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ON THE CONSTRUCTION OF THE GENERAL SOLUTION OF A CLASS OF COMPLEX NONAUTONOMOUS EQUATIONS

Yu. M. Okunev and M. V. Shamolin UDC 517.925; 531.01; 531.552

Abstract. In this paper, we give a survey of cases of integrability for some class of complex, linear, nonautonomous, ordinary, second-order differential equations. We perform a qualitative analysis of these cases and construct general solutions in the form of absolutely and uniformly converging series with respect to small parameters.

1. Preliminaries. Earlier [9, 11, 12, 21, 28–32] in the study of nonstationary motion with respect to the center of mass of a dynamically symmetric rigid body of spatial aerodynamical form with high carrier properties under the quadratic resistance of the medium, a nonautonomous linear complex second-order equation was obtained. In the framework of a linearized model that does not take into account damping moments of aerodynamical forces, one can observe the damping action of the lifting force and find restrictions on the aerodynamical coefficients under which angular oscillations of the body attenuate [17, 18, 22, 23, 26, 37, 38, 42, 43, 45–47, 49, 51, 52, 59, 60].

2. Complex nonautonomous linear second-order equation. Consider the following complex equation:

$$\ddot{y} + \left[(c_2 - c_1)v(t) - \left(2 - \frac{J_2}{J_1}\right)\omega_{s2} i\right]\dot{y} + \left[[k - (c_2 - c_1)c_1]v^2(t) - \left(1 - \frac{J_2}{J_1}\right)\omega_{s2}^2 - \left(1 - \frac{J_2}{J_1}\right)\omega_{s2}(c_2 - c_1)v(t)i\right]y = 0, \quad (2.1)$$

where k, c_1, c_2, J_1, J_2, and ω_{s2} are positive constants and the real-valued function of time $v = v(t)$ satisfies the equation

$$\dot{v} = -c_1 v^2, \quad (2.2)$$

which can be easily integrated:

$$v(t) = \frac{v_0}{1 + c_1 v_0 t}, \quad v_0 = v(0). \quad (2.3)$$

If the equality $J_2/J_1 = 0$ holds, then this equation has a general solution that can be expressed through a finite combination of elementary functions (see [7, 32, 36, 39–41, 48, 53, 57].

We examine Eq. (2.1) under the sufficiently general natural condition

$$\frac{J_2}{J_1} > 0. \quad (2.4)$$

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3. Reduced equation and the corresponding nonautonomous linear complex Hamiltonian system. Using the substitution

$$y(t) = u_0(t)z(t), \tag{3.1}$$

where

$$u_0(t) = (1 + c_1 v_0 t)^{(c_1-c_2)/2c_1} \times \exp\left[\left(1 - \frac{J_2}{2J_1}\right)\omega_{s2} i t\right], \tag{3.2}$$

Eq. (2.1) can be reduced to the form

$$\ddot{z} + f(t)z = 0, \tag{3.3}$$

where

$$f(t) = \left[k - (c_2 - c_1)c_1 - \frac{1}{4}(c_1 - c_2)^2\right] v^2(t) - \frac{1}{2}(c_2 - c_1)\dot{v}(t) + \frac{1}{2}(c_2 - c_1)\omega_{s2} i \frac{J_2}{J_1} v(t) + \frac{1}{4}\frac{J_2^2}{J_1^2}\omega_{s2}^2. \tag{3.4}$$

Using natural changes of variables, we transforn Eq. (3.3) to the following form:

$$w'' + \frac{1}{c_1^2}\left[k - \frac{c_2^2}{4} + b_1(c_2 - c_1)\frac{1}{v_0}e^{\tau} - \frac{b_1^2}{v_0^2}e^{2\tau}\right] w = 0, \tag{3.5}$$

where

$$z(t(\tau)) = w(\tau), \tag{3.6}$$

$$c_1\tau_1 = \ln(1 + c_1 v_0 t) = \tau, \tag{3.7}$$

$$(\dot{\ }) = \frac{d}{dt} = v(t)\frac{d}{d\tau_1} = v(t)c_1\frac{d}{d\tau} = v(t)c_1(\ '), \tag{3.8}$$

$$\frac{d^2}{dt^2} = -c_1\dot{v}(t)\frac{d}{d\tau} + c_1^2 v^2(t)\frac{d^2}{d\tau^2}, \tag{3.9}$$

$$b_1 = \frac{1}{2}\frac{J_2}{J_1}\omega_{s2} i. \tag{3.10}$$

We see that the parameter b_1 becomes purely imaginary.

