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TOPOLOGY ON POLYNUMBERS AND FRACTALS

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For the definition of Julia and Mandelbrot sets, it is necessary to define convergence to infinity of sequences of polynumbers (i.e., elements of a finite-dimensional associative algebra over real numbers, see [5, 6]). The notion of convergence to zero in a vector space allows one to define convergence to any point by using a translation. Infinite points can be introduced by compactifications of the space.

In the case of polynumbers, it is easier to define convergence of a sequence x_n to infinity as the convergence to zero of the sequence $1/x_n$. However, reciprocal values for polynumbers are not always defined, so we must also generalize the definition for this case. For the definition of fractals, we also must define admissible functions $f(x)$ whose iterations are calculated for constructing the corresponding fractal sets.

The notion of a converging sequence can be introduced without any topology, by the definition of all sequences converging to some point. The following natural requirements must be fulfilled.

(1) *Filter property.* If a sequence x_n converges to x and a new sequence y_n is obtained from the sequence x_n by adding or removing a finite number of terms, then the sequence y_n also converges to x.
(2) *Convergence property of a simple ultrafilter.* A sequence all of whose terms are equal to x converges to x.

Since we consider only finite-dimensional vector spaces over $\mathbb{R}$, it is natural to require the agreement of this definition with the axioms of vector spaces. The space of all sequences can be equipped with the natural structure of a vector space in which the sum of two sequences and the product of a sequence by a number are defined componentwise. Then the conditions of compatibility with the vector-space structure are as follows:

(3) If a sequence x_n converges to zero, then the sequence $y_n = x_n + x$ converges to x.
(4) If two sequences converge to zero, then their sum also converges to zero.
(5) If a sequence converges to zero, then the product of this sequence with any number also converges to zero.

To define the convergence structure, it suffices to define a set of sequences converging to zero and satisfying conditions (1), (2), (4), and (5). They form a linear subspace closed with respect to the operations

$$x_n \mapsto y_n = x_{n+1}$$

and

$$x_n \mapsto y_n = x_{n-1}, \quad n > 1, \quad y_1 \in \{0, 1\},$$

and are completely defined by these properties. In this way, a *filtered* vector space is defined (sometimes, the term *pseudo-topological space* is used, see [5]); it is a more general notion than the notion of a topological space.

There is a very large number of such spaces. The following requirement that strengthens property (5) is usually added:

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(6) If a sequence x_n converges to zero and a sequence of real numbers α_n satisfies the condition $|\alpha_n| \le 1$, then the sequence $\alpha_n x_n$ converges to zero.

This property is called the *star property* for filtered spaces. It essentially restricts the number of continuous structures. However, these conditions are also sufficiently general and the number of such structures of a set grows superexponentially (generally speaking, their cardinality is greater than $2^{\mathrm{Card}(X)}$; see [6]).

The necessity of a transition to filter spaces that are more general than topological spaces appears in the construction of differential calculus on infinite-dimensional vector spaces without norm (see [5]).

However, in this case, we can make a shift by filtered spaces that satisfy the following condition: if a set of filters converges to a point, then the filter, which is the intersection (as a subset of the set of all subsets of X) of all of these filters, also converges to this point. The number of such structures grows only exponentially as a function of the cardinality of the set X, and topological structures can be defined as "cooperations" (see [6]).

As in the case of topological spaces, such spaces can be introduced using three types of axioms (see [2]), namely, by converging filters, by the closure operation, and by neighborhoods. However, there are substantial differences between these notions in the general case; they are related to the fact that closed sets are not closed absolutely and open sets are not open absolutely. We will describe them in terms of the closure operation.

On the set of subsets of X, we can define a monotonic closure operation satisfying the following conditions:

(1) the closure $\bar{A}$ of a set A contains A itself;
(2) If $X \subseteq Y$, then for their closures we have the inclusion $\bar{X} \subseteq \bar{Y}$ (monotonicity of the closure operation);
(3) the closure of the empty set is empty;
(4) the closure of the union of sets A and B is the union of their closures.

Sets with such closure operations are called *quasi-topological* spaces.

