpha + \sin^2\alpha}{z\sin\alpha} = C_1 = \text{const}\,. \tag{10.35}$$

Remark 10.1. Consider system (10.21)–(10.23) with variable dissipation with zero mean (see [91]), which becomes conservative for $b = H_1$:

$$
\begin{aligned}
\alpha' &= -\left(1+b^2\right) z_3 + b\sin\alpha,\\
z_3' &= \sin\alpha\cos\alpha - \left(1+b^2\right) z^2 \frac{\cos\alpha}{\sin\alpha} - bz_3\cos\alpha,\\
z' &= \left(1+b^2\right) zz_3 \frac{\cos\alpha}{\sin\alpha} - bz\cos\alpha.
\end{aligned}
\tag{10.36}
$$

It possesses the following two analytic first integrals:

$$\left(1+b^2\right)\left(z_3^2+z^2\right) - 2bz_3\sin\alpha + \sin^2\alpha = C_1^* = \text{const}, \tag{10.37}$$

$$z\sin\alpha = C_2^* = \text{const}\,. \tag{10.38}$$

Obviously, the ratio of the two first integrals (10.37) and (10.38) is also a first integral of system (10.36). However, for $b \neq H_1$, none of the functions

$$(1+bH_1)\left(z_3^2+z^2\right) - (b+H_1)z_3\sin\alpha + \sin^2\alpha \tag{10.39}$$

and (10.38) is a first integral of system (10.21)–(10.23), but their ratio is a first integral of system (10.21)–(10.23) for any b and H_1.

We find the explicit form of the additional first integral of the third-order system (10.21)–(10.23). First, we transform the invariant relation (10.34) for $u_1 \neq 0$ as follows:

$$\left(u_2 - \frac{b+H_1}{2(1+bH_1)}\right)^2 + \left(u_1 - \frac{C_1}{2(1+bH_1)}\right)^2 = \frac{(b-H_1)^2 + C_1^2 - 4}{4(1+bH_1)^2}. \tag{10.40}$$

We see that the parameters of this invariant relation must satisfy the condition

$$(b-H_1)^2 + C_1^2 - 4 \geq 0, \tag{10.41}$$

and the phase space of system (10.21)–(10.23) is stratified into the family of surfaces defined by Eq. (10.40).

Thus, due to relation (10.34), the first equation of system (10.31) has the form

$$\tau\frac{du_2}{d\tau} = \frac{2(1+bH_1)u_2^2 - 2(b+H_1)u_2 + 2 - C_1U_1(C_1,u_2)}{b-(1+bH_1)u_2}, \tag{10.42}$$

where

$$
\begin{aligned}
U_1(C_1,u_2) &= \frac{1}{2(1+bH_1)}\{C_1 \pm U_2(C_1,u_2)\},\\
U_2(C_1,u_2) &= \sqrt{C_1^2 - 4(1+bH_1)\left(1-(b+H_1)u_2+(1+bH_1)u_2^2\right)},
\end{aligned}
$$

and the integration constant C_1 is defined by condition (10.41).

Therefore, the quadrature for the search for an additional first integral of system (10.21)–(10.23) becomes

$$\int\frac{d\tau}{\tau} = \int\frac{(b-(1+bH_1)u_2)du_2}{2(1-(b+H_1)u_2+(1+bH_1)u_2^2) - C_1\{C_1 \pm U_2(C_1,u_2)\}/(2(1+bH_1))}. \tag{10.43}$$

Obviously, the left-hand side (up to an additive constant) is equal to

$$\ln|\sin\alpha|. \tag{10.44}$$

If

$$u_2 - \frac{b+H_1}{2(1+bH_1)} = w_1, \quad b_1^2 = (b-H_1)^2 + C_1^2 - 4, \tag{10.45}$$

then the right-hand side of Eq. (10.43) becomes

$$
-\frac{1}{4}\int \frac{d\left(b_1^2-4(1+bH_1)w_1^2\right)}{\left(b_1^2-4(1+bH_1)w_1^2\right)\pm C_1\sqrt{b_1^2-4(1+bH_1)w_1^2}}
$$

$$
-(b-H_1)(1+bH_1)\int \frac{dw_1}{\left(b_1^2-4(1+bH_1)w_1^2\right)\pm C_1\sqrt{b_1^2-4(1+bH_1)w_1^2}}
$$

$$
=-\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2-4(1+bH_1)w_1^2}}{C_1}\pm 1\right|\pm\frac{b-H_1}{2}I_1, \quad (10.46)
$$

where

$$
I_1=\int\frac{dw_3}{\sqrt{b_1^2-w_3^2}(w_3\pm C_1)}, \quad w_3=\sqrt{b_1^2-4(1+bH_1)w_1^2}. \quad (10.47)
$$

In the calculation of the integral (10.47), the following three cases are possible:

I. $|b-H_1|>2$:

$$
I_1=-\frac{1}{2\sqrt{(b-H_1)^2-4}}\ln\left|\frac{\sqrt{(b-H_1)^2-4}+\sqrt{b_1^2-w_3^2}}{w_3\pm C_1}\pm\frac{C_1}{\sqrt{(b-H_1)^2-4}}\right|
$$

$$
+\frac{1}{2\sqrt{(b-H_1)^2-4}}\ln\left|\frac{\sqrt{(b-H_1)^2-4}-\sqrt{b_1^2-w_3^2}}{w_3\pm C_1}\mp\frac{C_1}{\sqrt{(b-H_1)^2-4}}\right|+\text{const}; \quad (10.48)
$$

II. $|b-H_1|<2$:

$$
I_1=\frac{1}{\sqrt{4-(b-H_1)^2}}\arcsin\frac{\pm C_1w_3+b_1^2}{b_1(w_3\pm C_1)}+\text{const}; \quad (10.49)
$$

III. $|b-H_1|=2$:

$$
I_1=\mp\frac{\sqrt{b_1^2-w_3^2}}{C_1(w_3\pm C_1)}+\text{const}\,. \quad (10.50)
$$

Returning to the variable

$$
w_1=\frac{z_2}{\sin\alpha}-\frac{b+H_1}{2(1+bH_1)}, \quad (10.51)
$$

we have the following final form of I_1:

I. $|b-H_1|>2$:

$$
I_1=-\frac{1}{2\sqrt{(b-H_1)^2-4}}\ln\left|\frac{\sqrt{(b-H_1)^2-4}\pm 2(1+bH_1)w_1}{\sqrt{b_1^2-4(1+bH_1)^2w_1^2}\pm C_1}\pm\frac{C_1}{\sqrt{(b-H_1)^2-4}}\right|
$$

$$
+\frac{1}{2\sqrt{(b-H_1)^2-4}}\ln\left|\frac{\sqrt{(b-H_1)^2-4}\mp 2(1+bH_1)w_1}{\sqrt{b_1^2-4(1+bH_1)^2w_1^2}\pm C_1}\mp\frac{C_1}{\sqrt{(b-H_1)^2-4}}\right|+\text{const}; \quad (10.52)
$$

II. $|b-H_1|<2$:

$$
I_1=\frac{1}{\sqrt{4-(b-H_1)^2}}\arcsin\frac{\pm C_1\sqrt{b_1^2-4(1+bH_1)^2w_1^2}+b_1^2}{b_1\left(\sqrt{b_1^2-4(1+bH_1)^2w_1^2}\pm C_1\right)}+\text{const}; \quad (10.53)
$$

III. $|b-H_1|=2$:

$$
I_1=\mp\frac{2(1+bH_1)w_1}{C_1\left(\sqrt{b_1^2-4(1+bH_1)^2w_1^2}\pm C_1\right)}+\text{const}\,. \quad (10.54)
$$

Thus, we have found an additional first integral for the third-order system (10.21)–(10.23) and we have the complete set of first integrals that are transcendental functions of their phase variables.

Remark 10.2. Formally, in the expression of the first integral found, we must substitute for C_1 the left-hand side of the first integral (10.34).

Then the obtained additional first integral has the following structure (similar to the transcendental first integral from planar dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha, \frac{z_3}{\sin\alpha}, \frac{z}{\sin\alpha}\right) = C_2 = \text{const}\,. \tag{10.55}$$

Thus, to integrate the sixth-order system (10.21)–(10.26), we have already found two independent first integrals. As was mentioned above, to integrate it completely, it suffices to find one first integral for the (potentially separated) system (10.24), (10.25) and an additional first integral that "attaches" Eq. (10.26).

To find a first integral of the (potentially separated) system (10.24), (10.25), we establish a correspondence between it and the following nonautonomous first-order equation:

$$\frac{dz_*}{d\beta_1} = \frac{1+z_*^2}{z_*}\frac{\cos\beta_1}{\sin\beta_1}. \tag{10.56}$$

After integration we obtain the required invariant relation

$$\frac{\sqrt{1+z_*^2}}{\sin\beta_1} = C_3 = \text{const}, \tag{10.57}$$

which in terms of the variables z_1 and z_2 has the form

$$\frac{\sqrt{z_1^2+z_2^2}}{z_1\sin\beta_1} = C_3 = \text{const}\,. \tag{10.58}$$

Further, to obtain an additional first integral that "attaches" Eq. (10.26), we establish a correspondence between Eqs. (10.26) and (10.24) and the following nonautonomous equation:

$$\frac{dz_*}{d\beta_2} = -\left(1+z_*^2\right)\cos\beta_1. \tag{10.59}$$

Since

$$C_3\cos\beta_1 = \pm\sqrt{C_3^2-1-z_*^2} \tag{10.60}$$

by (10.57), we have

$$\frac{dz_*}{d\beta_2} = \mp\frac{1}{C_3}\left(1+z_*^2\right)\sqrt{C_3^2-1-z_*^2}. \tag{10.61}$$

Integrating this relation, we arrive at the following quadrature:

$$\mp(\beta_2+C_4) = \int\frac{C_3dz_*}{(1+z_*^2)\sqrt{C_3^2-1-z_*^2}}, \quad C_4 = \text{const}\,. \tag{10.62}$$

Integration leads to the relation

$$\mp\tan(\beta_2+C_4) = \frac{C_3z_*}{\sqrt{C_3^2-1-z_*^2}}, \quad C_4 = \text{const}\,. \tag{10.63}$$

Expressed in terms of the variables z_1 and z_2 this invariant relation has the form

$$\mp\tan(\beta_2+C_4) = \frac{C_3z_2}{\sqrt{\left(C_3^2-1\right)z_1^2-z_2^2}}, \quad C_4 = \text{const}\,. \tag{10.64}$$

Finally, we have the following additional first integral that "attaches" Eq. (10.26):

$$\arctan\frac{C_3z_*}{\sqrt{C_3^2-1-z_*^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const} \tag{10.65}$$

or

$$\arctan \frac{C_3 z_2}{\sqrt{\left(C_3^2 - 1\right) z_1^2 - z_2^2}} \pm \beta_2 = C_4, \quad C_4 = \mathrm{const}\,. \tag{10.66}$$

Thus, in the case considered, the system of dynamical equations (8.3)–(8.6), (8.9)–(8.14) under condition (10.5) has eight invariant relations: the analytic nonintegrable constraint of the form (8.19), the cyclic first integrals of the form (8.17) and (8.18), the first integral of the form (10.35), the first integral expressed by relations (10.48)–(10.55), which is a transcendental function of the phase variables (in the sense of complex analysis) expressed through a finite combination of functions, and the transcendental first integrals of the form (10.57) (or (10.58)) and (10.65) (or (10.66)).

