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FINSLER SPACES, BINGLES, POLYINGLES, AND THEIR SYMMETRY GROUPS

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In [22], the notions of bingles and tringles in the space H_3 were introduced. Their definitions are based on an important additivity principle. However, the additivity principle itself is applicable only under a certain "coplanarity" condition. Therefore, based on this principle, one cannot compare bingles (and tringles, etc.) between incomparable angles, i.e., it is possible to take different values of missing coefficients of proportionality for different directions preserving the additivity principle.

The definition of bingles accepted in [22] possesses the following "strange" peculiarity: for any two vectors a and b from the same octant, there exists a vector c from this octant such that the bingles between a and c and between c and b are equal to zero. It is desirable that the definition of a bingle possesses the following property: if the bingle between a and b is equal to zero, then these vectors are proportional; in this case, the "strange" property cannot hold for any nonproportional vectors. Keeping the additivity principle, we can define the notion of a bingle so that the following, stronger metric property holds:

(1) $\varphi(a, b) \geq 0$;
(2) $\varphi(a, b) = 0 \Leftrightarrow a \parallel b$;
(3) $\varphi(a, b) \leq \varphi(a, c) + \varphi(c, b)$.

First, we consider the additivity principle in the most general case. Let X be a smooth manifold. Categories of smooth manifolds are studied within the framework of differential topology. Geometric considerations appear when one introduces a way of calculating lengths (areas, volumes, etc.) as additive functionals on one-dimensional (two-dimensional, three-dimensional, etc.) submanifolds of the manifold X.

Consider the one-dimensional case (length). Let

$$x(\tau), \quad 0 \leq \tau \leq 1,$$

be a smooth curve on a manifold X. The additivity of the length functional L means that

$$L\big(x(\tau)\big) = L\big(x_1(\tau)\big) + L\big(x_2(\tau)\big), \tag{1}$$

where the curve $x(\tau)$ is the sum of the curves $x_1(\tau)$ and $x_2(\tau)$:

$$x(\tau) = x_1(\tau) + x_2(\tau),$$

i.e., for some number $0 < \theta < 1$ we have

$$x(\tau) = \begin{cases} x_1\left(\dfrac{\tau}{\theta}\right), & 0 \leq \tau \leq \theta, \\ x_2\left(\dfrac{\tau - \theta}{1 - \theta}\right), & \theta \leq \tau \leq 1. \end{cases}$$

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Naturally, the length of a curve must be independent of its parametrization. These two requirements imply that

$$L\big(x(\tau)\big) = \int\limits_0^1 ds, \quad ds = g(x, dx), \quad g(x, a\,dx) = a\,g(x, dx),\ a > 0. \tag{2}$$

Finsler geometry also requires the possibility of expressing velocities (unit vectors of the tangent space) through momenta (tangent hyperplanes to the unit sphere) that are elements of the cotangent bundle, and vice versa. Functions of velocities become functions of momenta, while velocities and momenta themselves are related by the Legendre transforms that are tropical analogs of the Fourier transforms.

As is well known, the Legendre transform (see [3]) establishes a correspondence between any function of the variable x and a function of another variable p by the following rule:

$$f^*(p) = \max_x (px - f(x)). \tag{3}$$

Since the maximum in the last expression is attained exactly at the point x where the tangent hyperplane to the graph of the function $f(x)$ is orthogonal to the vector p, the Legendre transform can be interpreted as a transition to tangent variables. The inverse transform is defined by the same formula by interchanging the variables x and p. Surely, the variables x and p can be multi-dimensional; then the variable p belongs to the dual space and

$$px = \sum_i p_i x^i.$$

We can also consider the case where p and x are elements of functional spaces; then functionals play the role of functions. As is well known, in classical mechanics, the Lagrange transform establishes a correspondence between Lagrange functions depending on velocities and Hamilton functions depending on momenta. However, in classical mechanics the kinetic energy is a quadratic function of velocities and in this case the transition from the Lagrange function to the Hamilton function is relatively simple and can be performed without using the Legendre transform. In pure mathematics, the Legendre transform are used for obtaining various inequalities. For example, if $f(x)$ and $f^*(p)$ are related by Legendre transforms, then for any x and p, the following inequality holds:

$$\sum_i p_i x^i \le f(x) + f^*(p).$$

In particular, if a (Finsler) metric is defined by the formula

$$f(x) = \frac{\sum\limits_i x_i^a}{a}, \quad a \in \mathbb{R}$$

(here we omit the index i for brevity), then the corresponding metric in the dual space is defined as follows:

$$f^*(p) = \frac{\sum_i p_i^b}{b}, \quad \frac{1}{a} + \frac{1}{b} = 1,\ b \in \mathbb{R},$$

and for any vector x and any covector p, the corresponding inequality holds.