With such a description, topological spaces satisfy the following additional axiom: the closure of a closed set coincides with this set, i.e., the closure operation is the projection of the set of all subsets of the topological space onto the set of closed sets.

In contrast to classical topology, closed sets (closures of certain subsets) in a quasi-topological space are not quite closed, i.e., generally speaking, their closures are larger than they are themselves. Moreover, the projectivity condition for the closure operation is not fulfilled (an operation (morphism) is called a *projector* if it coincides with its square).

These features also appear in properties of open sets and neighborhoods. Nevertheless, these spaces, which are more general than topological spaces, are more convenient for applications in the theory of calculations with errors, for the construction of the differential calculus in infinite-dimensional spaces (see [5]), and for applications in theoretical physics.

The first two axioms are natural for the closure operation. The third axiom does not significantly affect the theory and is adopted basically for convenience.

We explain the fourth axiom. We call spaces satisfying the first three axioms *pseudo-topological* spaces. Such spaces form a fine continuous structure. The continuity of a mapping $f : A \to B$ is treated in the ordinary sense:

$$f(\overline{X}) \to \overline{f(X)}$$

for any subset X.

All these spaces can be described dually using the operation of taking the interior; this is similar to the description of topological spaces by sets of closed sets or by sets of open sets. The notion of limits is related to the notion of convergence.

For this purpose, we define the category of *pseudo-filter* spaces in which there is a relation between filters and points (convergence of a filter to a point) satisfying the following conditions:

(1) an ultra-filter $[x] \downarrow x$ converges to x (here and in what follows, we denote by $[x]$ the ultra-filter which is the set of all subsets containing the point x);
(2) if the property $F \downarrow x$ holds and F_1 is a filter subtler than F, $F \subseteq F_1$, then it also converges to x;
(3) the ultra-filter $[x]$ converges only to the point x.

Here the axioms approximately correspond to the axioms for pseudo-topologies. (However, the correspondence is not complete since quite different continuous structures are considered.) A continuous mapping is defined as a mapping that transforms converging filters into converging filters; more precisely, a mapping $f : A \to B$ is said to be continuous if $F \downarrow x$ implies $f(F) \downarrow f(x)$, where $f(F)$ is the set of all subsets that contain a certain set $f(X)$, $X \in F$.

It is easy to define the structural functor (i.e., the functor that does not change sets) $\mathcal{F}_1$ from the category of pseudo-filter spaces in the category of pseudo-topological spaces: we define the closure of a set X in a pseudo-filter space as the set of all points to which the set of all filters that are subtler than the filter $[X]$ converges. It turns out that the image of this functor does not contain all pseudo-topological spaces but consists only of quasi-topological spaces. The definition of continuity using filters approximately corresponds to the description of topology by neighborhoods.

In a pseudo-topological space, we define an open neighborhood U of a point x as the set that contains x and is the complement of the closure of a certain set. This allows one to define the functor $\mathcal{F}_2$ from the category of pseudo-topological spaces in the category of pseudo-filter spaces as follows: a filter F converges to a point x if and only if it is subtler than the filter of all neighborhoods of the point x. The image of this functor is the category of the quasi-filter spaces satisfying the following additional axiom:

(4) The filter which is the intersection (as the set of subsets) of all filters converging to the point x also converges to x.

Here the fourth axiom does not correspond to the fourth axiom of the quasi-topology. The property of filters themselves is an analog of this axiom: the intersection of two sets from a filter also belongs to the filter.

Is it easy to verify that the restrictions of these functors to quasi-topological and quasi-filter spaces are mutually inverse. This means that the filter description is another description of quasi-topological spaces.

Fröhlicher and Bucher also used quasi-topological spaces for the development of differential calculus in vector spaces without a norm (see [5]); however, they did not take into account the details mentioned above. The discussion of the need to use more general spaces than topological spaces for the construction of differential calculus from the point of view of category theory requires a separate study; in this paper, we briefly touch upon this issue.

By [6], differential structures form a sub-type of continuous structures. A differential structure is defined by the discriminating differentiable functions and by the definition of a projection from the set of differentiable functions to the set of polylinear mappings of tangent spaces. The coincidence of the double and ordinary closures is an algebraic rather than a topological property; it is used mostly in algebraic topology.