Theorem 10.1. *System* (8.3)–(8.6), (8.9)–(8.14) *under conditions* (8.19), (10.5), *and* (8.18) *possesses eight invariant relations* (*complete set*); *four of them are transcendental functions from the point of view of complex analysis. All the relations are expressed through finite combinations of elementary functions.*

10.4. Topological analogies. Consider the following fifth-order system:

$$\begin{gathered}
\ddot{\xi} + (b_* - H_{1*})\dot{\xi}\cos\xi + \sin\xi\cos\xi - \left[\dot{\eta}_1{}^2 + \dot{\eta}_2{}^2 \sin^2\eta_1\right] \frac{\sin\xi}{\cos\xi} = 0, \\
\ddot{\eta}_1 + (b_* - H_{1*})\dot{\eta}_1 \cos\xi + \dot{\xi}\dot{\eta}_1 \frac{1 + \cos^2\xi}{\cos\xi \sin\xi} - \dot{\eta}_2{}^2 \sin\eta_1 \cos\eta_1 = 0, \\
\ddot{\eta}_2 + (b_* - H_{1*})\dot{\eta}_2 \cos\xi + \dot{\xi}\dot{\eta}_2 \frac{1 + \cos^2\xi}{\cos\xi \sin\xi} + 2\dot{\eta}_1\dot{\eta}_2 \frac{\cos\eta_1}{\cos\eta_1} = 0,
\end{gathered} \tag{10.67}$$

where $b_* > 0$ and $H_{1*} > 0$. This system describes a fixed four-dimensional pendulum in a flow of a running medium for which the moment of forces depends on the angular velocity, i.e., a mechanical system in a nonconservative field (see [14, 15, 150]). Generally speaking, the order of this system must be equal to 6, but the phase variable η_2 is a cyclic variable, which leads to the stratification of the phase space and reduces the order of the system.

The phase space of this system is the tangent bundle

$$T\mathbf{S}^3 \left\{\dot{\xi}, \dot{\eta}_1, \dot{\eta}_2, \xi, \eta_1, \eta_2\right\} \tag{10.68}$$

of the three-dimensional sphere $\mathbf{S}^3\{\xi, \eta_1, \eta_2\}$. The equation that transforms system (9.64) into a system on the tangent bundle of the two-dimensional sphere

$$\dot{\eta}_2 \equiv 0 \tag{10.69}$$

and the equations of great circles

$$\dot{\eta}_1 \equiv 0, \quad \dot{\eta}_2 \equiv 0 \tag{10.70}$$

define families of integral manifolds.

It is easy to verify that system (10.67) is equivalent to a dynamical system with variable dissipation with zero mean on the tangent bundle (10.68) of the three-dimensional sphere. Moreover, the following theorem holds.

Theorem 10.2. *System* (8.3)–(8.6), (8.9)–(8.14) *under conditions* (8.19), (10.5), *and* (8.18) *is equivalent to the dynamical system* (10.67).

Proof. Indeed, it suffices to set $\alpha = \xi$, $\beta_1 = \eta_1$, $\beta_2 = \eta_2$, $b = -b_*$, and $H_1 = -H_{1*}$. □

On more general topological analogies, see [91].

CHAPTER 3

CASES OF INTEGRABILITY CORRESPONDING TO THE MOTION OF A RIGID BODY IN FOUR-DIMENSIONAL SPACE, II

In this chapter, we systematize results, both new results and results obtained earlier, concerning the study of the equations of motion of an axisymmetric four-dimensional (4D) rigid body in a field of nonconservative forces. These equations are taken from the dynamics of realistic rigid bodies of lesser dimension that interact with a resisting medium by laws of jet flow when the body is subjected to a nonconservative tracing force such that throughout the motion the center of mass of the body moves rectilinearly and uniformly; this means that in the system there exists a nonconservative couple of forces (see [5, 31, 36, 46, 53, 71, 77, 81, 88, 139, 152]).

Earlier, in [42, 81] the author proved the complete integrability of the equations of plane-parallel motion of a body in a resisting medium under the conditions of jet flow in the case where the system of dynamical equations possesses a first integral which is a transcendental (in the sense of the theory of functions of a complex variable) function of quasi-velocities. It was assumed that the interaction of the body with the medium is concentrated on a part of the surface of the body that has the form of a (one-dimensional) plate.

Subsequently (see [76, 77, 95]), the planar problem was generalized to the spatial (three-dimensional) case, where the system of dynamical equations possesses a complete set of transcendental first integrals. In this case, it was assumed that the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a planar (two-dimensional) disk.

In this chapter, we discuss results, both new results and results obtained earlier, concerning the case where the interaction of the medium with the body is concentrated on a part of the surface of the body that has the form of a three-dimensional disk and the force acts in the direction perpendicular to the disk. We systematize these results and formulate them in invariant form. We also introduce an additional dependence of the moment of a nonconservative force on the angular velocity; this dependence can be generalized to motion in higher-dimensional spaces.

11. General Problem of Motion Under a Tracing Force

Consider the motion of a homogeneous, dynamically symmetric (case (7.1)), rigid body with "front end face" (a three-dimensional disk interacting with a medium that fills four-dimensional space) in the field of a resistance force $\mathbf{S}$ under quasi-stationarity conditions (see [16, 17, 30, 35, 36, 42, 43, 89, 108, 126, 145, 152].

Let $(v, \alpha, \beta_1, \beta_2)$ be the (generalized) spherical coordinates of the velocity vector of the center D of the three-dimensional disk lying on the axis of symmetry of the body, let

$$\Omega = \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

be the tensor of angular velocity of the body, and let $Dx_1x_2x_3x_4$ be the coordinate system attached to the body such that the axis of symmetry CD coincides with the axis Dx_1 (recall that C is the center of mass), the axes Dx_2, Dx_3, and Dx_4 lie in the hyperplane of the disk, while I_1, I_2, $I_3 = I_2$, $I_4 = I_2$, and m are the characteristics of inertia and mass.

We adopt the following expansions in projections onto the axes of the coordinate system $Dx_1x_2x_3x_4$:

$$\begin{aligned} \mathbf{DC} &= \{-\sigma, 0, 0, 0\}, \\ \mathbf{v}_D &= \Big\{v\cos\alpha,\ v\sin\alpha\cos\beta_1,\ v\sin\alpha\sin\beta_1\cos\beta_2,\ v\sin\alpha\sin\beta_1\sin\beta_2\Big\}. \end{aligned} \tag{11.1}$$

In the case (7.1) we additionally have an expansion for the function of the influence of the medium on the four-dimensional body:

$$\mathbf{S} = \{-S, 0, 0, 0\} \tag{11.2}$$

i.e., in this case $\mathbf{F} = \mathbf{S}$.

Then the set of dynamical equations of motion of the body (including the Chaplygin analytic functions, [16, 17], see below) that describes the motion of the center of mass and corresponds to the space $\mathbb{R}^4$, in which the tangent forces of the influence of the medium on the three-dimensional disk vanish, takes the form

$$\begin{aligned} &\dot v\cos\alpha - \dot\alpha v\sin\alpha - \omega_6 v\sin\alpha\cos\beta_1 + \omega_5 v\sin\alpha\sin\beta_1\cos\beta_2 - \omega_3 v\sin\alpha\sin\beta_1\sin\beta_2 \\ &\qquad + \sigma\left(\omega_6^2 + \omega_5^2 + \omega_3^2\right) = -\frac{S}{m}, \end{aligned} \tag{11.3}$$

$$\begin{aligned} &\dot v\sin\alpha\cos\beta_1 + \dot\alpha v\cos\alpha\cos\beta_1 - \dot\beta_1 v\sin\alpha\sin\beta_1 + \omega_6 v\cos\alpha - \omega_4 v\sin\alpha\sin\beta_1\cos\beta_2 \\ &\qquad + \omega_2 v\sin\alpha\sin\beta_1\sin\beta_2 - \sigma(\omega_4\omega_5 + \omega_2\omega_3) - \sigma\dot\omega_6 = 0, \end{aligned} \tag{11.4}$$

$$\begin{aligned} &\dot v\sin\alpha\sin\beta_1\cos\beta_2 + \dot\alpha v\cos\alpha\sin\beta_1\cos\beta_2 + \dot\beta_1 v\sin\alpha\cos\beta_1\cos\beta_2 - \dot\beta_2 v\sin\alpha\sin\beta_1\sin\beta_2 \\ &\qquad - \omega_5 v\cos\alpha + \omega_4 v\sin\alpha\cos\beta_1 - \omega_1 v\sin\alpha\sin\beta_1\sin\beta_2 - \sigma(-\omega_1\omega_2 + \omega_4\omega_6) + \sigma\dot\omega_5 = 0, \end{aligned} \tag{11.5}$$

$$\begin{aligned} &\dot v\sin\alpha\sin\beta_1\sin\beta_2 + \dot\alpha v\cos\alpha\sin\beta_1\sin\beta_2 + \dot\beta_1 v\sin\alpha\cos\beta_1\sin\beta_2 + \dot\beta_2 v\sin\alpha\sin\beta_1\cos\beta_2 \\ &\qquad + \omega_3 v\cos\alpha - \omega_2 v\sin\alpha\cos\beta_1 + \omega_1 v\sin\alpha\sin\beta_1\cos\beta_2 + \sigma(\omega_2\omega_6 + \omega_1\omega_5) - \sigma\dot\omega_3 = 0, \end{aligned} \tag{11.6}$$

where

$$S = s(\alpha)v^2, \quad \sigma = CD, \quad v > 0. \tag{11.7}$$

Further, the auxiliary matrix (7.11) for the calculation of the moment of the resistance force takes the form