Such transforms in both directions are possible if the function is convex or concave. The dimensions of the domains of the corresponding variables for smooth functions coincide if and only if they strictly convex or concave.

Tropical mathematics is based on the replacement of multiplication by addition and replacement of addition by the binary operation $\min(a, b)$ or $\max(a, b)$; in this case, distributivity is not violated. Although this extraordinary mathematics gave nothing to conventional science, it deserves attention due to new points of view on usual things. For example, the spectral Fourier transform in tropical mathematics is reduced to the Legendre transform.

It may well turn that in Finsler geometry tropical ideas can lead to new results. The first case (of convexity of a metric) is realized if and only if the triangle inequality is valid:

$$g(x, a+b) \leq g(x, a) + g(x, b), \tag{4}$$

and equality occurs only in the case where the vectors a and b are parallel. Now it becomes clear that the length functional determines a metric if and only if the function $g(x, a)$ is positive for all nonzero vectors a. Then Finsler geometry becomes an appropriate generalization of Euclidean geometry.

The second case is realized if and only if the inverse triangle inequality holds for any two measurable vectors from the same connected component:

$$g(x, a+b) \geq g(x, a) + g(x, b), \tag{5}$$

and equality holds only for parallel vectors of nonzero length. The second case is a Finsler generalization of Minkowski space and is of great interest in the physical context.

Note that inequality (5) cannot hold for all vectors. Therefore, for each point x, one can consider the set of all admissible vectors that are said to be measurable.

The set of measurable vectors is constrained by the following conditions:

(1) if vectors a and b are measurable, then for any positive numbers x and y, the vector $xa + yb$ is also measurable;
(2) the set of measurable vectors has maximal dimension, i.e., there exists a basis consisting only of measurable vectors.

The first condition, in particular, implies that the set of measurable vectors is a convex cone. In inequality (5), we must additionally stipulate that strong equality is impossible in the case of nonparallel measurable vectors of nonzero length (respectively, from the inner domain of measurable vectors) or assume that boundary vectors of zero length are either nonmeasurable or parallel to all measurable vectors (thus, we extend the notion of parallelism).

We also note that Finsler generalizations do not generate other pseudo-Euclidean metrics with other signatures. In this case, we can assume that measurability is defined only for "positive" vectors, i.e., we are restricted to one connected component where measurability is defined; the corresponding vectors are said to be time-like. Thus, the notion of length is defined only for time-like world lines.

The validity of conditions (4) or (5) is also necessary for the existence and uniqueness of a geodesic emanating from a given point in a given measurable direction. In the first case, geodesics yield a minimum whereas in the second case a maximum.

As an example, consider metrics associated with hyper-complex numbers H_n, i.e., metrics that are invariant with respect to automorphisms of the algebra H_n, which coincides with the symmetric group S_n. This class includes metrics defined by symmetric k-order polynomials of n variables. It turns out that all nondegenerate metrics of this type belong to the first class, where the triangle inequality holds, or to the second case, where property (5) holds. In the general case, the proof of this assertion is difficult since one must apply gradual rotations of the bingle between a and b.

For example, we verify the last assertion for the Berwald–Moor metric. In this case, measurable vectors are n-tuples of positive numbers and inequality (5) is equivalent to the well-known Minkowski inequality [4]:

$$\left(\prod_i (a_i + b_i)\right)^{1/n} \geq \left(\prod_i a_i\right)^{1/n} + \left(\prod_i b_i\right)^{1/n}.$$