Equation (3.5) is equivalent to the following complex, linear, nonautonomous, Hamiltonian system with one degree of freedom:

$$\tilde{w}' = -\frac{\partial H(\tilde{w}, w, \tau)}{\partial w}, \quad w' = \frac{\partial H(\tilde{w}, w, \tau)}{\partial \tilde{w}} \tag{3.11}$$

with the Hamiltonian

$$H(\tilde{w}, w, \tau) = \frac{\tilde{w}^2}{2} + \left[\frac{k}{c_1^2} - \frac{c_2^2}{4c_1^2} + \varepsilon\frac{c_2 - c_1}{c_1}e^{\tau} - \varepsilon^2 e^{2\tau}\right]\frac{w^2}{2}, \tag{3.12}$$

where

$$\varepsilon = \frac{b_1}{c_1 v_0} = \frac{1}{2}\frac{J_2}{J_1}\frac{\omega_{s2}}{c_1 v_0} i. \tag{3.13}$$

Thus, we consider a linear, complex Hamiltonian system with the purely imaginary parameter ε.

4. Complete integral and the Jacobi equation. For integration of the system (3.11), it suffices to find its complete integral

$$S = S(\tau, w, \tilde{\alpha}),$$

where $\tilde{\alpha}$ is an arbitrary complex constant, as a solution of the complex Hamilton–Jacobi equation:

$$\frac{\partial S(\tau, w, \tilde{\alpha})}{\partial \tau} + H\left(\tilde{w}, \frac{\partial S(\tau, w, \tilde{\alpha})}{\partial w}, \tau\right) = 0 \tag{4.1}$$

(see [1–6, 8, 10, 13–16, 19, 20, 24, 25, 27, 33–35, 44, 50, 54–56, 58])

We search for a function $S = S(\tau, w, \tilde{\alpha})$ in the form

$$S = S(\tau, w, \tilde{\alpha}) = S_0(\tau, \tilde{\alpha}) \frac{w^2}{2}; \tag{4.2}$$

moreover, the partial differential equation (4.1) is reduced to the following Riccati ordinary differential equation (see [10, 16, 34]):

$$\frac{dS_0(\tau, \tilde{\alpha})}{d\tau} + S_0^2(\tau, \tilde{\alpha}) + \left[\frac{k}{c_1^2} - \frac{c_2^2}{4c_1^2} + \varepsilon \frac{c_2 - c_1}{c_1} e^{\tau} - \varepsilon^2 e^{2\tau}\right] = 0. \tag{4.3}$$

5. Some particular solutions of the Riccati equation. Equation (4.3) is a Riccati equation, and its general solution, in general, cannot be expressed through a finite combination of elementary functions (see [10, 16, 34]). However, it is known that if one knows a particular solution of this equation, then the general solution can also be obtained.

First, for simplicity, we search for a particular solution in the form

$$S_0 = \alpha + \beta e^{\tau}. \tag{5.1}$$

Then the coefficients α and β are defined by the following (in general, incompatible) complex algebraic equations:

$$\begin{aligned} &\beta^2 = \varepsilon^2, \\ &\alpha^2 = -\frac{k'}{c_1^2}, \\ &\beta + 2\alpha\beta + \frac{c_2 - c_1}{c_1}\varepsilon = 0, \end{aligned} \tag{5.2}$$

where $k' = k - c_2^2/4$.