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} & x_{4N} \\ -S & 0 & 0 & 0 \end{pmatrix}; \tag{11.8}$$

then the set of dynamical equations that describes the motion of the body about the center of mass and corresponds to the Lie algebra so(4) takes the form

$$(\lambda_4 + \lambda_3)\dot\omega_1 + (\lambda_3 - \lambda_4)(\omega_3\omega_5 + \omega_2\omega_4) = 0, \tag{11.9}$$

$$(\lambda_2 + \lambda_4)\dot\omega_2 + (\lambda_2 - \lambda_4)(\omega_3\omega_6 - \omega_1\omega_4) = 0, \tag{11.10}$$

$$(\lambda_4 + \lambda_1)\dot\omega_3 + (\lambda_4 - \lambda_1)(\omega_2\omega_6 + \omega_1\omega_5) = x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2, \tag{11.11}$$

$$(\lambda_3 + \lambda_2)\dot\omega_4 + (\lambda_2 - \lambda_3)(\omega_5\omega_6 + \omega_1\omega_2) = 0, \tag{11.12}$$

$$(\lambda_1 + \lambda_3)\dot\omega_5 + (\lambda_3 - \lambda_1)(\omega_4\omega_6 - \omega_1\omega_3) = -x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2, \tag{11.13}$$

$$(\lambda_1 + \lambda_2)\dot\omega_6 + (\lambda_1 - \lambda_2)(\omega_4\omega_5 + \omega_2\omega_3) = x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) s(\alpha)v^2. \tag{11.14}$$

Thus, the phase space of the tenth-order system (11.3)–(11.6), (11.9)–(11.14) is the direct product of the four-dimensional manifold and the Lie algebra so(4):

$$\mathbb{R}^1 \times \mathbf{S}^3 \times \mathrm{so}(4). \tag{11.15}$$

Note that system (11.3)–(11.6), (11.9)–(11.14), due to the existing dynamical symmetry

$$I_2 = I_3 = I_4, \tag{11.16}$$

possesses the cyclic first integrals

$$\omega_1 \equiv \omega_1^0 = \mathrm{const}, \quad \omega_2 \equiv \omega_2^0 = \mathrm{const}, \quad \omega_4 \equiv \omega_4^0 = \mathrm{const}\,. \tag{11.17}$$

Henceforth, we will consider the dynamics of the system on zero levels:

$$\omega_1^0 = \omega_2^0 = \omega_4^0 = 0. \tag{11.18}$$

If we consider a more general problem on the motion of a body under a tracing force $\mathbf{T}$ lying on the straight line $CD = Dx_1$ that assumes that throughout the motion the condition

$$\mathbf{V}_C \equiv \mathrm{const} \tag{11.19}$$

(here $\mathbf{V}_C$ is the velocity of the center of mass, see also [91]) is satisfied, then system (11.3)–(11.6), (11.9)–(11.14) contains zero instead of F_x, since a nonconservative couple of forces acts on the body:

$$T - s(\alpha)v^2 \equiv 0, \quad \sigma = DC. \tag{11.20}$$

For this purpose, obviously, we must select the value of the tracing force T in the form

$$T = T_v(\alpha, \Omega) = s(\alpha)v^2, \quad \mathbf{T} \equiv -\mathbf{S}. \tag{11.21}$$

The case (11.21) of the choice of the value T of the tracing force is a particular case of the separation of an independent fifth-order subsystem after a certain transformation of the sixth-order system (11.3)–(11.6), (11.9)–(11.14).

Indeed, let the following condition for T hold:

$$T = T_v(\alpha, \beta_1, \beta_2, \Omega) = \sum_{\substack{i,j=0,\\ i\le j}}^{4} \tau_{i,j}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\Omega_i\Omega_j = T_1\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)v^2, \quad \Omega_0 = v. \tag{11.22}$$

We introduce new quasi-velocities into the system. For this purpose, we transform ω_3, ω_5, and ω_6 by a composition of two rotations:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \mathbf{T}_1(-\beta_1) \circ \mathbf{T}_3(-\beta_2) \begin{pmatrix} \omega_3 \\ \omega_5 \\ \omega_6 \end{pmatrix}, \tag{11.23}$$

where

$$\mathbf{T}_1(\beta_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta_1 & -\sin\beta_1 \\ 0 & \sin\beta_1 & \cos\beta_1 \end{pmatrix}, \quad \mathbf{T}_3(\beta_2) = \begin{pmatrix} \cos\beta_2 & -\sin\beta_2 & 0 \\ \sin\beta_2 & \cos\beta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{11.24}$$

Thus, the following relations hold:

$$\begin{aligned} z_1 &= \omega_3\cos\beta_2 + \omega_5\sin\beta_2, \\ z_2 &= -\omega_3\cos\beta_1\sin\beta_2 + \omega_5\cos\beta_1\cos\beta_2 + \omega_6\sin\beta_1, \\ z_3 &= \omega_3\sin\beta_1\sin\beta_2 - \omega_5\sin\beta_1\cos\beta_2 + \omega_6\cos\beta_1. \end{aligned} \tag{11.25}$$

System (11.3)–(11.6), (11.9)–(11.14) in the cases (11.16)–(11.18) and (11.22) can be rewritten in the form

$$\dot{v} + \sigma\left(z_1^2 + z_2^2 + z_3^2\right)\cos\alpha - \sigma\frac{v^2}{2I_2}s(\alpha)\sin\alpha\cdot\Gamma_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = \frac{T_1\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)v^2 - s(\alpha)v^2}{m}\cos\alpha, \tag{11.26}$$

$$\dot{\alpha}v + z_3 v - \sigma\left(z_1^2 + z_2^2 + z_3^2\right)\sin\alpha - \sigma\frac{v^2}{2I_2}s(\alpha)\cos\alpha\cdot\Gamma_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = \frac{s(\alpha)v^2 - T_1\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)v^2}{m}\sin\alpha, \tag{11.27}$$

$$\dot{\beta}_1\sin\alpha - z_2\cos\alpha - \frac{\sigma v}{2I_2}s(\alpha)\cdot\Delta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = 0, \tag{11.28}$$

$$\dot{\beta}_2\sin\alpha\sin\beta_1 + z_1\cos\alpha - \frac{\sigma v}{2I_2}s(\alpha)\cdot\Theta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = 0, \tag{11.29}$$

$$\dot{\omega}_3 = \frac{v^2}{2I_2}x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha), \tag{11.30}$$

$$\dot{\omega}_5 = -\frac{v^2}{2I_2}x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha), \tag{11.31}$$

$$\dot{\omega}_6 = \frac{v^2}{2I_2}x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha). \tag{11.32}$$

Introducing new dimensionless phase variables and a new differentiation by the formulas

$$z_k = n_1 v Z_k,\ k = 1, 2, 3, \quad \langle\dot{}\rangle = n_1 v\langle\,'\rangle,\ n_1 > 0,\ n_1 = \text{const}, \tag{11.33}$$

we reduce system (11.26)–(11.32) to the following form:

$$v' = v\Psi(\alpha, \beta_1, \beta_2, Z), \tag{11.34}$$

$$\alpha' = -Z_3 + \sigma n_1\left(Z_1^2 + Z_2^2 + Z_3^2\right)\sin\alpha + \frac{\sigma}{2I_2 n_1}s(\alpha)\cos\alpha\cdot\Gamma_v\left(\alpha, \beta_1, \beta_2, n_1 Z\right) - \frac{T_1\left(\alpha, \beta_1, \beta_2, n_1 Z\right) - s(\alpha)}{m n_1}\sin\alpha, \tag{11.35}$$

$$Z_3' = \frac{s(\alpha)}{2I_2 n_1^2}\cdot\Gamma_v\left(\alpha, \beta_1, \beta_2, n_1 Z\right) - \left(Z_1^2 + Z_2^2\right)\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma}{2I_2 n_1}Z_2\frac{s(\alpha)}{\sin\alpha}\cdot\Delta_v\left(\alpha, \beta_1, \beta_2, n_1 Z\right) + \frac{\sigma}{2I_2 n_1}Z_1\frac{s(\alpha)}{\sin\alpha}\cdot\Theta_v\left(\alpha, \beta_1, \beta_2, n_1 Z\right) - Z_3\cdot\Psi\left(\alpha, \beta_1, \beta_2, Z\right), \tag{11.36}$$

$$\begin{aligned}Z_2' = &-\frac{s(\alpha)}{2I_2 n_1^2}\cdot\Delta_v\left(\alpha,\beta_1,\beta_2,n_1 Z\right) + Z_2 Z_3\frac{\cos\alpha}{\sin\alpha} + Z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}\\ &+\frac{\sigma}{2I_2 n_1}Z_3\frac{s(\alpha)}{\sin\alpha}\cdot\Delta_v\left(\alpha,\beta_1,\beta_2,n_1 Z\right) - \frac{\sigma}{2I_2 n_1}Z_1\frac{s(\alpha)}{\sin\alpha}\cdot\Theta_v\left(\alpha,\beta_1,\beta_2,n_1 Z\right)\\ &- Z_2\cdot\Psi\left(\alpha,\beta_1,\beta_2,Z\right),\end{aligned}\tag{11.37}$$

$$\begin{aligned}Z_1' = &\frac{s(\alpha)}{2I_2 n_1^2}\cdot\Theta_v\left(\alpha,\beta_1,\beta_2,n_1 Z\right) + Z_1 Z_3\frac{\cos\alpha}{\sin\alpha} - Z_1 Z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}\\ &-\frac{\sigma}{2I_2 n_1}\frac{s(\alpha)}{\sin\alpha\sin\beta_1}\cdot\Theta_v\left(\alpha,\beta_1,\beta_2,n_1 Z\right)\cdot\left[Z_3\sin\beta_1 - Z_2\cos\beta_1\right]\\ &- Z_1\cdot\Psi\left(\alpha,\beta_1,\beta_2,Z\right),\end{aligned}\tag{11.38}$$