For purely continuous structures, this property is an obstacle for the continuity (in the sense of [6]) of the functors defined; it also creates certain obstacles for the definition of differential structures in spaces without norms. We briefly discuss obstacles created by this property for the definition of continuous structures in the Finsler geometry of Minkowski type (with a concave indicatrix).

For simplicity, we consider the two-dimensional space in which neighborhoods of a point (x_0, y_0) are sets containing the sets $|(x - x_0)(y - y_0)| < a$, $a > 0$, for some a.

Is it easy to verify that a nonempty set which is a neighborhood of each its points coincides with the whole plane. Therefore, there does not exist a topological structure (except for the trivial structure containing only two open sets, the empty set and the space itself) which is consistent with the Finsler metric of Minkowski type.

At the same time, these neighborhoods define a quasi-topology consistent with the Finsler geometry. Quasi-topologies are also consistent with roughly rounded structures and structures of fuzzy sets. These structures are not consistent with the topological property of coincidence of double and single closures. These facts indicate that the restriction by topological spaces in the definition of continuous structures is unnatural. As we see below, the same topology is not consistent with the continuity of such numbers in the sets of polynumbers with divisors of zero.

The set of polynumbers is equipped with multiplication, which defines an algebraic norm as the determinant of the multiplication matrix that represents an element x. (Recall that polynumbers are finite-dimensional associative algebras.) For polynumbers, the determinant of the pre-multiplication matrix coincides with the determinant of the post-multiplication matrix (they are the determinants of mutually transpose matrixes). An algebraic norm satisfies the condition

$$|xy| = |x| \cdot |y|.$$

However, there are many functions satisfying only this property. For example, on the set of hypercomplex numbers H_n with a basis consisting of diagonal matrixes we can define the following property:

$$|x| = |x_1|^{a_1} \cdots |x|^{a_n}.$$

In the definition of a continuous structure consistent with the algebraic norm, we can introduce the following semi-norm:

$$\|x\| = (|(|x|)|)^a,$$

where a is some number. The continuous structure itself is independent of a, so we choose it from the condition

$$\|\delta x\| = |\delta| \cdot \|x\|. \tag{1}$$

A semi-norm can also be defined using a continuous structure. For any vector (polynumber) x, we define a positive number $\theta = \|x\|$ such that the sequence

$$x_n = y^n, \quad y = \alpha x, \quad |\alpha| < \frac{1}{\theta},$$

converges to zero. If such a number exists for any vector, then the continuous structure is said to be *consistent with the semi-norm.* Clearly, the semi-norm of the zero vector is equal to zero. Vectors with semi-norm 1 represent the indicatrix of some (symmetric with respect to change of sign) Finsler geometry.

A continuous structure in which there exists the corresponding semi-norm $\theta = \|x\|$ defines a topology consistent with the semi-norm; however, it is possible that the continuous structure itself is not defined by any topology.

Now we consider the definitions of the Mandelbrot and Julia sets for commutative polynumbers. In this case, the algebra of polynumbers itself is the direct sum of several algebras of real and complex numbers:

$$\mathbb{P} = \mathbb{R}^k + \mathbb{C}^m, \tag{2}$$

where functions are calculated componentwise. Therefore, we must examine iterations of the mappings

$$z_{n+1} = f(z_n, c) \tag{3}$$

in algebras of real and complex numbers. In algebras of real numbers this process is easily calculated in terms of the behavior of orbits. For this purpose, we calculate stationary points (one-point orbits) $z = f(z, c)$ and also k-point orbits satisfying the condition

$$z = f^{(k)}(z, c).$$

Points for which the numbers of points in the orbit are divisors of k also can be solutions of the last equation. The other points have infinite orbit. Usually, they are contained in an attractor (finite or infinite) of a stationary point or a finite cycle, which is defined as a stationary point of the kth iteration of the function.