Equations (5.2) are compatible under at least one of the following conditions for the coefficient k:

$$k = c_1(c_2 - c_1), \tag{5.3}$$

$$k = 0. \tag{5.4}$$

The cases (5.3) and (5.4), respectively, are characterized by the existence of the following particular solutions of Eq. (4.3):

$$S_{01} = \left(\frac{c_2}{2c_1} - 1\right) - \varepsilon e^{\tau}, \tag{5.5}$$

$$S_{02} = -\frac{c_2}{2c_1} + \varepsilon e^{\tau}. \tag{5.6}$$

6. Calculation of a complete integral of the Hamiltonian system in the case (5.3)**.** If condition (5.3) holds, then Eq. (4.3) takes the form

$$\frac{dS_0(\tau)}{d\tau} + S_0^2(\tau) + \left[-\left(1 - \frac{c_2}{2c_1}\right)^2 + \varepsilon \frac{c_2 - c_1}{c_1} e^{\tau} - \varepsilon^2 e^{2\tau} \right] = 0 \tag{6.1}$$

(we omit the complex constant $\tilde{\alpha}$). We will search for the general solution of Eq. (6.1) in the form

$$S_0 = S_{01} + z. \tag{6.2}$$

Indeed, this form is convenient since in the case (6.2) Eq. (6.1) is transformed into the Bernoulli equation

$$z' + 2S_{01}z + z^2 = 0, \tag{6.3}$$

which, in turn, can be easily reduced to the following linear inhomogeneous equation:

$$\frac{d}{d\tau}\left(\frac{1}{z}\right) - 2S_{01}\left(\frac{1}{z}\right) - 1 = 0. \tag{6.4}$$

The solution of Eq. (6.4) is expressed as follows:

$$\frac{1}{z} = \exp\left\{\left(-2 + \frac{c_2}{c_1}\right)\tau - 2\varepsilon \exp\{\tau\}\right\} \cdot \left[\int\limits_0^{\tau} \exp\left\{\left(2 - \frac{c_2}{c_1}\right)\xi + 2\varepsilon \exp\{\xi\}\right\} d\xi + \tilde{\alpha}\right], \tag{6.5}$$

where $\tilde{\alpha}$ is an arbitrary complex constant.

We write equality (6.5) in the form

$$z = \frac{\exp\{(2 - c_2/c_1)\tau + 2\varepsilon \exp\{\tau\}\}}{\tilde{\alpha} + \int\limits_0^{\tau} \exp\{(2 - c_2/c_1)\xi + 2\varepsilon \exp\{\xi\}\} d\xi}. \tag{6.6}$$

Thus, the general solution of Eq. (6.1) has the form

$$S_0(\tau, \tilde{\alpha}) = \left(\frac{c_2}{2c_1} - 1\right) - \varepsilon e^{\tau} + \frac{\Phi_1(\tau)}{\tilde{\alpha} + \int_0^{\tau} \Phi_1(\xi) d\xi}, \tag{6.7}$$

where

$$\Phi_1(\tau) = \exp\left\{\left(2 - \frac{c_2}{c_1}\right)\tau + 2\varepsilon \exp\{\tau\}\right\}. \tag{6.8}$$

So, in the case (5.3) we have obtained a complete integral of the Hamiltonian system (3.11), (3.12) corresponding to Eq. (3.5).

It is known (see [34, 54, 55]) that if the complete integral (4.2) is known, then integrals of the Hamiltonian system (3.11), (3.12) can be found from the following equalities:

$$\begin{aligned} \frac{\partial S(\tau, w, \tilde{\alpha})}{\partial w} &= \tilde{w}, \\ \frac{\partial S(\tau, w, \tilde{\alpha})}{\partial \tilde{\alpha}} &= -\tilde{\beta}' = \text{const}. \end{aligned} \tag{6.9}$$

From the second of Eqs. (6.9) we have

$$\frac{\partial S_0(\tau, \tilde{\alpha})}{\partial \tilde{\alpha}} \frac{w^2}{2} = -\frac{\Phi_1(\tau)}{\left[\tilde{\alpha} + \int\limits_0^{\tau} \Phi_1(\xi) d\xi\right]^2} \frac{w^2}{2} = -\frac{\tilde{\beta}_1}{2} = \text{const}. \tag{6.10}$$