$$\beta_1' = Z_2\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma}{2I_2 n_1}\frac{s(\alpha)}{\sin\alpha}\cdot\Delta_v\left(\alpha,\beta_1,\beta_2,n_1 Z\right),\tag{11.39}$$

$$\beta_2' = -Z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1} + \frac{\sigma}{2I_2 n_1}\frac{s(\alpha)}{\sin\alpha\sin\beta_1}\cdot\Theta_v\left(\alpha,\beta_1,\beta_2,n_1 Z\right),\tag{11.40}$$

where

$$\begin{aligned}\Psi(\alpha,\beta_1,\beta_2,Z) = &-\sigma n_1\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha + \frac{\sigma}{2I_2 n_1}s(\alpha)\sin\alpha\cdot\Gamma_v\left(\alpha,\beta_1,\beta_2,n_1 Z\right)\\ &+\frac{T_1\left(\alpha,\beta_1,\beta_2,n_1 Z\right) - s(\alpha)}{m n_1}\cos\alpha,\end{aligned}\tag{11.41}$$

$$\begin{aligned}\Gamma_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = &\,x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\sin\beta_1\sin\beta_2 + x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\sin\beta_1\cos\beta_2\\ &+ x_{2N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\cos\beta_1,\end{aligned}\tag{11.42}$$

$$\begin{aligned}\Delta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = &\,x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\cos\beta_1\sin\beta_2 + x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\cos\beta_1\cos\beta_2\\ &- x_{2N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\sin\beta_1,\end{aligned}\tag{11.43}$$

$$\Theta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\cos\beta_2 - x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)\sin\beta_2.\tag{11.44}$$

We see that the seventh-order system (11.34)–(11.40) contains an independent sixth-order subsystem (11.35)–(11.40), which can be separately examined in its own six-dimensional phase space.

In particular, this method of separation of an independent sixth-order subsystem can also be applied under condition (11.21).

Here and in what follows, the dependence on the group of variables $(\alpha,\beta_1,\beta_2,\Omega/v)$ is meant as a composite dependence on $(\alpha,\beta_1,\beta_2,z_1/v,z_2/v,z_3/v)$ (and further of $(\alpha,\beta_1,\beta_2,n_1Z_1,n_1Z_2,n_1Z_3)$) due to (11.25) and (11.33).

12. Case Where the Moment of a Nonconservative Force Is Independent of the Angular Velocity

12.1. Reduced system. As in the choice of Chaplygin analytic functions (see [16, 17]), we select the dynamical functions s, x_{2N}, x_{3N}, and x_{4N} in the following form:

$$
\begin{gathered}
s(\alpha) = B\cos\alpha, \\
x_{2N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = x_{2N0}(\alpha,\beta_1,\beta_2) = A\sin\alpha\cos\beta_1, \\
x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = x_{3N0}(\alpha,\beta_1) = A\sin\alpha\sin\beta_1\cos\beta_2, \\
x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = x_{4N0}(\alpha,\beta_1,\beta_2) = A\sin\alpha\sin\beta_1\sin\beta_2,
\end{gathered}
\tag{12.1}
$$

where $A, B > 0$ and $v \neq 0$. We see that in the system considered, the moment of nonconservative forces is independent of the angular velocity and depends only on the angles α, β_1, and β_2. The functions $\Gamma_v(\alpha,\beta_1,\beta_2,\Omega/v)$, $\Delta_v(\alpha,\beta_1,\beta_2,\Omega/v)$, and $\Theta_v(\alpha,\beta_1,\beta_2,\Omega/v)$ in system (11.34)–(11.40) have the following form:

$$
\Gamma_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = A\sin\alpha, \quad \Delta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) \equiv \Theta_v\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) \equiv 0. \tag{12.2}
$$

Then, due to conditions (11.19) and (12.1), the transformed dynamical part of the equations of motion (system (11.34)–(11.40)) becomes the analytic system

$$v' = v\Psi(\alpha,\beta_1,\beta_2,Z), \tag{12.3}$$

$$\alpha' = -Z_3 + b\left(Z_1^2+Z_2^2+Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha, \tag{12.4}$$

$$Z_3' = \sin\alpha\cos\alpha - \left(Z_1^2+Z_2^2\right)\frac{\cos\alpha}{\sin\alpha} + bZ_3\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha - bZ_3\sin^2\alpha\cos\alpha, \tag{12.5}$$

$$Z_2' = Z_2Z_3\frac{\cos\alpha}{\sin\alpha} + Z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + bZ_2\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha - bZ_2\sin^2\alpha\cos\alpha, \tag{12.6}$$

$$Z_1' = Z_1Z_3\frac{\cos\alpha}{\sin\alpha} - Z_1Z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + bZ_1\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha - bZ_1\sin^2\alpha\cos\alpha, \tag{12.7}$$

$$\beta_1' = Z_2\frac{\cos\alpha}{\sin\alpha}, \tag{12.8}$$

$$\beta_2' = -Z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \tag{12.9}$$

where

$$\Psi(\alpha,\beta_1,\beta_2,Z) = -b\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha + b\sin^2\alpha\cos\alpha$$

and the dimensionless parameter b and the constant n_1 are chosen as follows:

$$b = \sigma n_0, \quad n_0^2 = \frac{AB}{2I_2}, \quad n_1 = n_0. \tag{12.10}$$

Thus, system (12.3)–(12.9) can be considered on its own seven-dimensional phase manifold

$$W_1 = \mathbb{R}_+^1\{v\} \times T\mathbf{S}^3\Big\{Z_1,\ Z_2,\ Z_3,\ 0<\alpha<\pi,\ 0<\beta_1<\pi,\ 0\le\beta_2<2\pi\Big\}, \tag{12.11}$$

i.e., on the direct product of the number half-line and the tangent bundle of the three-dimensional sphere $\mathbf{S}^3\big\{0<\alpha<\pi,\ 0<\beta_1<\pi,\ 0\le\beta_2<2\pi\big\}$.

We see that the seven-dimensional system (12.3)–(12.9) contains the independent sixth-order system (12.4)–(12.9) on its own six-dimensional manifold.

For the complete integration of system (12.3)–(12.9) we need, in general, six independent first integrals. However, after the change of variables

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \to \begin{pmatrix} Z \\ Z_* \end{pmatrix}, \quad Z = \sqrt{Z_1^2 + Z_2^2}, \quad Z_* = Z_2/Z_1, \tag{12.12}$$

system (12.4)–(12.9) splits as follows:

$$\alpha' = -Z_3 + b\left(Z^2 + Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha, \tag{12.13}$$

$$Z_3' = \sin\alpha\cos\alpha - Z^2\frac{\cos\alpha}{\sin\alpha} + bZ_3\left(Z^2 + Z_3^2\right)\cos\alpha - bZ_3\sin^2\alpha\cos\alpha, \tag{12.14}$$

$$Z' = ZZ_3\frac{\cos\alpha}{\sin\alpha} + bZ\left(Z^2 + Z_3^2\right)\cos\alpha - bZ\sin^2\alpha\cos\alpha, \tag{12.15}$$

$$Z_*' = (\pm)Z\sqrt{1 + Z_*^2}\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{12.16}$$

$$\beta_1' = (\pm)\frac{ZZ_*}{\sqrt{1 + Z_*^2}}\frac{\cos\alpha}{\sin\alpha}, \tag{12.17}$$

$$\beta_2' = (\mp)\frac{Z}{\sqrt{1 + Z_*^2}}\frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{12.18}$$

We see that the sixth-order system also splits into independent subsystems of lower orders: system (12.13)–(12.15) of order 3 and system (12.16), (12.17) (after the change of the independent variable) of order 2. Thus, for the complete integrability of system (12.3), (12.13)–(12.18) it suffices to specify two independent first integrals of system (12.13)–(12.15), one first integral of system (12.16), (12.17), and two additional first integrals that “attach” Eqs. (12.18) and (12.3).

Note that system (12.13)–(12.15) can be considered on the tangent bundle $T\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$.

12.2. Complete list of first integrals. System (12.13)–(12.15) has the form of a system that appears in the dynamics of a three-dimensional (3D) rigid body in a nonconservative field.

Note that, by (11.19), the value of the velocity of the center of mass is a first integral of system (11.26)–(11.32) (under condition (11.21)); namely, the function of phase variables

$$\Psi_0(v, \alpha, \beta_1, \beta_2, z_1, z_2, z_3) = v^2 + \sigma^2\left(z_1^2 + z_2^2 + z_3^2\right) - 2\sigma z_3 v\sin\alpha = V_C^2 \tag{12.19}$$

is constant on phase trajectories of the system (here z_1, z_2, and z_3 are chosen due to (11.25)).

Due to a nondegenerate change of the independent variable (for $v \neq 0$), system (12.3), (12.13)–(12.18) also possesses an analytic integral, namely, the function of phase variables

$$\Psi_1(v, \alpha, \beta_1, \beta_2, Z, Z_*, Z_3) = v^2\left(1 + b^2\left(Z^2 + Z_3^2\right) - 2bZ_3\sin\alpha\right) = V_C^2 \tag{12.20}$$

is constant on phase trajectories of the system.

Equality (12.20) allows one to find the dependence of the velocity of the characteristic point of the rigid body (the center D of the disk) on the other phase variables without solving system (12.3), (12.13)–(12.18); namely, for $V_C \neq 0$ we have the relation

$$v^2 = \frac{V_C^2}{1 + b^2\left(Z^2 + Z_3^2\right) - 2bZ_3\sin\alpha}. \tag{12.21}$$

Since the phase space

$$W_2 = \mathbb{R}_+^1\{v\} \times T\mathbf{S}^3\Big\{Z,\ Z_*,\ Z_3,\ 0 < \alpha < \pi,\ 0 < \beta_1 < \pi,\ 0 \le \beta_2 < 2\pi\Big\} \tag{12.22}$$

of system (12.3), (12.13)–(12.18) has dimension 7 and contains asymptotic limit sets, Eq. (12.20) defines a unique analytic (even continuous) first integral of system (12.3), (12.13)–(12.18) in the whole phase space (see [3, 4, 8, 11, 38, 39, 56, 69, 91]).