For the symmetric metric

$$|a| = \left(\sum_i |a_i|^\alpha \right)^{1/\alpha},$$

the following inequality is known:

$$(1-\alpha)\big[(a+b) - |a| - |b|\big] \geq 0,$$

where equality (for positive vectors) holds only in the case where the vectors are proportional (parallel). Thus, the transition from Euclidean Finsler geometry ($\alpha > 1$) to Minkowski-type geometry ($\alpha < 1$) implies the replacement of the triangle inequality by the inverse inequality, which must hold for vectors with positive components, more precisely, when

$$a_i b_i \geq 0 \ \forall i.$$

If we divide the metric by the constant $n^{1/\alpha}$ and let α tend to zero, then we obtain the Berwals–Moor metric as a particular case of such a metric for $\alpha = 0$. Unfortunately, in the monograph [6] devoted to Finsler geometry, the two special types of Finsler spaces described above and the corresponding triangle inequalities (4) and (5) are not emphasized. In the first case, the signature of the metric (which can also be defined for non-quadratic metrics) is $\underbrace{+ + \cdots +}_{n \text{ times}}$ and in the second case is $+\underbrace{- - \cdots -}_{n-1 \text{ times}}$.

To prove that other signatures are impossible, consider the simplest pseudo-Euclidean metric with a different signature:

$$ds^2 = dx_1^2 + dx_2^2 - dx_3^2.$$

The indicatrix of this metric is a hyperboloid of one sheet, which is neither convex nor concave surface. We calculate the coordinates of the dual space (see [6]):

$$p_1 = \frac{dx_1}{ds}, \quad p_2 = \frac{dx_2}{ds}, \quad p_3 = -\frac{dx_3}{ds}, \quad ds = \sqrt{dx_1^2 + dx_2^2 - dx_3^2}. \tag{6}$$

It is easy to see that there is a unique functional dependence between the coordinates:

$$p_1^2 + p_2^2 - p_3^2 = 1,$$

and that this space completely corresponds to the definition of a Finsler space by [6]. However, in this space geodesics are not defined: one can move from point $(0,0,0)$ to point (x_1, x_2, x_3) in many ways lying in the vertical plane containing the initial point and the endpoint, and the length of the curve can equal zero. Thus, this space is not a Finsler space of Euclidean type.

Also, we can easily verify that in this space, there exist arbitrarily long paths, i.e., the space is not a Finsler space of Minkowski type.

In fact, this refers to all pseudo-Euclidean geometries with signatures different from the signatures of Euclidean space and Minkowski space.

It is interesting that in courses of geometry and physics this elementary fact that geometries of other signatures are unsuitable for the definition of geodesics from the variational principle is not mentioned. Thus, the variational method of determining distances based on the additive principle (the definition of a minimum or maximum of some integral taken along a curve connecting two points) leads to the convexity or concavity of the indicatrix. The uniqueness condition for geodesics requires strict convexity or concavity, which can be given by the triangle inequalities (4) and (5) with equality conditions only for parallel vectors. In addition, for (twice) smooth indicatrices, these conditions also provide the requirement of coincidence of the dimension of the tangential variables on a sphere with dimension $n-1$.

Pavlov and his disciples study commutative polynumber Finsler geometries. In this case, all commutative sets of polynumbers are direct sums of some copies of the sets of real and complex numbers.

We show that if the set of complex numbers is contained in a set of polynumbers as a proper subalgebra, then for this set of polynumbers, neither Finsler geometry of Euclidean type nor a geometry of Minkowski type can be defined.

In the sum $\mathbb{R}+\mathbb{C}$, we consider two vectors. Let a nonzero vector $a=(1, a_2+ia_3)$ be measurable. Then the vector $b=(b_1, a_2+ia_3)$ is also a measurable nonzero vector if b_1 is close to 1. Therefore, for any positive t the vector

$$a+tb=(t+1)a+(b_1-1, 0+i0)$$

is also measurable and has the length

$$l(t)=\sqrt[3]{(1+tb_1)(t+1)^2\left(a_2^2+a_3^2\right)}.$$

Since

$$|a|=\sqrt[3]{a_2^2+a_3^2}, \quad |b|=c|a|, \quad c=\sqrt[3]{b_1},$$

we have

$$l(t)=t|b|+\frac{1+2c^3}{3c^3}|a|+O\left(\frac{1}{t}\right),$$

for large t and

$$l(t)=|a|+\frac{2+c^3}{3c}|b|+O(t)$$

for small t. This shows that if $c<1$, then for large t the inequality triangle holds while for t the opposite inequality holds. For $c>1$, the opposite situation holds.

It is easy to show that the last property holds for any algebra of commutative polynumbers containing complex numbers as a proper subalgebra.