Relation (6.10) allows one to find the dependence of the solution of the reduced equation (3.5) on the independent variable τ and on two arbitrary complex constants $\tilde{\alpha}$ and $\tilde{\beta}$ in the case (5.3):

$$w(\tau) = \tilde{\beta}\left[\exp\left\{\left(\frac{c_2}{2c_1} - 1\right)\tau - \varepsilon\exp\{\tau\}\right\}\right] \cdot \left[\tilde{\alpha} + \int\limits_0^\tau \Phi_1(\xi)d\xi\right], \tag{6.11}$$

where, as above,

$$\Phi_1(\tau) = \exp\left\{\left(2 - \frac{c_2}{c_1}\right)\tau + 2\varepsilon\exp\{\tau\}\right\}.$$

7. Calculation of a complete integral of the Hamiltonian system in the case (5.4)**.** Under condition (5.4), Eq. (4.3) takes the following form (we omit the complex constant $\tilde{\alpha}$):

$$\frac{dS_0(\tau)}{d\tau} + S_0^2(\tau) + \left[-\frac{c_2^2}{4c_1^2} + \varepsilon\frac{c_2 - c_1}{c_1}e^\tau - \varepsilon^2 e^{2\tau}\right] = 0. \tag{7.1}$$

We will search for the general solution of Eq. (7.1) in the form

$$S_0 = S_{02} + z. \tag{7.2}$$

Indeed, this is conveniaent since in the case (7.2), Eq. (6.1) is transformed into the Bernoulli equation

$$z' + 2S_{02}z + z^2 = 0, \tag{7.3}$$

which, in turn, can be easily reduced to the linear inhomogeneous equation

$$\frac{d}{d\tau}\left(\frac{1}{z}\right) - 2S_{02}\left(\frac{1}{z}\right) - 1 = 0. \tag{7.4}$$

The solution of Eq. (7.4) is represented in the following form:

$$\frac{1}{z} = \exp\left\{-\frac{c_2}{c_1}\tau + 2\varepsilon\exp\{\tau\}\right\} \cdot \left[\int\limits_0^\tau \exp\left\{\frac{c_2}{c_1}\xi - 2\varepsilon\exp\{\xi\}\right\}d\xi + \tilde{\alpha}\right], \tag{7.5}$$

where $\tilde{\alpha}$ is an arbitrary complex constant.

We write Eq. (7.5) in the form

$$z = \frac{\exp\{c_2\tau/c_1 - 2\varepsilon\exp\{\tau\}\}}{\tilde{\alpha} + \int\limits_0^\tau \exp\{c_2\xi/c_1 - 2\varepsilon\exp\{\xi\}\}d\xi}. \tag{7.6}$$

Thus, the general solution of Eq. (7.1) has the form

$$S_0(\tau, \tilde{\alpha}) = -\frac{c_2}{2c_1} + \varepsilon e^\tau + \frac{\Phi_2(\tau)}{\tilde{\alpha} + \int\limits_0^\tau \Phi_2(\xi)d\xi}, \tag{7.7}$$

where

$$\Phi_2(\tau) = \exp\left\{\frac{c_2}{c_1}\tau - 2\varepsilon\exp\{\tau\}\right\}. \tag{7.8}$$

Thus, in the case (5.4) we have obtained the complete integral of the Hamiltonian system (3.11), (3.12) corresponding to Eq. (3.5).

It is known (see [34, 54, 55]) that if the complete integral (4.2) is known, then integrals of the Hamiltonian system (3.11), (3.12) can be found from the following equalities:

$$\begin{gathered}\frac{\partial S(\tau, w, \tilde{\alpha})}{\partial w} = \tilde{w}, \\ \frac{\partial S(\tau, w, \tilde{\alpha})}{\partial \tilde{\alpha}} = -\tilde{\beta}' = \text{const.}\end{gathered} \tag{7.9}$$

From the second equation (7.9) we have

$$\frac{\partial S_0(\tau, \tilde{\alpha})}{\partial \tilde{\alpha}} \frac{w^2}{2} = -\frac{\Phi_2(\tau)}{\left[\tilde{\alpha} + \int\limits_0^\tau \Phi_2(\xi) d\xi\right]^2} \frac{w^2}{2} = -\frac{\tilde{\beta}_2}{2} = \text{const.} \tag{7.10}$$