This can be easily illustrated by polynomial functions. A polynomial of zero degree has a unique orbit and empty Mandelbrot and Julia sets. A first-degree polynomial

$$f(x, c) = ax + c$$

for $|a| > 1$ has a stable stationary point at infinity and an unstable stationary point

$$x_* = \frac{c}{1-a}. \tag{4}$$

Therefore, the Mandelbrot set in this case consists of the unique zero point and the Julia set consists of the unique point x_*.

If we consider the set of hypercomplex numbers H_k, we must clarify a continuous structure in this space. We find that the ith component is defined by the formula

$$x_{ni} = a_i^n (x_{0i} - x_{*i}) + x_{*i}. \tag{5}$$

Remark 1. If we introduce the product topology (which is a Euclidean topology), sequences of points from an orbit tend to infinity if at least one of these components tends to infinity. This occurs when $|a_i| > 1$ or $a_i = 1$ and $c_i \neq 0$ for at least one of the components.

Remark 2. One can try to introduce a topology with respect to a semi-norm assuming that a numerical sequence converges to zero (respectively, to infinity) if and only if the semi-norms of terms of the sequence converge to zero (respectively, to infinity). In this case, all orbits recede to infinity if the absolute value of the product $a_1 a_2 \dots a_k$ is greater than 1 or is equal to 1 but for at least one of the components we have $a_i = 1$, $c_i \neq 0$.

In other cases, the sequence does not tend to infinity. This defines a "topology" that does not satisfy even the weakest separability axiom. For this "topology" (we use the quotes since, as a rule, such continuous structures do not correspond to any topology), any function $H_k \to H_m$ one of whose components is constant is continuous. This "topology" is the most unnatural for these spaces.

In this case, the Mandelbrot set coincides with the whole space if the absolute value of the product of the coefficients is less than 1 or is equal to 1 but there are no components such that $a_i = 1$, $c_i \neq 0$.

In the opposite case, the Mandelbrot set consists of the union of the hyperplanes $c_i = 0$ (the set of points one of whose coordinates is zero). The Julia set in the first case is the empty set and in the second case is the union of hyperplanes $x_i = x_{*i}$. A detailed analysis shows that this continuous structure is not a topological space.

Remark 3. The most natural continuous structure consistent with the semi-norm is a quasi-topology whose base of neighborhoods of zero is defined by the set

$$\left| \prod_{x_i \neq 0} x_i \right| < \frac{1}{N}.$$

Then on sections with one constant (perhaps, zero) coordinate, neighborhoods cut the same sets as neighborhoods in the set H_{k-1}. In particular, a sequence in H_2 one of whose components is zero converges to some point if and only if the other component also converges.

In the previous unnatural quasi-topology, any sequence converges to zero since its semi-norm is zero. The obtained quasi-topology is not a Hausdorff topology, but it satisfies the separability axiom $\mathbf{T}_1$.

Within the physical framework, points possessing several (but not all) identical coordinates correspond to events belonging to the same light cone. These points were not separable with respect to the old semi-norm, however, because of the specificity of the new semi-norm and the corresponding quasi-topology, they are separable as in $\mathbf{T}_1$.

Finsler spaces of Euclidean type are defined by a usual metric and a usual topology. We show, using the example of polynumbers, that quasi-topologies for Finsler spaces of Minkowski type can be defined by a pseudo-metric described below.

First, we introduce on $\mathbb{R}^n$ a lexicographic order: we assume that $(a_1, \ldots, a_n) > (b_1, \ldots, b_n)$ if $a_j = b_j$, $j < i$, and $a_i > b_i$, $i \leq n$.

A *pseudo-norm* on H_n is a function $|\ldots| : H_n \to \mathbb{R}^n_+$ defined componentwise as follows:

$$|h_1, \ldots, h_n| = \left(|h_1 \ldots h_n|^{1/n} \left| \max \prod h_j \right|^{1/(n-1)}, \ldots, |\max(h_j)| \right),$$

where the ith position is occupied by $(\max(|h_{j_1} \ldots h_{j_k}|))^{1/k}$ over all sets of $k = n + 1 - i$ different coordinates h_j.