Thus, Finsler geometry of Minkowski type is realized only on a direct sum of some (clearly, more than one) copies of the algebra of real numbers, while Finsler geometry of Euclidean type is realized on irreducible algebras without divizors of zero consisting of real numbers, complex numbers, and—in the noncommutative case—quaternions. Note that in this case, nonassociative algebras are not considered.

Summarizing the discussion of Finsler spaces, we specify the new terminology. A metric Finsler space is a smooth manifold equipped with the positive length functional defined by (2). In this case, the metric function $g(x,a)$ vanishes only for $a=0$ and satisfies the triangle inequality (4), which becomes an equality only for parallel vectors, since homogeneity implies the equality

$$g(x,a+b)=g(x,a)+g(x,b)$$

for $b=ra$, $r>0$.

A Finsler–Minkowski space is a smooth manifold on which the length functional is defined by formula (2). In this case, the metric function satisfies the "triangle anti-inequality" (5), which becomes an equality only for vectors of positive length a and b that differ by a positive factor. Precisely these spaces are important in physics.

Now we consider a general Finsler space. We distinguish one of the coordinates a_0 of a vector and denote the ratio of all other coordinates to a_0 by

$$v_i=\frac{a_i}{a_0}, \quad i=1,\dots,n-1.$$

Then from (2) we obtain the decomposition

$$g(x,a)=c(x)a_0+c_1(x)v_1+\cdots+c_{n-1}(x)v_{n-1}+f(x,v). \tag{7}$$

Passing to proper time

$$dt=c(x)dx_0+\sum_i c_i(x)dx_i,$$

we can eliminate the components of the velocity in a first-order decomposition. Further, assuming that the Hessian is nondegenerate, we uniquely distinguish the spatial coordinates such that the metric has the form

$$g(x, dx) = dt\left(1 + \frac{v^2}{2} + O\left(v^3\right)\right) \tag{8}$$

in the first case and the form

$$g(x, dx) = dt\left(1 - \frac{v^2}{2} + O\left(v^3\right)\right) \tag{9}$$

in the second case. Other signatures are impossible due to conditions (4) or (5). All other coordinates that reduce the metric to the same form are related to rotations in the first case and to Lorentz boosts in the second case.

Up to third order, isometries always exist. More exact isometries, in general, become anisotropic and/or nonlinear and may be absent altogether except for the identical isometry. Relations (8) and (9) can be locally obtained near any direction defined by a unit vector, but, unlike the case of quadratic metrics, the coordinates of the hyperplane

$$\sum_i p_i da_i = 0$$

do not correspond to the coordinates of the vector a. For example, for a metric of rank k, the coordinates of a hyperplane are defined as polynomials of degree $k-1$ of the coordinates of the vector

$$p_i = g_{j_1 \dots j_{k-1} i} a^{j_1} \dots a^{j_{k-1}}$$

(as usual, summation over repeated upper and lower indices is meant).

The notion of locality significantly differs from the traditional, where considerations near a chosen point are assumed. In our case, all vectors emanate from the same point, and locality relates to the set of vectors under consideration with directions "close" to a given direction a. Therefore, Finsler geometry has infinitely many degrees of freedom at each point and hence can serve as a bridge to the unification of the general theory of relativity with quantum mechanics.

The normalization of the Euclidean metric on the tangent space of the sphere in conditions (8) and (9) is defined by the normalization inherited from the Finsler space, with respect to which the length of the vector a is equal to 1. This allows one to define the length of a curve on the sphere. In the metric case length is defined by the formula

$$dr^2 = g^2(x, dx) - dt^2, \quad |dx| \ll dt,$$

and geodesics coincide with that defined by the induced metric. In Finsler spaces of Minkowski type we obtain the following:

$$dr^2 = dt^2 - g^2(x, dx), \quad dt = 1,\ |dx| \ll 1,$$

and geodesics on the sphere are defined by the minimum of this new functional on the sphere, which becomes a Riemannian space. Here the first argument is a constant (a point does not vary) and the sphere is defined by the endpoints of direction vectors dx.