Relation (7.10) allows one to find the dependence of the solution of the reduced equation (3.5) on the independent variable τ and on two arbitrary complex constants $\tilde{\alpha}$ and $\tilde{\beta}$ in the case (5.4):

$$w(\tau) = \tilde{\beta}\left[\exp\left\{-\frac{c_2}{2c_1}\tau + \varepsilon \exp\{\tau\}\right\}\right] \cdot \left[\tilde{\alpha} + \int\limits_0^\tau \Phi_2(\xi) d\xi\right], \tag{7.11}$$

where, as above,

$$\Phi_2(\tau) = \exp\left\{\frac{c_2}{c_1}\tau - 2\varepsilon \exp\{\tau\}\right\}.$$

Thus, the solution of the initial equation (2.1) in the cases (5.3) and (5.4) can be found from formulas (7.9) and (6.9) after reduction of the auxiliary variables w and τ to the initial variables y and t.

8. Reduction to initial variables. We return to the initial variables and rewrite the solutions of the complex differential equations found above.

The required solution of Eq. (2.1) is expressed by the formula

$$y(t) = w(t) u_1(t) u_2(t), \tag{8.1}$$

where

$$u_1(t) = (1 + c_1 v_0 t)^{\frac{c_1 - c_2}{2c_1}} \cdot \exp\left\{\left(1 - \frac{J_2}{2J_1}\right)\omega_{s2} i t\right\}, \tag{8.2}$$

$$u_2(t) = \exp\left\{\frac{\tau}{2}\right\} = (1 + c_1 v_0 t)^{1/2}. \tag{8.3}$$

We obtain the following intermediate result:

$$y(t) = w(t)\left[(1 + c_1 v_0 t)^{\frac{2c_1 - c_2}{2c_1}} \cdot \exp\left\{\left(1 - \frac{J_2}{2J_1}\right)\omega_{s2} i t\right\}\right]. \tag{8.4}$$

Recalling the representation

$$e^\tau = 1 + c_1 v_0 t \tag{8.5}$$

of the independent variable for the cases (5.3) and (5.4), respectively, we obtain

$$w(t) = \tilde{\beta}(1 + c_1 v_0 t)^{\frac{c_2 - 2c_1}{2c_1}} \cdot \exp\{-\varepsilon(1 + c_1 v_0 t)\} \cdot \left[\tilde{\alpha} + \int\limits_0^t \Psi_1(\xi) d\xi\right], \tag{8.6}$$

where

$$\Psi_1(t) = (1 + c_1 v_0 t)^{\frac{2c_1 - c_2}{c_1}} \exp\{2\varepsilon(1 + c_1 v_0 t)\},$$

and

$$w(t) = \tilde{\beta}(1 + c_1 v_0 t)^{-\frac{c_2}{2c_1}} \cdot \exp\{\varepsilon(1 + c_1 v_0 t)\} \cdot \left[\tilde{\alpha} + \int\limits_0^t \Psi_2(\xi) d\xi \right], \tag{8.7}$$

where

$$\Psi_2(t) = (1 + c_1 v_0 t)^{\frac{c_2}{c_1}} \exp\{-2\varepsilon(1 + c_1 v_0 t)\}.$$

Thus, returning to the initial notation and taking into account the relation

$$\varepsilon = \frac{b_1}{c_1 v_0} = \frac{1}{2} \frac{J_2}{J_1} \frac{\omega_{s2}}{c_1 v_0} i, \tag{8.8}$$

we finally obtain for the cases (5.3) and (5.4), respectively,

$$y(t) = \tilde{\beta} \exp\left\{ \left(1 - \frac{J_2}{J_1}\right) \omega_{s2} i t \right\} \cdot \left[\tilde{\alpha} + \int\limits_0^t \Psi_1(\xi) d\xi \right], \tag{8.9}$$

$$y(t) = \tilde{\beta}(1 + c_1 v_0 t)^{\frac{c_1 - c_2}{c_1}} \cdot \exp\{\omega_{s2} i t\} \cdot \left[\tilde{\alpha} + \int\limits_0^t \Psi_2(\xi) d\xi \right]. \tag{8.10}$$