This pseudo-norm possesses the following properties. It is nonnegative and vanishes only in the case where all coordinates are zero. It satisfies the inverse triangle inequality (see [1]):

$$\left|(u_1, \ldots, u_n) + (v_1, \ldots, v_n)\right| \geq |u_1, \ldots, u_n| + |(v_1, \ldots, v_n)|$$

for any two commensurable vectors (i.e., vectors such that $u_i v_i \geq 0$ for all i); equality holds if and only if the vectors define the same direction (are proportional with a positive coefficient in the case of nonzero vectors).

We also note that in physics extremums are taken only over commensurable paths (when all tangent vectors are commensurable) and instead of the minimum in Minkowski space one finds the maximum. Thus, the length of an interval of commensurable paths for the case of straight-line paths attains the maximum. The pseudo-norm is also a homogeneous function of degree 1.

By the pseudo-norm introduced above, we can define the pseudo-metric $\rho(u, v) = |u - v|$ with similar properties and (pseudo)neighborhoods in the quasi-topology.

In this case, the Mandelbrot set consists of those c such that for the set of nonzero components $c_i \neq 0$, the absolute value of the product of the coefficients a_i is less than or equal to 1 but among them there is no coefficient such that $a_j = 1$. The Julia set is empty only if the absolute values of all coefficients do not exceed 1 or, in the case where $a_i = 1$, we have $c_i = 0$. If it is nonempty, then it is the union of planes determined by the equations

$$x_{i_1} = x_{*i_1}, \quad \ldots, \quad x_{i_l} = x_{*i_l}, \tag{6}$$

so that the absolute value of the product of the coefficients a_j that do not belong to the fixed values (6) does not exceed 1 (if there is 1 among them, then we have the corresponding relation $c_j = 0$), and the supplement of any a_{i_j} from the list (6) in the product makes the absolute value of this product greater than or equal to 1. Moreover, the supplemented coefficient satisfies the properties $a_{i_j} = 1$ and $c_{i_j} \neq 0$.

A detailed analysis also shows that this continuous structure cannot be defined by any topology since open neighborhoods are not neighborhoods of all their points. However, this structure is defined by a quasi-topology in the sense discussed above. We can show that in polynumber sets with divisors of zero, there are no topologies consistent with semi-norms. The best continuous structure for them is a quasi-topology of the form specified above.

We see that even in the case of a first-degree polynomial we must clarify the topology of the space and examine a variety of situations typical for higher-order polynomials. These examples also show that the construction of the Julia sets in [6] contains serious errors, and one cannot construct Julia sets only by computer simulations. To explain this situation, we consider the following example in H_2 with functions of the form

$$x_{n+1} = 0.6(x_n - x_*) + x_*, \quad y_{n+1} = 2(y_n - y_*) + y_* \tag{7}$$

(we denote by x and y the two components of a polynumber).

For any C, we can easily calculate the set of points (x_0, y_0) for which $\|(x_n, y_n)\| = C$. This set consists of branches of hyperbolas with axes $x = x_*$ and $y = y_*$. This occurs also after the millionth iteration. However, we cannot conclude that the Julia set consists of hyperbolas. In the case where functions are nonlinear, these sets of hyperbolas "breed," and the Julia set seems to be a set with a large number of "whiskers." We consider this after examining quadratic functions.

In the case of polynomials of higher degrees, using affine changes of components, one can reduce the problem to the form where the leading coefficient equals 1 while the next coefficient vanishes. For polynomials of second degree this leads to the following recurrence:

$$x_{n+1,i} = x_{n,i}^2 + c_i. \tag{8}$$

In this case all features of higher-order cases appear: chaotic trajectories, limit cycles, and stable and unstable stationary points. Moreover, the infinite point (which is a singular point) is a stable stationary point with respect to each component. After examining this case, it is easy to study any polynomial function.

In the case (8), in addition to the infinite stationary point, we have the following relations:

$$x_{*i} = \frac{1 - \sqrt{1 - 4c_i}}{2}, \quad x_{**i} = \frac{1 + \sqrt{1 - 4c_i}}{2}. \tag{9}$$

If $c_i > 1/4$, then the sequence x_{ni} tends to infinity *superexponentially* for any initial value. If $c_i = 1/4$, then the sequence x_{ni} tends to infinity superexponentially except for the case where the initial point satisfies the condition $|x_{0i}| \leq 1/2$. This stationary point is unstable from the right and is stable from the left. Therefore, if

$$|x_{0i}| \leq x_{**i}, \tag{10}$$

then the sequence x_{ni} tends to x_{**i}, otherwise, it tends to infinity.

