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POLYNUMBERS, NORMS, METRICS, AND POLYINGLES

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CONTENTS

1. Vector Spaces and Algebras of Endomorphisms

Let $\mathbb{R}^n$ be a vector space of dimension n over the field of real numbers. This means that $\mathbb{R}^n$ is equipped with a commutative and associative operation called addition with respect to which $\mathbb{R}^n$ is a group and an operation of multiplication of elements of $\mathbb{R}^n$ by real numbers (this operation is always assumed to be a distributive linear transform). Moreover, there exist n elements $e_1, e_2, \dots, e_n$ such that any element $x \in \mathbb{R}^n$ can be uniquely represented in the form

$$x = \sum_{i=1}^{n} x^i e_i = x^i e_i, \quad x^i \in \mathbb{R},$$

Here we accept the Einstein summation convention: when an index variable appears twice in a single term, it implies summation of that term over all the values of the index (which is usually clear from the context).

An additive mapping $f : \mathbb{R}^n \to \mathbb{R}^k$ is a homomorphism (of commutative groups with respect to addition) between the spaces $\mathbb{R}^n$ and $\mathbb{R}^k$. A linear mapping $f : \mathbb{R}^n \to \mathbb{R}^k$ is a homomorphism (of modules over $\mathbb{R}$) between the spaces $\mathbb{R}^n$ and $\mathbb{R}^k$. In the finite-dimensional case, the notions of a linear mapping and a continuous additive mapping coincide.

Definitions of multiplication in modules, rings, and algebras include distributivity, which means the additivity of the multiplication transform from the left and from the right. Instead, we require the linearity of multiplication (from the left and from the right), which is a stronger condition. Linear mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ are called endomorphisms of the space $\mathbb{R}^n$. They form not only a vector space of dimension n^2, but also an associative algebra M_n with unity. The multiplication operation in this algebra is not commutative.

If f is a linear mapping from $\mathbb{R}^n$ to M_n, then in the space $\mathbb{R}^n$ we can define multiplication by the rule

$$x * y = f(x)(y).$$

However, for an arbitrary linear mapping, the obtained structure of an algebra on $\mathbb{R}^n$ is neither commutative nor associative. The associativity condition is equivalent to the condition

$$f(x * y) = f(x)f(y),$$

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i.e., the condition that f is a mapping of rings. Then f maps $\mathbb{R}^n$ into an n-dimensional subalgebra of the algebra M_n.

If in an algebra there is no preimage of the identity matrix, we can introduce such an element increasing the dimension of the algebra by 1.

We propose a physical interpretation of these constructions in the framework of the special theory of relativity (STR). By $\mathbb{R}^n$ we mean coordinates of events in some inertial reference system. A transition to another inertial reference system corresponds to a linear (invertible) transform of coordinates of events. The commutativity of the algebra is equivalent to the commutativity of the Lorentz addition of velocities, when the reference systems 3 and 4 coincide (the reference system 3 is a system that moves with velocity $\mathbf{u}$ with respect to the reference system 1, which, in turn, moves with velocity $\mathbf{v}$, and the reference system 4 is a system that moves with velocity $\mathbf{v}$ with respect to the reference system 2, which, in turn, moves with velocity $\mathbf{u}$). In principle, noncommutativity of multiplication in an algebra is allowed.

Velocities correspond to the indicatrix, so that multiplication (or Lorentz addition) of velocities determines a Lie group of dimension $n-1$.

Actually, in the STR, all transforms of a transition to another inertial reference system are not closed with respect to multiplication. If we take all possible sums and products of such elements, then we obtain a subalgebra M_4 that commutes with the operator of multiplication by the matrix I, which in an appropriate basis has the form

$$I = \begin{pmatrix} i & O \\ O & i \end{pmatrix},$$

where

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is isomorphic to the algebra $M_2(\mathbb{C})$ of dimension 8 with respect to the set of real numbers $\mathbb{R}$.