9. Construction of the general solution of the equation in the form of a series with respect to a small parameter. We introduce the following notation:

$$k_1 = \frac{4k - c_2^2}{4c_1^2}, \quad c_0 = \frac{c_2 - c_1}{c_1}. \tag{9.1}$$

Then Eq. (3.5) becomes

$$w'' + \left[k_1 + \varepsilon c_0 e^{\tau} - \varepsilon^2 e^{2\tau}\right] w = 0. \tag{9.2}$$

We search for a solution of this equation in the form of a series in ε:

$$w(\tau, \varepsilon) = \sum_{n=0}^{\infty} w_n(\tau) \varepsilon^n. \tag{9.3}$$

Substituting solution (9.3) into Eq. (9.2), we obtain the following infinite system of linear differential equations:

$$w_0'' + k_1 w_0 = 0, \tag{9.4}$$

$$w_1'' + k_1 w_1 + c_0 e^{\tau} w_0 = 0, \tag{9.5}$$

$$w_n'' + k_1 w_n + c_0 e^{\tau} w_{n-1} - e^{2\tau} w_{n-2} = 0, \quad n \geq 2. \tag{9.6}$$

System (9.4)–(9.6) defines an infinite system of recurrent linear inhomogeneous equation. We search for its solution in the form

$$w_n(\tau) = w_{nO}(\tau) + w_{nH}(\tau), \quad n \geq 0, \tag{9.7}$$

where $w_{nO}(\tau)$ is a solution of the corresponding homogeneous equation

$$w_{nO}'' + k_1 w_{nO} = 0, \quad n \geq 0, \tag{9.8}$$

and $w_{nH}(\tau)$ is a solution of the corresponding inhomogeneous equation

$$w_{nH}'' + k_1 w_{nH} + c_0 e^{\tau} w_{n-1} - e^{2\tau} w_{n-2} = 0, \quad n \geq 0. \tag{9.9}$$

Formally

$$w_{-2}(\tau) \equiv w_{-1}(\tau) \equiv 0$$

and

$$w_{0H}(\tau) \equiv 0$$

by Eq. (9.4).

Obviously, for $n \geq 0$, the general solution of the homogeneous equation (9.8) has the form

$$w_{nO} = C_1^n e^{\omega_0^+ \tau} + C_2^n e^{\omega_0^- \tau}, \tag{9.10}$$

where

$$\omega_0^{\pm} = 0 \pm \sqrt{k_1} i. \tag{9.11}$$

Moreover, we will henceforth need the values

$$\omega_s^{\pm} = s \pm \sqrt{k_1} i, \quad s \in \mathbb{N}; \tag{9.12}$$

they satisfy the relation

$$\omega_s^{\pm 2} + k_1 = s^2 \pm 2s\sqrt{k_1} i. \tag{9.13}$$

The equation for w_{1H} becomes

$$w''_{1H} + k_1 w_{1H} + c_0 \left[C_1^0 e^{\omega_1^+ \tau} + C_2^0 e^{\omega_1^- \tau} \right] = 0. \tag{9.14}$$

We will search for its particular solution in the form

$$w_{1H}(\tau) = A_1^1 e^{\omega_1^+ \tau} + A_2^1 e^{\omega_1^- \tau}. \tag{9.15}$$

The constants A_k^1, $k = 1, 2$, are defined by the equalities

$$\begin{aligned} A_1^1\left(C_1^0\right) &= -\frac{c_0 C_1^0}{\omega_1^{+2} + k_1}, \\ A_2^1\left(C_2^0\right) &= -\frac{c_0 C_2^0}{\omega_1^{-2} + k_1}. \end{aligned} \tag{9.16}$$

Then the general solution of Eq. (9.5) has the form

$$w_1(\tau) = C_1^1 e^{\omega_0^+ \tau} + C_2^1 e^{\omega_0^- \tau} + A_1^1\left(C_1^0\right) e^{\omega_1^+ \tau} + A_2^1\left(C_2^0\right) e^{\omega_1^- \tau}. \tag{9.17}$$