We examine the existence of other (additional) first integrals of system (12.3), (12.13)–(12.18). Its phase space is stratified into surfaces

$$\Big\{(v, \alpha, \beta_1, \beta_2, Z, Z_*, Z_3) \in W_2 : V_C = \text{const}\Big\}; \tag{12.23}$$

the dynamics on these surfaces is determined by the first integrals of system (12.3), (12.13)–(12.18).

First, we establish a correspondence between the independent third-order subsystem (12.13)–(12.15) and the nonautonomous second-order system

$$\begin{aligned} \frac{dZ_3}{d\alpha} &= \frac{\sin\alpha\cos\alpha + bZ_3\left(Z^2+Z_3^2\right)\cos\alpha - bZ_3\sin^2\alpha\cos\alpha - Z^2\cos\alpha/\sin\alpha}{-Z_3 + b\left(Z^2+Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha}, \\ \frac{dZ}{d\alpha} &= \frac{bZ\left(Z^2+Z_3^2\right)\cos\alpha - bZ\sin^2\alpha\cos\alpha + ZZ_3\cos\alpha/\sin\alpha}{-Z_3 + b\left(Z^2+Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha}. \end{aligned} \tag{12.24}$$

Applying the substitution $\tau = \sin\alpha$, we rewrite system (12.24) in algebraic form:

$$\begin{aligned} \frac{dZ_3}{d\tau} &= \frac{\tau + bZ_3\left(Z^2+Z_3^2\right) - bZ_3\tau^2 - Z^2/\tau}{-Z_3 + b\tau\left(1-\tau^2\right) + b\tau\left(Z^2+Z_3^2\right)}, \\ \frac{dZ}{d\tau} &= \frac{bZ\left(Z^2+Z_3^2\right) - bZ\tau^2 + ZZ_3/\tau}{-Z_3 + b\tau\left(1-\tau^2\right) + b\tau\left(Z^2+Z_3^2\right)}. \end{aligned} \tag{12.25}$$

Further, introducing the homogeneous variables by the formulas

$$Z = u_1\tau, \quad Z_3 = u_2\tau, \tag{12.26}$$

we reduce system (12.25)) to the following form:

$$\begin{aligned} \tau\frac{du_2}{d\tau} + u_2 &= \frac{1 - bu_2\tau^2 + bu_2\left(u_1^2+u_2^2\right)\tau^2 - u_1^2}{-u_2 + b\tau^2\left(u_1^2+u_2^2\right) + b\left(1-\tau^2\right)}, \\ \tau\frac{du_1}{d\tau} + u_1 &= \frac{bu_1\left(u_1^2+u_2^2\right)\tau^2 - bu_1\tau^2 + u_1u_2}{-u_2 + b\tau^2\left(u_1^2+u_2^2\right) + b\left(1-\tau^2\right)}, \end{aligned} \tag{12.27}$$

which is equivalent to

$$\begin{aligned} \tau\frac{du_2}{d\tau} &= \frac{1 - bu_2 + u_2^2 - u_1^2}{-u_2 + b\tau^2\left(u_1^2+u_2^2\right) + b\left(1-\tau^2\right)}, \\ \tau\frac{du_1}{d\tau} &= \frac{2u_1u_2 - bu_1}{-u_2 + b\tau^2\left(u_1^2+u_2^2\right) + b\left(1-\tau^2\right)}. \end{aligned} \tag{12.28}$$

We establish a correspondence between the second-order system (12.28) and the nonautonomous first-order equation

$$\frac{du_2}{du_1} = \frac{1 - bu_2 + u_2^2 - u_1^2}{2u_1u_2 - bu_1}, \tag{12.29}$$

which is easily transformed to exact-differential form:

$$d\left(\frac{u_2^2 + u_1^2 - bu_2 + 1}{u_1}\right) = 0. \tag{12.30}$$

Thus, Eq. (12.29) has the first integral

$$\frac{u_2^2 + u_1^2 - bu_2 + 1}{u_1} = C_1 = \text{const}, \tag{12.31}$$

which in terms of the previous variables has the form

$$\frac{Z_3^2 + Z^2 - bZ_3 \sin\alpha + \sin^2\alpha}{Z \sin\alpha} = C_1 = \text{const}\,. \tag{12.32}$$

Remark 12.1. Consider system (12.13)–(12.15) with variable dissipation with zero mean (see [91]), which becomes conservative for $b = 0$:

$$\begin{aligned} \alpha' &= -Z_3, \\ Z_3' &= \sin\alpha\cos\alpha - Z^2\frac{\cos\alpha}{\sin\alpha}, \\ Z' &= ZZ_3\frac{\cos\alpha}{\sin\alpha}. \end{aligned} \tag{12.33}$$

This system possesses two analytic first integrals of the form

$$Z_3^2 + Z^2 + \sin^2\alpha = C_1^* = \text{const}, \tag{12.34}$$

$$Z\sin\alpha = C_2^* = \text{const}\,. \tag{12.35}$$

Obviously, the ratio of the two first integrals (12.34) and (12.35) is also a first integral of system (12.33). However, for $b \neq 0$, none of the functions

$$Z_3^2 + Z^2 - bZ_3\sin\alpha + \sin^2\alpha \tag{12.36}$$

and (12.35) is a first integral of system (12.13)–(12.15), but their ratio is a first integral system (12.13)–(12.15) for any b.

Further, we find an additional first integral of the third-order system (12.13)–(12.15). First, we transform the invariant relation (12.31) for $u_1 \neq 0$ as follows:

$$\left(u_2 - \frac{b}{2}\right)^2 + \left(u_1 - \frac{C_1}{2}\right)^2 = \frac{b^2 + C_1^2}{4} - 1. \tag{12.37}$$

We see that the parameters of this invariant relation must satisfy the condition

$$b^2 + C_1^2 - 4 \geq 0, \tag{12.38}$$

and the phase space of system (12.13)–(12.15) is stratified into a family of surfaces defined by (12.37).

Thus, due to relation (12.31), the first equation of system (12.28) takes the form

$$\tau\frac{du_2}{d\tau} = \frac{1 - bu_2 + u_2^2 - U_1^2(C_1, u_2)}{-u_2 + b\,(1-\tau^2) + b\tau^2\left(U_1^2(C_1, u_2) + u_2^2\right)}, \tag{12.39}$$

where

$$U_1(C_1, u_2) = \frac{1}{2}\left\{C_1 \pm \sqrt{C_1^2 - 4\left(u_2^2 - bu_2 + 1\right)}\right\} \tag{12.40}$$

and the integration constant C_1 is defined by condition (12.38), or the form of the Bernoulli equation:

$$\frac{d\tau}{du_2} = \frac{(b - u_2)\tau - b\tau^3\left(1 - U_1^2(C_1, u_2) - u_2^2\right)}{1 - bu_2 + u_2^2 - U_1^2(C_1, u_2)}. \tag{12.41}$$

Using (12.40), we can transform Eq. (12.41) into the form of a nonhomogeneous linear equation:

$$\frac{dp}{du_2} = \frac{2(u_2 - b)p + 2b\left(1 - U_1^2(C_1, u_2) - u_2^2\right)}{1 - bu_2 + u_2^2 - U_1^2(C_1, u_2)}, \quad p = \frac{1}{\tau^2}. \tag{12.42}$$

This means that we can find another transcendental first integral in explicit form (i.e., in the form of a finite combination of quadratures). Moreover, the general solution of Eq. (12.42) depends on an arbitrary constant C_2. We omit complete calculations but note that the general solution of the homogeneous linear equation obtained from (12.42) in the particular case $b = C_1 = 2$ has the form

$$p = p_0(u_2) = C\left[\sqrt{1-(u_2-1)^2} \pm 1\right] \exp\left[\sqrt{\frac{1 \mp \sqrt{1-(u_2-1)^2}}{1 \pm \sqrt{1-(u_2-1)^2}}}\right], \quad C = \text{const}\,. \tag{12.43}$$

Remark 12.2. Formally, in the expression of the first integral thus found, we must substitute for C_1 the left-hand side of the first integral (12.31).

Then the obtained additional first integral has the following structure (similar to the transcendental first integral from planar dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha,\ \frac{z_3}{\sin\alpha},\ \frac{z}{\sin\alpha}\right) = C_2 = \text{const}\,. \tag{12.44}$$

Thus, for integration of the sixth-order system (12.13)–(12.18) we already have two independent first integrals. For the complete integration, it suffices to find one first integral for the (potentially separated) system (12.16), (12.17) and an additional first integral that "attaches" Eq. (12.18).

To find a first integral of the (potentially separated) system (12.16), (12.17), we establish a correspondence between it and the following nonautonomous first-order equation:

$$\frac{dZ_*}{d\beta_1} = \frac{1+Z_*^2}{Z_*}\frac{\cos\beta_1}{\sin\beta_1}. \tag{12.45}$$

After integration we obtain the required invariant relation

$$\frac{\sqrt{1+Z_*^2}}{\sin\beta_1} = C_3 = \text{const}; \tag{12.46}$$

in terms of the variables Z_1 and Z_2 it has the form

$$\frac{\sqrt{Z_1^2+Z_2^2}}{Z_1\sin\beta_1} = C_3 = \text{const}\,. \tag{12.47}$$

Further, to find an additional first integral that "attaches" Eq. (12.18), we establish a correspondence between Eqs. (12.18) and (12.16) and the following nonautonomous equation:

$$\frac{dZ_*}{d\beta_2} = -\left(1+Z_*^2\right)\cos\beta_1. \tag{12.48}$$

Since, due to (12.46),

$$C_3\cos\beta_1 = \pm\sqrt{C_3^2-1-Z_*^2}, \tag{12.49}$$

we have

$$\frac{dZ_*}{d\beta_2} = \mp\frac{1}{C_3}\left(1+Z_*^2\right)\sqrt{C_3^2-1-Z_*^2}. \tag{12.50}$$

Integrating this relation, we obtain the following quadrature:

$$\mp(\beta_2+C_4) = \int\frac{C_3 dZ_*}{(1+Z_*^2)\sqrt{C_3^2-1-Z_*^2}}, \quad C_4 = \text{const}\,. \tag{12.51}$$

Another integration leads to the relation

$$\mp\tan(\beta_2+C_4) = \frac{C_3 Z_*}{\sqrt{C_3^2-1-Z_*^2}}, \quad C_4 = \text{const}\,. \tag{12.52}$$

In the variables Z_1 and Z_2, this invariant relation has the form

$$\mp\tan(\beta_2+C_4) = \frac{C_3 Z_2}{\sqrt{\left(C_3^2-1\right)Z_1^2 - Z_2^2}}, \quad C_4 = \text{const}\,. \tag{12.53}$$

Finally, we have the following form of the additional first integral that "attaches" Eq. (12.18):

$$\arctan\frac{C_3 Z_*}{\sqrt{C_3^2 - 1 - Z_*^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const}, \tag{12.54}$$

or

$$\arctan\frac{C_3 Z_2}{\sqrt{\left(C_3^2 - 1\right) Z_1^2 - Z_2^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const}\,. \tag{12.55}$$

Thus, in the case considered the system of dynamical equations (11.3)–(11.6), (11.9)–(11.14) under condition (12.1) has eight invariant relations: the analytic nonintegrable constraint of the form (11.19) corresponding to the analytic first integral (12.19), the cyclic first integrals of the form (11.17) and (11.18), the first integral of the form (12.32). Moreover, there exists the first integral that can be found from Eq. (12.42); it is a transcendental function of phase variables (in the sense of complex analysis). Finally, we have the transcendental first integrals of the form (12.46) (or (12.47)) and (12.54) (or (12.55)).