If $c_i < 1/4$, we have two finite stationary points defined by Eqs. (9). The upper stationary point is always unstable (from both directions). The lower stationary point is stable if and only if

$$-\frac{3}{4} \leq c_i < \frac{1}{4}.$$

In this case, if the strong inequality (10) holds, then the sequence tends to the lower stationary point. If an equality holds in (10), then the sequence after the zero term coincides with the upper unstable stationary point. If inequality (10) does not hold, then the sequence always superexponentially tends to infinity.

Note that superexponential convergence to zero (to the lower stationary point) occurs only in the case $c_i = 0$; therefore, the semi-norm of a number can remain bounded even in the case where several components tend to infinity. In the case of a polynomial of third or higher degree, this requires convergence to zero with respect to at least one coordinate with the same degree.

If the inequalities $-2 \leq c_i < -3/4$ hold, the sequence remains bounded by (10) (if this inequality is fulfilled for the initial value).

Now we consider the case where $c_i < -2$. In this case and also in the case where condition (10) does not hold, the sequence tends superexponentially to infinity. However, everything said above holds for the majority of initial conditions when condition (10) holds.

We introduce the notation $I_0 = [-x_{**i}, x_{**i}]$. Indeed, if

$$x_{0i} \in I_1 = (y_1, z_1), \quad -y_1 = z_1 = \sqrt{-c_i - x_{**i}},$$

then the next term of the sequence falls outside of the interval I_0 and the sequence tends to infinity.

Further, for any subinterval $I_k \supset (y_k, z_k)$ for which the kth term falls outside of the interval I_0, we find two symmetric intervals of level $(k+1)$

$$\left(\sqrt{-c_i + y_k}, \sqrt{-c_i + z_k}\right), \quad \left(-\sqrt{-c_i + z_k}, -\sqrt{-c_i + y_k}\right)$$

such that if the initial value belongs to these intervals, then all terms of the sequence up to the kth term inclusively remains in the interval I_0, whereas the $(k+1)$th term falls outside of this interval. Therefore, intervals of each subsequent kth level do not intersect with intervals of previous levels and consist of 2^{k-1} subintervals located inside the complements of intervals of previous levels. Thus, all remaining points form a nowhere dense Cantor set in I_0.

Having clarified the structure of sequences of iterations of components, we can find the Mandelbrot and Julia sets depending on the continuous structure of the space H_k.

First, we find the Julia set. As was found above, if $c_i \in [-2, 1/4]$, then the sequence of the ith component remains bounded, since otherwise this component tends to infinity. Therefore, the product of these intervals belongs to the Mandelbrot set for any continuous structure. Let us assume that the continuous structure for polynumbers satisfies the following condition: if at least one coordinate tends to infinity (superexponentially), then the sequence of polynumbers tends to infinity. Then the Mandelbrot set does not have any other points. In particular, this is valid for the (Euclidean) product topology.

For strange continuous structures that do not satisfy this condition, we must clarify the notion of the Mandelbrot set. By definition, it is a set of parameters $c = (c_1, \dots, c_k)$ such that the iteration sequence with zero initial values remains in a bounded domain. In the most unnatural topology (where boundedness is understood as the boundedness of the product of all components, see above), the Mandelbrot set, in addition to the product of intervals, also contains hyperplanes, where one of the components is identically zero.

In continuous structures where coordinates with identical zero are not taken into account, the Mandelbrot set remains the product of these intervals. In particular, this is valid for the natural topology introduced for the set H_k.

Now we define the Julia set. In [3, 4] it is defined as the boundary of the set $J(f)$ of initial-value points for which the sequence tends to infinity. However, this definition is not quite accurate. From the meaning of the unstability of the dynamics of iterations and in agreement with the definition of the Mandelbrot set and the relation between these sets, a more exact definition is as follows: $J(f)$ *is the boundary of the sets of initial-value points when the sequence of iterations remains bounded.*

According to this definition, we do not need the definition of convergence to infinity. The difference between these definitions is that in the second definition the Julia set is less by a set of initial-value points for which the sequence of iterations is unbounded but does not tend to infinity.