Now we consider the notion of k-dimensional volumes for k-dimensional submanifolds of X. We can assume that the parameters determining the surface run over the simplex

$$\tau_1 + \tau_2 + \dots + \tau_k = 1, \quad \tau_i \geq 0.$$

We denote the vertices of the simplex by $A_0, A_1, \dots, A_k$. Let O be a point of this simplex; then the additivity condition can be written in the form

$$V_k(A_0, A_1, \dots, A_k) = \sum_{i=0}^{k} V_k\big(\dots, A_{i-1}, O, A_{i+1}, \dots\big). \tag{10}$$

The sum on the left-hand side consists of k-dimensional volumes of partitions of the simplex when the ith vertex is replaced by the point O. Using also the natural condition of independence of k-dimensional volumes of the parametrization, we find that k-dimensional volumes can be calculated as the integrals

$$V_k = \int dv_k, \quad dv_k = g(x, J_k) d\tau_1 \wedge d\tau_2 \cdots \wedge d\tau_k, \tag{11}$$

where J_k is a set of k-dimensional minors $\partial x_i / \partial \tau_j$. Note that g_k is a homogeneous function of degree one (as in the case $k = 1$) with respect to these minors.

In Riemannian geometry, the definition of length for curves automatically implies the definitions of measures for areas, volumes, etc. In the present paper, we restrict ourselves to the study of the two-dimensional case (area); measures in higher dimensions (e.g., volume) are defined similarly.

As is well known, the area is the sum of absolute values of some bilinear skew-symmetric form on elementary pairs of vectors. All skew-symmetric forms of n vectors in an n-dimensional space are defined up to a constant factor. In the Riemannian case, this factor is chosen so that for the orthonormed basis the area is equal to 1. However, in Minkowski space this normalization leads to problems. A vector orthogonal to another vector with endpoints on the sphere can be nonmeasurable; moreover, all vectors orthogonal to the time vector $(1, 0, 0, 0)$ are nonmeasurable. In measuring vectors that are tangent to the indicatrix, using an isometry (see [22]) we reduce the problem to the measurement of lengths of such nonmeasurable vectors. In this sense, in calculating bingles, we consider, rather than the area of the sector of the unit sphere between the vectors a and b, the length of the geodesic arc on the sphere connecting them.

Note that in calculating angles (bingles, tringles, etc.) between vectors at the same point we can assume that the space is a flat Finsler space if the metric is invariant with respect to translations. In this case, the radius of a sphere equal to 1 has only conditional character since we do not calculate induced distances on the sphere from the Finsler space considered, and only calculate rotation angles between vectors at a given point.

First, we consider bingles. As was noted above, we cam assume that the Finsler space is flat. The sphere is defined by the condition

$$g^2(x, dx) = f(y) = 1, \quad y = dx.$$

Therefore, the direction dx becomes a position of a point y on the sphere y, and on the sphere a Riemannian metric is defined. To describe this metric near the direction of a vector a of length 1, we introduce the covector defining the tangent hypersurface at the point a:

$$p = \operatorname{grad} f = \left(\frac{\partial f(y)}{\partial y^i} \right) \bigg|_{y=a}.$$

Tangent vectors z to the sphere at a point a are annihilated by this covector:

$$p_i z^i = 0.$$

We pass to a coordinate system in which the vector y has the form

$$y = at + z^i e_i,$$

where e_i are tangent vectors to the sphere, $i = 1, \ldots, n-1$. Expanding the function $f(y)$ in a series up to second order, we obtain

$$1 = f(y) = f(a) + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial z^i z^j} dz^i dz^j.$$

This defines a Riemannian metric on the sphere in local coordinates:

$$dr^2 = g_{ij}dz^i dz^j, \quad g_{ij}(a) = \pm\frac{1}{2}\frac{\partial^2 f}{\partial z^i z^j}.$$

In the last expression, the sign "+" corresponds to the metric Finsler case and the sign "−" to the case of a Finsler–Minkowski space. Thus, the metric on the sphere needed for defining angles (bingles, tringles, etc.) is always a positive-definite Riemannian metric.

Using bingles, one can introduce new coordinates on the sphere. Performing this procedure on a sphere or a pseudosphere, we obtain local isomorphisms between them:

$$A : \mathbf{S}_\Phi \to \mathbf{S}_*,$$

where $\mathbf{S}_\Phi$ is the sphere of directions of the Finsler space and $\mathbf{S}_*$ is an $(n-1)$-dimensional sphere (for the metric Finsler spaces) or a pseudosphere (for Finsler–Minkowski spaces).