Theorem 12.1. *System* (11.3)–(11.6), (11.9)–(11.14) *under conditions* (11.19), (12.1), (11.18), *and* (11.17) *possesses eight invariant relations* (*complete set*), *four of which are transcendental functions* (*from the point of view of complex analysis*). *Moreover, seven of these eight relations are expressed through finite combinations of elementary function.*

12.3. Topological analogies. We show that there exists another mechanical and topological analogy.

Theorem 12.2. *The first integral* (12.32) *of system* (11.3)–(11.6), (11.9)–(11.14) *under conditions* (11.19), (12.1), (11.18), *and* (11.17) *is constant on phase trajectories of system* (9.10)–(9.15).

Proof. Indeed, the first integral (12.32) can be obtained by a change of coordinates by means of (12.31), whereas the first integral (9.31) can be obtained by a change of coordinates by means of (9.30). But relations (12.31) and (9.30) coincide. The theorem is proved. □

Thus, we have the following topological and mechanical analogies in the sense explained above:

(1) motion of a free rigid body in a nonconservative field with a tracing force (under a nonintegrable constraint);
(2) motion of a fixed physical pendulum in a flow of a running medium (a nonconservative field);
(3) rotation of a rigid body about the center of mass, which, in turn, moves rectilinearly and uniformly in a nonconservative field.

For more general topological analogies, see also [91].

13. Case Where the Moment of a Nonconservative Force Depends on the Angular Velocity

13.1. Introduction of the dependence on the angular velocity and the reduced system. In this chapter, we continue to study the dynamics of a four-dimensional rigid body in four-dimensional space. The present section, like the analogous section of Chap. 2, is devoted to the study of motion in the case where the moment of forces depends on the tensor of angular velocity. Thus, we introduce this dependence as in the previous chapter. This also allows us to introduce this dependence for multi-dimensional bodies.

Let $x = (x_{1N}, x_{2N}, x_{3N}, x_{4N})$ be the coordinates of the application point N of the nonconservative force (influence of the medium) on the three-dimensional disk and let $Q = (Q_1, Q_2, Q_3, Q_4)$ be the components of the force $\mathbf{S}$ of the influence of the medium independent of the tensor of angular velocity.

We consider only the linear dependence of the function $(x_{1N}, x_{2N}, x_{3N}, x_{4N})$ on the tensor of angular velocity since this introduction itself is not obvious (see [14, 15]).

We adopt the following dependence:

$$x = Q + R, \tag{13.1}$$

where $R = (R_1, R_2, R_3, R_4)$ is a vector-valued function containing the components of the tensor of angular velocity. The dependence of the functions R on the components of the tensor of angular velocity is gyroscopic:

$$R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} = -\frac{1}{v} \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}, \tag{13.2}$$

where (h_1, h_2, h_3, h_4) are some positive parameters (cf. [91]).

Since $x_{1N} \equiv 0$, we have

$$x_{2N} = Q_2 - h_1 \frac{\omega_6}{v}, \quad x_{3N} = Q_3 + h_1 \frac{\omega_5}{v}, \quad x_{4N} = Q_4 - h_1 \frac{\omega_3}{v}. \tag{13.3}$$

As in to the choice of the Chaplygin analytic functions (see [16, 17]),

$$\begin{aligned} Q_2 &= A \sin\alpha \cos\beta_1, \\ Q_3 &= A \sin\alpha \sin\beta_1 \cos\beta_2, \\ Q_4 &= A \sin\alpha \sin\beta_1 \sin\beta_2, \end{aligned} \tag{13.4}$$

where $A > 0$, and we select the dynamical functions s, x_{2N}, x_{3N}, and x_{4N} in the following form:

$$\begin{aligned} s(\alpha) &= B\cos\alpha, \ B > 0, \\ x_{2N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= A \sin\alpha \cos\beta_1 - h\frac{\omega_6}{v}, \quad h = h_1 > 0, \ v \neq 0, \\ x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= A \sin\alpha \sin\beta_1 \cos\beta_2 + h\frac{\omega_5}{v}, \quad h = h_1 > 0, \ v \neq 0, \\ x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= A \sin\alpha \sin\beta_1 \sin\beta_2 - h\frac{\omega_3}{v}, \quad h = h_1 > 0, \ v \neq 0. \end{aligned} \tag{13.5}$$

This shows that in this problem, there is an additional damping (but accelerating in certain domains of the phase space) moment of a nonconservative force (i.e., there is a dependence of the moment on the components of the tensor of angular velocity). By the dynamical symmetry of the body, $h_2 = h_3 = h_4$.

The functions $\Gamma_v(\alpha, \beta_1, \beta_2, \Omega/v)$, $\Delta_v(\alpha, \beta_1, \beta_2, \Omega/v)$, and $\Theta_v(\alpha, \beta_1, \beta_2, \Omega/v)$ in system (11.35)–(11.40) have the following form:

$$\begin{aligned} \Gamma_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= A\sin\alpha - \frac{h}{v} z_3, \\ \Delta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= \frac{h}{v} z_2, \\ \Theta_v\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) &= -\frac{h}{v} z_1. \end{aligned} \tag{13.6}$$

By conditions (11.19) and (13.5), the transformed dynamical part of the equations of motion (system (11.34)–(11.40)) becomes the following analytic system:

$$v' = v\Psi(\alpha, \beta_1, \beta_2, Z), \tag{13.7}$$

$$\alpha' = -Z_3 + b\left(Z_1^2 + Z_2^2 + Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha - bH_1 Z_3 \cos^2\alpha, \tag{13.8}$$

$$Z_3' = \sin\alpha\cos\alpha - (1+bH_1)\left(Z_1^2+Z_2^2\right)\frac{\cos\alpha}{\sin\alpha} + bZ_3\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha - bZ_3\sin^2\alpha\cos\alpha + bH_1Z_3^2\sin\alpha\cos\alpha - H_1Z_3\cos\alpha, \tag{13.9}$$

$$Z_2' = (1+bH_1)\,Z_2Z_3\frac{\cos\alpha}{\sin\alpha} + (1+bH_1)\,Z_1^2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + bZ_2\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha - bZ_2\sin^2\alpha\cos\alpha + bH_1Z_2Z_3\sin\alpha\cos\alpha - H_1Z_2\cos\alpha, \tag{13.10}$$

$$Z_1' = (1+bH_1)\,Z_1Z_3\frac{\cos\alpha}{\sin\alpha} - (1+bH_1)\,Z_1Z_2\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1} + bZ_1\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha - bZ_1\sin^2\alpha\cos\alpha + bH_1Z_1Z_3\sin\alpha\cos\alpha - H_1Z_1\cos\alpha, \tag{13.11}$$

$$\beta_1' = (1+bH_1)\,Z_2\frac{\cos\alpha}{\sin\alpha}, \tag{13.12}$$

$$\beta_2' = -(1+bH_1)\,Z_1\frac{\cos\alpha}{\sin\alpha\sin\beta_1}, \tag{13.13}$$

where

$$\Psi(\alpha,\beta_1,\beta_2,Z) = -b\left(Z_1^2+Z_2^2+Z_3^2\right)\cos\alpha + b\sin^2\alpha\cos\alpha - bH_1Z_3\sin\alpha\cos\alpha;$$

as above, the dimensionless parameters b and H_1 and the constant n_1 are chosen as follows:

$$b = \sigma n_0, \quad n_0^2 = \frac{AB}{2I_2}, \quad H_1 = \frac{Bh}{2I_2n_0}, \quad n_1 = n_0. \tag{13.14}$$

Thus, system (13.7)–(13.13) can be considered on its seven-dimensional phase manifold

$$W_1 = \mathbb{R}_+^1\{v\} \times T\mathbf{S}^3\Big\{Z_1,\ Z_2,\ Z_3,\ 0<\alpha<\pi,\ 0<\beta_1<\pi,\ 0\le\beta_2<2\pi\Big\}, \tag{13.15}$$

i.e., on the direct product of the number half-line and the tangent bundle of the three-dimensional sphere $\mathbf{S}^3\{0<\alpha<\pi,\ 0<\beta_1<\pi,\ 0\le\beta_2<2\pi\}$.

We see that the seventh-order system (13.7)–(13.13) contains the independent sixth-order system (13.8)–(13.13) on its own six-dimensional manifold.