In the classical case of the complex plane and rational functions, these definitions coincide. They also coincide in the case of natural quasi-topologies on H_k for rational functions; however, due to the choice of "bad" functions or an unnatural quasi-topology, they may differ by the set specified above. Obviously, we need to clarify the notion of boundary.

In natural situations, the set $J(f)$ is open, so that its complement (the Fatou set) is closed, and the notion of the boundary in these condition is irrelevant to the definition of the Julia set. However, there exist cases where the properties specified above are not fulfilled and, perhaps, some clarifications in the definition of the Julia set in terms of boundary are needed: it must be defined using the operator of (unnatural) closure of the set itself and its closure.

In the case of the Euclidean topology, where for at least one component the inequality $c_i > 1/4$ holds, the set $J(f)$ coincides with the whole space and hence the Julia set is empty. If, moreover, none of the coefficients vanishes ($c_j \neq 0$), then the Julia set is also empty for the unnatural and natural quasi-topologies compatible with the semi-norm considered above.

We denote by $J_i = [-x_{**i}, x_{**i}]$ either the set of points for $-2 \leq c_i \leq 1/4$ or the Cantor set constructed above for $c_i < -2$.

In the Euclidean topology, the Julia set coincides with the boundary of the product $J = \prod_i J_i$ of these sets. The boundary of the product is defined traditionally as

$$\partial J = \sum_i \partial J_i \times \prod_{j \neq i} J_j$$

(here the sum of sets is meant as the union); moreover, the boundary of the Cantor set coincides with itself and the boundary of an interval consists of two points. This notation can be extended to the case where certain coefficients satisfy the inequalities $c_i > 1/4$ and $J_i = \emptyset$ (the product of a set and the empty set is empty and the boundary of the empty set is empty).

In the case where none of the coefficients c_i vanishes, the Julia sets are the same as in the case of the Euclidean topology, since no exponential convergence whatsoever can compensate superexponential divergence at infinity with respect to at least one coordinate.

If there are zero coefficients in topologies related to a semi-norm, additional subsets of the Julia set besides the set specified above appear.

To illustrate the appearance of additional branches in the Julia set in this case we consider the following example in H_2:

$$x_{n+1} = x_n^2, \quad y_{n+1} = y_n^2 + c.$$

The first recurrence can be easily solved: $x_n = x_0^{2^n}$.

It is easy to verify that for any initial value y_0 that does not belong to the set J_2 corresponding to the y-component, the following inequality holds for some combinations of parameters:

$$a(y_0) = \exp\left[\lim_{n\to\infty} \frac{\ln(y_n)}{2^n}\right] > 1.$$

This function is continuous outside the interval $[-y_{**}, y_{**}]$. Therefore, if $|x_0|a(y_0) < 1$, then the semi-norm tends to zero. If the opposite inequality holds, then the semi-norm tends to infinity. Hence, the set of points $|x_0|a(y_0) = 1$ similar to a hyperbola is added to the Julia set.

In the case of the natural topology, we have the set whose first coordinate is zero and whose second coordinate runs through the complement to the set J_2.

More general cases of polynomials of higher degrees can be considered similarly.

Now we explain the failure of the computer simulation used in [4] for the construction of the Julia set. In [4], the cases $c = -1.3 + 0 \cdot j$ ($c_1 = c_2 = -1.3$) and $c = -1 + 0.2 \cdot j$ ($c_1 = -0.8$, $c_2 = -1.2$) were considered, i.e., the cases where $-2 < c_i < -3/4$ and the orbits remain bounded in the interval $[-x_{**i}, x_{**i}]$ if the initial values are contained in this interval or superexponentially tend to infinity otherwise.

One of the components, being bounded in the interval $[-x_{**i}, x_{**i}]$, performs a chaotic motion inside this interval since, due to the instability of the lower and upper stationary points, the inequality $c_i < -3/4$ holds.