Any isometry of a Finsler space (we can speak only of isometry of directions at a given point) isometrically transforms the sphere. Therefore, under such transforms, all angles (bingles, tringles, etc.) are preserved. By definition, they are also invariant under scaling.

Denote by G_k the group of transforms preserving k-ingles. Then for transforms F generated by small translations we obtain a local transform AFA^{-1} of the sphere or the pseudosphere preserving $(k-1)$-dimensional areas. Therefore, in the case $k < n$, this group coincides with the motion group of the sphere in the metric case or the motion group of the pseudosphere (i.e., the Lorentz group) in the case of a Finsler–Minkowski space.

For n-ingles, when $(n-1)$-dimensional volumes in an $(n-1)$-dimensional sphere (pseudosphere) are measured, this group is substantially wider since any divergence-free field on this sphere generates a transform preserving volumes. Therefore, it is infinite-dimensional. In addition, one can also use scaling transforms that turn a vector a into $c(a)a$ (multiplication by a positive factor depending on the vector itself). Nevertheless, it may occur that there is no linear transforms preserving bingles, tringles, etc. (except for the identity transform) among them.

There is another way of defining tringles and higher ingles in arbitrary Finsler spaces. For this purpose, we expand the function $g^k(x, dx)$ up to the kth degree and obtain a form of degree k of translations in the tangent space and calculate k-dimensional volumes of sectors by using such a metric of rank k. Since expansion terms of all degrees coincide under an isometry, such k-ingles are also conformally invariant. They also possess the additivity property. In this case, 1-ingles are defined by linear expansions and coincide with the lengths of vectors.

As above, bingles are proportional to quadratic forms on vectors tangent to the sphere. However, the isometry group for higher k-ingles can turn out to be narrower than for the previous definition based on the Riemannian metric on the sphere.

Now we consider a specific metric of the Berwald–Moor space, which is a Finsler–Minkowski space. In this case, multiplication by hyper-complex numbers with norm 1 is an isometry. Therefore, the sphere is a commutative Lie group of dimension $n-1$. By construction, the metric is invariant under the action of this group. For the metric defined by a metric tensor of rank k we have

$$\begin{aligned} f(y) &= \left(1 + k(a, a, \ldots, dy) + \frac{k(k-1)}{2}(a, a, \ldots, dy, dy)\right)^{2/k} + O\left(dy^3\right) \\ &= 1 + 2(a, a, \ldots, dy) + (k-1)(a, a, \ldots, dy, dy) + (2-k)(a, a, \ldots, dy)^2 + O\left(dy^3\right). \end{aligned}$$

This implies that the Riemannian metric for this Finsler–Minkowski space has the form

$$dr^2 = g_{ij}dz^i dz^j = -(k-1)(a, a, \ldots, e_i, e_j), \tag{12}$$

where e_i are basis vectors in the tangent space of the sphere at the point a.

Note that if the initial metric is quadratic ($k = 2$), then it coincides with the corresponding induced metric on the tangent space up to sign and is independent of the point a. Assuming that $k = n$, where n is the dimension of the space, and using the Berwald–Moor metric, we obtain the invariance of the metric considered.

We find the coordinates of an orthonormal system of tangent vectors at the point $\mathbf{1} = (1, 1, \dots, 1)$. Up to a constant factor, we can take

$$e_i = c\big[(1, 1, \dots, 1) + d(1, 0, \dots, 0) + r(0, \dots, 0, 1, 0, \dots, 0)\big].$$

From the orthogonality of vectors we obtain

$$d = \pm\sqrt{n-1}, \quad r = -n - d, \quad c = \sqrt{\frac{n}{n^2 + n - 2 + 2nd}}.$$

Then, to find the bingle between two positive vectors a and b reduced to the same norm, it suffices to represent them in the form

$$b = a\exp(c), \quad c = \left(\ln\frac{b_1}{a_1},\ \ln\frac{b_2}{a_2},\ \dots,\ \ln\frac{b_n}{a_n}\right)$$

and calculate the Euclidean length c as the square root of the sum of squares of coordinates in the orthonormal basis e_i.

Similarly, one can calculate tringles as areas of triangles in logarithmic coordinates and polyingles as the corresponding volumes in this space of logarithms. The symmetry group of bingles etc., without scaling, coincides with the Lorentz group. As above, for n-ingles the symmetry group is infinite-dimensional (see [1, 2, 5, 7–21, 23–27]).

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