For complete integration of system (13.7)–(13.13), in general, we need six independent first integrals. However, after the change of variables

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \to \begin{pmatrix} Z \\ Z_* \end{pmatrix}, \quad Z = \sqrt{Z_1^2+Z_2^2}, \quad Z_* = Z_2/Z_1, \tag{13.16}$$

system (13.8)–(13.13) splits as follows:

$$\alpha' = -Z_3 + b\left(Z^2+Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha - bH_1Z_3\cos^2\alpha, \tag{13.17}$$

$$Z_3' = \sin\alpha\cos\alpha - (1+bH_1)\,Z^2\frac{\cos\alpha}{\sin\alpha} + bZ_3\left(Z^2+Z_3^2\right)\cos\alpha - bZ_3\sin^2\alpha\cos\alpha + bH_1Z_3^2\sin\alpha\cos\alpha - H_1Z_3\cos\alpha, \tag{13.18}$$

$$Z' = (1+bH_1)\,ZZ_3\frac{\cos\alpha}{\sin\alpha} + bZ\left(Z^2+Z_3^2\right)\cos\alpha - bZ\sin^2\alpha\cos\alpha + bH_1ZZ_3\sin\alpha\cos\alpha - H_1Z\cos\alpha, \tag{13.19}$$

$$Z_*' = (\pm)\,(1+bH_1)\,Z\sqrt{1+Z_*^2}\frac{\cos\alpha}{\sin\alpha}\frac{\cos\beta_1}{\sin\beta_1}, \tag{13.20}$$

$$\beta_1' = (\pm)\,(1 + bH_1)\,\frac{ZZ_*}{\sqrt{1 + Z_*^2}}\frac{\cos\alpha}{\sin\alpha}, \tag{13.21}$$

$$\beta_2' = (\mp)\,(1 + bH_1)\,\frac{Z}{\sqrt{1 + Z_*^2}}\frac{\cos\alpha}{\sin\alpha\sin\beta_1}. \tag{13.22}$$

We see that the sixth-order system splits into independent subsystems of lower orders: system (13.17)–(13.19) of order 3 and system (13.20), (13.21) (after the change of the independent variable) of order 2. This, for the complete integrability of system (13.7), (13.17)–(13.22), it suffices to specify two independent first integrals of system (13.17)–(13.19), one first integral of system (13.20), (13.21), and two additional first integrals that "attach" Eqs. (13.22) and (13.7).

Note that system (13.17)–(13.19) can be considered on the tangent bundle $T\mathbf{S}^2$ of the two-dimensional sphere $\mathbf{S}^2$.

13.2. Complete list of first integrals. System (13.17)–(13.19) has the form of a system of equations that appears in the dynamics of a three-dimensional (3D) rigid body in a nonconservative field.

Note that, by (11.19), the value of the velocity of the center of mass is a first integral of system (11.26)–(11.32) (under condition (11.21)); namely, the function of phase variables

$$\Psi_0(v, \alpha, \beta_1, \beta_2, z_1, z_2, z_3) = v^2 + \sigma^2\left(z_1^2 + z_2^2 + z_3^2\right) - 2\sigma z_3 v\sin\alpha = V_C^2 \tag{13.23}$$

is constant on phase trajectories of this system (the values of z_1, z_2, and z_3 are taken by virtue of (11.25)).

Due to the nondegenerate change of the independent variable (for $v \neq 0$), system (13.7), (13.17)–(13.22) also possesses an analytic integral, namely, the function of phase variables

$$\Psi_1(v, \alpha, \beta_1, \beta_2, Z, Z_*, Z_3) = v^2\left(1 + b^2\left(Z^2 + Z_3^2\right) - 2bZ_3\sin\alpha\right) = V_C^2 \tag{13.24}$$

is constant on phase trajectories of this system.

Equality (13.24) allows one to find the dependence of the velocity of the characteristic point of the rigid body (the cemter D of the disk) on the other phase variables without solving system (13.7), (13.17)–(13.22); namely, for $V_C \neq 0$ we have

$$v^2 = \frac{V_C^2}{1 + b^2\left(Z^2 + Z_3^2\right) - 2bZ_3\sin\alpha}. \tag{13.25}$$

Since the phase space

$$W_2 = \mathbb{R}_+^1\{v\} \times T\mathbf{S}^3\Big\{Z,\ Z_*,\ Z_3,\ 0 < \alpha < \pi,\ 0 < \beta_1 < \pi,\ 0 \leq \beta_2 < 2\pi\Big\} \tag{13.26}$$

of system (13.7), (13.17)–(13.22) has dimension 7 and contains asymptotic limit sets, we see that Eq. (13.24) determines the unique analytic (even continuous) first integral of system (13.7), (13.17)–(13.22) on the whole phase space (cf. [38, 91]).

We examine the existence of other (additional) first integrals of system (13.7), (13.17)–(13.22). Its phase space is stratified into surfaces

$$\Big\{(v, \alpha, \beta_1, \beta_2, Z, Z_*, Z_3) \in W_2 : V_C = \text{const}\Big\}; \tag{13.27}$$

the dynamics on these surfaces is determined by the first integrals of system (13.7), (13.17)–(13.22).

First, we establish a correspondence between the independent third-order subsystem (13.17)–(13.19) and the nonautonomous second-order system

$$
\begin{aligned}
\frac{dZ_3}{d\alpha} &= \frac{R_2(\alpha, Z, Z_3)}{-Z_3 + b\left(Z^2 + Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha - bH_1 Z_3\cos^2\alpha},\\
\frac{dZ}{d\alpha} &= \frac{R_1(\alpha, Z, Z_3)}{-Z_3 + b\left(Z^2 + Z_3^2\right)\sin\alpha + b\sin\alpha\cos^2\alpha - bH_1 Z_3\cos^2\alpha},\\
R_2(\alpha, Z, Z_3) &= \sin\alpha\cos\alpha + bZ_3\left(Z^2 + Z_3^2\right)\cos\alpha - bZ_3\sin^2\alpha\cos\alpha\\
&\quad - (1 + bH_1)Z^2\frac{\cos\alpha}{\sin\alpha} + bH_1 Z_3^2\sin\alpha\cos\alpha - H_1 Z_3\cos\alpha,\\
R_1(\alpha, Z, Z_3) &= bZ\left(Z^2 + Z_3^2\right)\cos\alpha - bZ\sin^2\alpha\cos\alpha\\
&\quad + (1 + bH_1)ZZ_3\frac{\cos\alpha}{\sin\alpha} + bH_1 ZZ_3\sin\alpha\cos\alpha - H_1 Z\cos\alpha.
\end{aligned}
\tag{13.28}
$$

Using the substitution $\tau = \sin\alpha$, we rewrite system (13.28) in algebraic form

$$
\begin{aligned}
\frac{dZ_3}{d\tau} &= \frac{\tau + bZ_3\left(Z^2 + Z_3^2\right) - bZ_3\tau^2 - (1 + bH_1)Z^2/\tau + bH_1 Z_3^2\tau - H_1 Z_3}{-Z_3 + b\tau\left(1 - \tau^2\right) + b\tau\left(Z^2 + Z_3^2\right) - bH_1 Z_3\left(1 - \tau^2\right)},\\
\frac{dZ}{d\tau} &= \frac{bZ\left(Z^2 + Z_3^2\right) - bZ_1\tau^2 + (1 + bH_1)ZZ_3/\tau + bH_1 ZZ_3\tau - H_1 Z}{-Z_3 + b\tau\left(1 - \tau^2\right) + b\tau\left(Z^2 + Z_3^2\right) - bH_1 Z_3\left(1 - \tau^2\right)}.
\end{aligned}
\tag{13.29}
$$

Further, introducing homogeneous variables by the formulas

$$
Z = u_1\tau, \quad Z_3 = u_2\tau, \tag{13.30}
$$

we transform system (13.29) into the following form:

$$
\begin{aligned}
\tau\frac{du_2}{d\tau} + u_2 &= \frac{1 - bu_2\tau^2 + bu_2\left(u_1^2 + u_2^2\right)\tau^2 - (1 + bH_1)u_1^2 - H_1 u_2 + bH_1 u_2^2\tau^2}{-u_2 + b\tau^2\left(u_1^2 + u_2^2\right) + b\left(1 - \tau^2\right) - bH_1 u_2\left(1 - \tau^2\right)},\\
\tau\frac{du_1}{d\tau} + u_1 &= \frac{bu_1\left(u_1^2 + u_2^2\right)\tau^2 - bu_1\tau^2 + (1 + bH_1)u_1 u_2 - H_1 u_1 + bH_1 u_1 u_2}{-u_2 + b\tau^2\left(u_1^2 + u_2^2\right) + b\left(1 - \tau^2\right) - bH_1 u_2\left(1 - \tau^2\right)},
\end{aligned}
\tag{13.31}
$$

which is equivalent to

$$
\begin{aligned}
\tau\frac{du_2}{d\tau} &= \frac{1 - (b + H_1)u_2 + (1 + bH_1)u_2^2 - (1 + bH_1)u_1^2}{-u_2 + b\tau^2\left(u_1^2 + u_2^2\right) + b\left(1 - \tau^2\right) - bH_1 u_2\left(1 - \tau^2\right)},\\
\tau\frac{du_1}{d\tau} &= \frac{2(1 + bH_1)u_1 u_2 - (b + H_1)u_1}{-u_2 + b\tau^2\left(u_1^2 + u_2^2\right) + b\left(1 - \tau^2\right) - bH_1 u_2\left(1 - \tau^2\right)}.
\end{aligned}
\tag{13.32}
$$

We establish a correspondence between the second-order system (13.32) and the first-order nonautonomous equation

$$
\frac{du_2}{du_1} = \frac{1 - (b + H_1)u_2 + (1 + bH_1)u_2^2 - (1 + bH_1)u_1^2}{2(1 + bH_1)u_1 u_2 - (b + H_1)u_1}, \tag{13.33}
$$

which can be easily transformed into exact-differential form:

$$
d\left(\frac{(1 + bH_1)u_2^2 + (1 + bH_1)u_1^2 - (b + H_1)u_2 + 1}{u_1}\right) = 0. \tag{13.34}
$$

Therefore, Eq. (13.33) has the first integral

$$
\frac{(1 + bH_1)u_2^2 + (1 + bH_1)u_1^2 - (b + H_1)u_2 + 1}{u_1} = C_1 = \text{const}, \tag{13.35}
$$

which in terms of the previous variables has the form

$$\frac{(1+bH_1)Z_3^2+(1+bH_1)Z^2-(b+H_1)Z_3\sin\alpha+\sin^2\alpha}{Z\sin\alpha}=C_1=\text{const}\,. \tag{13.36}$$