Further, we have the equation for w_{2H}:

$$\begin{aligned} w''_{2H} + k_1 w_{2H} + c_0 &\left[C_1^1 e^{\omega_1^+ \tau} + C_2^1 e^{\omega_1^- \tau} \right] \\ &+ \left[c_0 A_1^1\left(C_1^0\right) - C_1^0 \right] e^{\omega_2^+ \tau} + \left[c_0 A_2^1\left(C_2^0\right) - C_2^0 \right] e^{\omega_2^- \tau} = 0. \end{aligned} \tag{9.18}$$

We will search for its particular solution in the form

$$w_{2H}(\tau) = A_1^1\left(C_1^1\right) e^{\omega_1^+ \tau} + A_2^1\left(C_2^1\right) e^{\omega_1^- \tau} + A_1^2 e^{\omega_2^+ \tau} + A_2^2 e^{\omega_2^- \tau}. \tag{9.19}$$

The constants A_k^2, $k = 1, 2$, are defined by the equalities

$$\begin{aligned} A_1^2\left(C_1^0\right) &= -\frac{c_0 A_1^1\left(C_1^0\right) - C_1^0}{\omega_2^{+2} + k_1}, \\ A_2^2\left(C_2^0\right) &= -\frac{c_0 A_2^1\left(C_2^0\right) - C_2^0}{\omega_2^{-2} + k_1}. \end{aligned} \tag{9.20}$$

Then the general solution of Eq. (9.6) for $n = 2$ has the form

$$\begin{aligned} w_2(\tau) = C_1^2 e^{\omega_0^+ \tau} &+ C_2^2 e^{\omega_0^- \tau} + A_1^1\left(C_1^1\right) e^{\omega_1^+ \tau} \\ &+ A_2^1\left(C_2^1\right) e^{\omega_1^- \tau} + A_1^2\left(C_1^0\right) e^{\omega_2^+ \tau} + A_2^2\left(C_2^0\right) e^{\omega_2^- \tau}. \end{aligned} \tag{9.21}$$

Further, we have the equation for w_{3H}:

$$w''_{3H} + k_1 w_{3H} + c_0 \left[C_1^2 e^{\omega_1^+ \tau} + C_2^2 e^{\omega_1^- \tau}\right] + \left[c_0 A_1^1 \left(C_1^1\right) - C_1^1\right] e^{\omega_2^+ \tau} + \left[c_0 A_2^1 \left(C_2^1\right) - C_2^1\right] e^{\omega_2^- \tau} + \left[c_0 A_1^2 \left(C_1^0\right) - A_1^1 \left(C_1^0\right)\right] e^{\omega_3^+ \tau} + \left[c_0 A_2^2 \left(C_2^0\right) - A_2^1 \left(C_2^0\right)\right] e^{\omega_3^- \tau} = 0. \tag{9.22}$$

We will search for its particular solution in the form

$$w_{3H}(\tau) = A_1^1 \left(C_1^2\right) e^{\omega_1^+ \tau} + A_2^1 \left(C_2^2\right) e^{\omega_1^- \tau} + A_1^2 \left(C_1^1\right) e^{\omega_2^+ \tau} + A_2^2 \left(C_2^1\right) e^{\omega_2^- \tau} + A_1^3 e^{\omega_3^+ \tau} + A_2^3 e^{\omega_3^- \tau}. \tag{9.23}$$

The constants A_k^3, $k = 1, 2$, are defined by the equalities

$$A_1^3 \left(C_1^0\right) = -\frac{c_0 A_1^2 \left(C_1^0\right) - A_1^1 \left(C_1^0\right)}{\omega_3^{+2} + k_1}, \quad A_2^3 \left(C_2^0\right) = -\frac{c_0 A_2^2 \left(C_2^0\right) - A_2^1 \left(C_2^0\right)}{\omega_3^{-2} + k_1}. \tag{9.24}$$

Then the general solution of Eq. (9.6) for $n = 3$ has the form

$$w_3(\tau) = C_1^3 e^{\omega_0^+ \tau} + C_2^3 e^{\omega_0^- \tau} + A_1^1 \left(C_1^2\right) e^{\omega_1^+ \tau} + A_2^1 \left(C_2^2\right) e^{\omega_1^- \tau} + A_1^2 \left(C_1^1\right) e^{\omega_2^+ \tau} + A_2^2 \left(C_2^1\right) e^{\omega_2^- \tau} + A_1^3 \left(C_1^0\right) e^{\omega_3^+ \tau} + A_2^3 \left(C_2^0\right) e^{\omega_3^- \tau}. \tag{9.25}$$