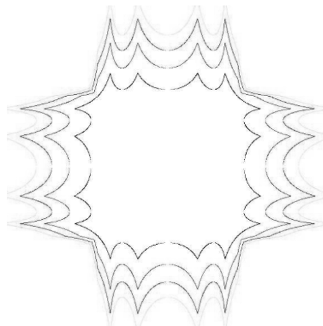

Fig. 1

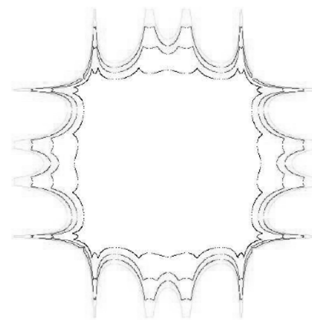

Fig. 2

In the case where the initial value satisfies the equation

$$f^{(k)}(x) = f\left(f^{(k-1)}(x)\right) = 0, \quad f^{(1)}(x) = f(x) = x^2 + c,$$

possessing 2^k real roots from the corresponding interval, the kth term is equal to 0. If the initial value is very close to one of these solutions, then the kth term is close to zero and, therefore, the semi-norm of the kth term can be small and the impression is created that the terms do not tend to infinity. Such values exist for any number.

The main mistake here is that the authors did not calculate the norm of the $(k+1)$th term, which is large for large values of k. Therefore, these points must be excluded from the list of points with bounded orbits. We also note that if the coefficients satisfy the condition

$$g^{(k)}(c) = 0, \quad g(c) = c^2 + c,$$

then for this countable number of values of the coefficients, the values of the components periodically vanish; examining the norms of terms of the iterative sequence of polynumbers, we will be able to guess at the boundedness of the sequence only after a sufficiently large number of steps.

Moreover, trajectories (orbits) obtained for these special values of coefficients are superstable k-cycles, i.e., in the case where an initial point is sufficiently close to a point of the cycle, the orbit superexponentially tends to this cycle passing through the zero point.

This leads to the fact that for these special values of coefficients, the Julia sets presented in [4] actually differ from the Julia sets defined by us more precisely by meaning (see [7–27].

We draw the domains defined by the following inequalities:

$$I_{nk} = \{(x_0, y_0) : |x_n y_n| < kA, \ |x_{n+1} y_{n+1}| < kA\}, \ A = x_{**} y_{**}.$$

For small $0 < k < 1$, the domains can be not simply-connected and they are not important for separation of the domain of initial values for which the iteration sequence (orbit) is bounded from the domain of initial values for which the iteration sequence is unbounded.

In Figs. 1–**??** (here $c_1 = c_2 = -1.3$ and $n = 1, 2, 3, 4, 5$), we illustrate the cases where $k = 1$, 2, 3, and 4. These figures show that the "tails" are cut off. If an infinite tail exists for the domain $|x_n y_n| < kA$, it is cut off by the boundary of the domain $|x_{n+1} y_{n+1}| < kA$ and vice versa.

Thus, the qualitative behavior is possible in the case where $-2 \le c_1, c_2 < 1/4$. These domains have finite (cut off) tails and the domains themselves converge to the rectangles $|x_0| < x_{**}$, $|y_0| < y_{**}$.

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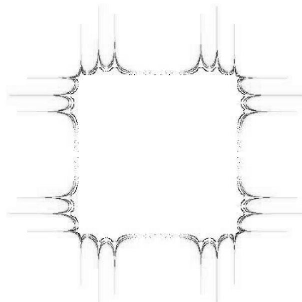

Fig. 3

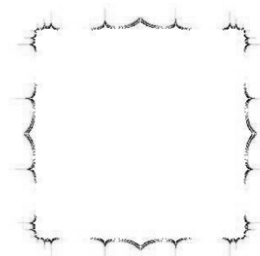

Fig. 4

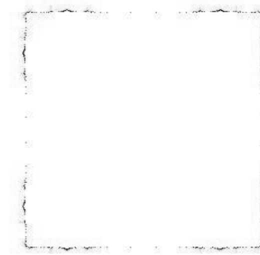

Fig. 5

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