Remark 13.1. Consider system (13.17)–(13.19) with variable dissipation with zero mean (see [91]), which becomes conservative for $b = H_1$:

$$\begin{aligned}
\alpha' &= -Z_3+b\left(Z^2+Z_3^2\right)\sin\alpha+b\sin\alpha\cos^2\alpha-b^2Z_3\cos^2\alpha,\\
Z_3' &= \sin\alpha\cos\alpha-\left(1+b^2\right)Z^2\frac{\cos\alpha}{\sin\alpha}+bZ_3\left(Z^2+Z_3^2\right)\cos\alpha-bZ_3\sin^2\alpha\cos\alpha\\
&\quad+b^2Z_3^2\sin\alpha\cos\alpha-bZ_3\cos\alpha,\\
Z' &= \left(1+b^2\right)ZZ_2\frac{\cos\alpha}{\sin\alpha}+bZ\left(Z^2+Z_3^2\right)\cos\alpha-bZ\sin^2\alpha\cos\alpha+b^2ZZ_3\sin\alpha\cos\alpha-bZ\cos\alpha.
\end{aligned} \tag{13.37}$$

It possesses two analytic first integrals

$$\left(1+b^2\right)\left(Z_3^2+Z^2\right)-2bZ_3\sin\alpha+\sin^2\alpha=C_1^*=\text{const}, \tag{13.38}$$

$$Z\sin\alpha=C_2^*=\text{const}\,. \tag{13.39}$$

Obviously, the ratio of the two first integrals (13.38) and (13.39) is also a first integral of system (13.37). However, for $b \neq H_1$, neither of the functions

$$(1+bH_1)\left(Z_3^2+Z^2\right)-(b+H_1)Z_3\sin\alpha+\sin^2\alpha \tag{13.40}$$

and (13.39) is a first integral of system (13.17)–(13.19), but their ratio is a first integral of system (13.17)–(13.19) for all b and H_1.

We find the explicit form of an additional first integral of the third-order system (13.17)–(13.19). For this purpose, we transform the invariant relation (13.35) for $u_1 \neq 0$ as follows:

$$\left(u_2-\frac{b+H_1}{2(1+bH_1)}\right)^2+\left(u_1-\frac{C_1}{2(1+bH_1)}\right)^2=\frac{(b-H_1)^2+C_1^2-4}{4(1+bH_1)^2}. \tag{13.41}$$

We see that the parameters of this invariant relation must satisfy the condition

$$(b-H_1)^2+C_1^2-4\geq 0, \tag{13.42}$$

and that the phase space of system (13.17)–(13.19) is stratified into the family of surfaces determined by Eq. (13.41).

Thus, by relation (13.35), the first equation of system (13.32) has the form

$$\tau\frac{du_2}{d\tau}=\frac{1-(b+H_1)u_2+(1+bH_1)u_2^2-(1+bH_1)U_1^2(C_1,u_2)}{-u_2+b\left(1-\tau^2\right)+b\tau^2\left(U_1^2(C_1,u_2)+u_2^2\right)-bH_1u_2\left(1-\tau^2\right)}, \tag{13.43}$$

where

$$U_1(C_1,u_2)=\frac{1}{2}\left\{C_1\pm\sqrt{C_1^2-4(1+bH_1)\left(1-(b+H_1)u_2+(1+bH_1)u_2^2\right)}\right\} \tag{13.44}$$

and the integration constant C_1 is defined by condition (13.42) or the form of the Bernoulli equation:

$$\frac{d\tau}{du_2}=\frac{(b-(1+bH_1)u_2)\tau-b\tau^3\left(1-U_1^2(C_1,u_2)-u_2^2-H_1u_2\right)}{1-(b+H_1)u_2+(1+bH_1)u_2^2-(1+bH_1)U_1^2(C_1,u_2)}. \tag{13.45}$$

Using (13.44), we can easily transform Eq. (13.45) into a nonhomogeneous linear equation

$$\frac{dp}{du_2}=\frac{2((1+bH_1)u_2-b)p+2b\left(1-H_1u_2-u_2^2-U_1^2(C_1,u_2)\right)}{1-(b+H_1)u_2+(1+bH_1)u_2^2-(1+bH_1)U_1^2(C_1,u_2)},\quad p=\frac{1}{\tau^2}. \tag{13.46}$$

This means that there exists another transcendental first integral in explicit form (i.e., through a finite combination of quadratures). Moreover, the general solution of Eq. (13.46) depends on an arbitrary constant C_2. We omit the complete calculations but note that the general solution of the homogeneous linear equation obtained from (13.46) in the particular case

$$|b - H_1| = 2, \quad C_1 = \frac{1 - A_1^4}{1 + A_1^4}, \quad A_1 = \frac{1}{2}(b + H_1)$$

has the following solution:

$$p = p_0(u_2) = C[1 - A_1 u_2]^{2/(1+A_1^4)} \left| \frac{\sqrt{C_1^2 - 4A_1^2(1 - A_1 u_2)^2} \pm C_1}{\sqrt{C_1^2 - 4A_1^2(1 - A_1 u_2)^2} \mp C_1} \right|^{\pm A_1^4/(1+A_1^4)}$$
$$\times \exp \frac{2(A_1 - b)}{\left(1 + A_1^4\right) A_1(A_1 u_2 - 1)}, \quad C = \text{const}\,. \tag{13.47}$$

Remark 13.2. Formally, in the expression of the first integral thus found, we must substitute for C_1 the left-hand side of the first integral (13.35).

Then the additional first integral obtained has the following structure (similar to the transcendental first integral from planar dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha, \frac{Z_3}{\sin\alpha}, \frac{Z}{\sin\alpha}\right) = C_2 = \text{const}\,. \tag{13.48}$$

Thus, for the integration of the sixth-order system (13.17)–(13.22) we already have two independent first integrals. For the complete integrability, as was noted above, it suffices to find one first integral for the (potentially separated) system (13.20), (13.21) and an additional first integral that "attaches" Eq. (13.22).

To find the first integral of the (potentially separated) system (13.20), (13.21), we establish a correspondence between it and the following nonautonomous first-order equation:

$$\frac{dZ_*}{d\beta_1} = \frac{1 + Z_*^2}{Z_*}\frac{\cos\beta_1}{\sin\beta_1}. \tag{13.49}$$

After integration, this leads to the required invariant relation

$$\frac{\sqrt{1 + Z_*^2}}{\sin\beta_1} = C_3 = \text{const}, \tag{13.50}$$

which in terms of the variables Z_1 and Z_2 has the form

$$\frac{\sqrt{Z_1^2 + Z_2^2}}{Z_1 \sin\beta_1} = C_3 = \text{const}\,. \tag{13.51}$$

Further, to find an additional first integral that "attaches" Eq. (13.22), we establish a correspondence between Eqs. (13.22) and (13.20) and the following nonautonomous equation:

$$\frac{dZ_*}{d\beta_2} = -\left(1 + Z_*^2\right)\cos\beta_1. \tag{13.52}$$

Since, by (13.50),

$$C_3 \cos\beta_1 = \pm\sqrt{C_3^2 - 1 - Z_*^2}, \tag{13.53}$$

we have

$$\frac{dZ_*}{d\beta_2} = \mp\frac{1}{C_3}\left(1 + Z_*^2\right)\sqrt{C_3^2 - 1 - Z_*^2}. \tag{13.54}$$

Integrating this relation, we obtain the following quadrature:

$$\mp(\beta_2 + C_4) = \int \frac{C_3 dZ_*}{(1 + Z_*^2)\sqrt{C_3^2 - 1 - Z_*^2}}, \ C_4 = \text{const}\,. \tag{13.55}$$

Another integration leads to the relation

$$\mp\tan(\beta_2 + C_4) = \frac{C_3 Z_*}{\sqrt{C_3^2 - 1 - Z_*^2}}, \quad C_4 = \text{const}\,. \tag{13.56}$$

In terms of the variables Z_1 and Z_2, this invariant relation becomes

$$\mp\tan(\beta_2 + C_4) = \frac{C_3 Z_2}{\sqrt{\left(C_3^2 - 1\right) Z_1^2 - Z_2^2}}, \quad C_4 = \text{const}\,. \tag{13.57}$$

Finally, we have the additional first integral that "attaches" Eq. (13.22):

$$\arctan \frac{C_3 Z_*}{\sqrt{C_3^2 - 1 - Z_*^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const} \tag{13.58}$$

or

$$\arctan \frac{C_3 Z_2}{\sqrt{\left(C_3^2 - 1\right) Z_1^2 - Z_2^2}} \pm \beta_2 = C_4, \quad C_4 = \text{const}\,. \tag{13.59}$$

Thus, in the case considered, the system of dynamical equations (11.3)–(11.6), (11.9)–(11.14) under condition (13.5) has eight invariant relations: the analytic nonintegrable constraint of the form (11.19) corresponding to the analytic first integral (13.23), the cyclic first integrals of the form (11.17) and (11.18), the first integral of the form (13.36); moreover, there is a first integral that can be found from Eq. (13.46) (it is a transcendental function of phase variables in the sense of complex analysis), and, finally, transcendental first integrals of the form (13.50) (or (13.51)) and (13.58) (or (13.59)).

Theorem 13.1. *System* (11.3)–(11.6), (11.9)–(11.14) *under conditions* (11.19), (13.5), (11.18), *and* (11.17) *possesses eight invariant relations* (*complete set*), *four of which are transcendental functions* (*from the point of view of complex analysis*). *Moreover, at least seven of these eight relations are expressed through finite combinations of elementary functions.*

13.3. Topological analogies. We show that there exists another mechanical and topological analogy.

Theorem 13.2. *The first integral* (13.36) *of system* (11.3)–(11.6), (11.9)–(11.14) *under conditions* (11.19), (13.5), (11.18), *and* (11.17) *is constant on phase trajectories of system* (10.14)–(10.19).

Proof. Indeed the first integral (13.36) can be obtained by a change of coordinates by means of relation (13.35), whereas the first integral (10.35) can be obtained by a change of coordinates by means of relation (10.34). But relations (13.35) and (10.34) coincide. The theorem is proved. □

Thus, we have the following topological and mechanical analogies in the sense explained above:

(1) motion of a free rigid body in a nonconservative field with a tracing force (under a nonintegrable constraint);
(2) motion of a fixed physical pendulum in a flow of a running medium (nonconservative field);
(3) rotation of a rigid body about the center of mass that moves rectilinearly and uniformly in a nonconservative field.

On more general topological analogies, see also [91].

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