It is easy to prove that the general solution of the equation for w_n has the form

$$w_n(\tau) = C_1^n e^{\omega_0^+ \tau} + C_2^n e^{\omega_0^- \tau} + \sum_{s=1}^{n} \left[A_1^s \left(C_1^{n-s}\right) e^{\omega_s^+ \tau} + A_2^s \left(C_2^{n-s}\right) e^{\omega_s^- \tau}\right], \tag{9.26}$$

where, by (9.12),

$$A_1^s(p) = -\frac{c_0 A_1^{s-1}(p) - A_1^{s-2}(p)}{s^2 + 2s\sqrt{k_1} i}, \quad A_2^s(p) = -\frac{c_0 A_2^{s-1}(p) - A_2^{s-2}(p)}{s^2 - 2s\sqrt{k_1} i}, \quad s \geq 1; \tag{9.27}$$

moreover,

$$A_k^0(p) \equiv p, \quad A_k^{-1}(p) \equiv 0, \quad k = 1, 2. \tag{9.28}$$

10. Convergence theorem. Now we examine the convergence of the functional series (9.3) of two variables τ and ε with the general term (9.26)–(9.28) depending on τ. For this purpose, we estimate it on an arbitrarily large segment of length L for $\tau \in [0, L]$.

Indeed, since the initial conditions are chosen from a bounded set of the complex plane, there exist positive constants M_k, $k = 1, \ldots, 4$, such that for any $\tau \in [0, L]$ we have the estimate

$$|w_n(\tau)| \leq |C_1^n| + |C_2^n| + \sum_{s=1}^{n} \left[\left(\left|A_1^s \left(C_1^{n-s}\right)\right| + \left|A_2^s \left(C_2^{n-s}\right)\right|\right) e^{s\tau}\right]$$
$$\leq M_1 + \sum_{s=1}^{n} \left[\left(\left|A_1^s \left(C_1^{n-s}\right)\right| + \left|A_2^s \left(C_2^{n-s}\right)\right|\right) e^{sL}\right] \leq M_1 + n e^{nL} \sum_{s=1}^{n} \left[\left|A_1^s \left(C_1^{n-s}\right)\right| + \left|A_2^s \left(C_2^{n-s}\right)\right|\right]$$
$$\leq M_1 + n e^{nL} \sum_{s=1}^{n} \left[M_2 \prod_{s=1}^{n} \frac{1}{s^2} + M_3 \prod_{s=1}^{n} \frac{1}{s^2}\right] \leq M_1 + M_4 \frac{n^2 e^{nL}}{(n!)^2} = M_1 + M_4 a_n, \tag{10.1}$$

where the real-valued numerical series

$$\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} \frac{n^2 e^{nL}}{(n!)^2} \tag{10.2}$$

converges.

Indeed,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{e^L}{(n+1)^2} = 0. \tag{10.3}$$

Further, we calculate the convergence radius R of the complex power series

$$\sum_{n=0}^{\infty} a_n \varepsilon^n. \tag{10.4}$$

We have the equalities

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^2 e^{nL}}{(n!)^2}} = e^L \lim_{n\to\infty} \left(\frac{n}{n!}\right)^{2/n} = 0. \tag{10.5}$$

Thus, the convergence radius is equal to ∞.

Theorem (convergence theorem)**.** *The power series* (9.3) *with the general term* (9.26)–(9.28) *converges absolutely and uniformly for any* $\varepsilon \in \mathbb{C}$ *on the segment* $[0, L]$ *for any* $L > 0$.

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Yu. M. Okunev
Institute of Mechanics of the M. V. Lomonosov Moscow State University, Moscow, Russia
E-mail: `common@imec.msu.ru`
M. V. Shamolin
Institute of Mechanics of the M. V. Lomonosov Moscow State University, Moscow, Russia
E-mail: `shamolin@imec.msu.ru`