In the STR, multiplication (Lorentz addition) is defined independently of multiplication in $M_2(\mathbb{C})$ (the algebra obtained is neither a subalgebra nor a quotient algebra). The algebra obtained contains the rotation group (with respect to multiplication) G. Moreover, for any transform $f(x)$ and any element $g \in G$, there exists another element $g_1 \in G$ such that

$$gf(x) = f(x)g_1,$$

and the condition

$$f(x_1)g_1 = f(x_2)g_2, \quad x_1 \neq 0,$$

implies

$$x_1 = x_2, \quad g_1 = g_2.$$

For any two elements x_1 and x_2, there exist $x_3, g \in G$ such that

$$f(x_1)f(x_2) = f(x_3)g.$$

This allows one to define multiplication of velocities similar to multiplication in the quotient group $x_1 * x_2 = x_3$, which defines multiplication up to Thomson rotations. We obtain

$$\ln(f(x_3)) = \ln(f(x_1)) + \ln(f(x_2)) \mod J,$$

where J is the ideal corresponding to G in the Lie algebra of the group of invertible transforms $M_2(\mathbb{C})$.

However, this multiplication is not related to the addition operation by the distributivity law. The genuine algebra of the SRT is the Clifford algebra $\mathrm{Cl}(3) \cong M_2(\mathbb{C})$, which also has the biquaternion representation (see [13]). Correspondingly, there is nothing wrong in the fact that the dimension of the algebra is greater than the dimension of the representation space. Generally speaking, multiplications that correspond to velocities do not define the algebra completely. We must also introduce multiplications by λv, $\lambda \in \mathbb{R}_+$, and several multiplications that correspond to the reversal of orientation of time and/or space and permutation of directions in space.

In polynumbers (i.e., finite-dimensional associative algebras over $\mathbb{R}$ with unities), the operation of exponential mapping can be introduced:

$$\exp(0) = 1, \quad \exp(a1) = \exp(a)1, \quad a \in \mathbb{R}.$$

In this case, if elements x and y commute, then

$$\exp(x + y) = \exp(x)\exp(y).$$

Images of the exponential mapping are vectors (subsequently, we identify a vector with its matrix representation) corresponding to positive eigenvalues.

First, we consider the case of a commutative algebra. In this case, all transforms $f(x)$ have the same basis of eigenvectors and can be diagonalized in this basis, which reduces the algebra to the direct sum of the algebras of real and complex numbers, $\mathbb{R}^{n-2k} + \mathbb{C}^k$, where k is the number of pairs of complex conjugate eigenvectors for the basis. Therefore, any function of vectors invariant with respect to a transition to another basis is a symmetric function on eigenvalues (i.e., a function of diagonal elements in a special basis). This is also valid in the noncommutative case.

Groups of coordinate transformations act on functions of vectors (elements of the algebra). There is the canonical Galois correspondence between transformation groups and the sets of functions that remain invariant under the actions of these groups (see [5]). The minimal transformation group of interest is the group of automorphisms of the set of polynumbers, i.e., the group of invertible linear mappings for which

$$a(xy) = a(x)a(y).$$

From the point of view of physics, functions that are not preserved even under the action of automorphisms of the algebra are not of interest. A wider group is obtained by complementing automorphisms with antiautomorphisms (which can be called odd automorphisms), i.e., invertible linear transformations for which

$$a(xy) = a(y)a(x).$$

Clearly, the product of two antiautomorphisms is an automorphism. Therefore, functions that are preserved under automorphisms but are not preserved under the extension of the group by the antiautomorphisms described above, can be called odd functions, whereas functions that are also preserved under antiautomorphism can be called even functions. A wider group is obtained if we consider all invertible linear transformations (module automorphisms instead of algebra automorphisms). Function that are preserved under the action of this group are said to be invariant.

There are no invariant vectors. Unit vectors are preserved under both automorphisms and antiautomorphisms, i.e., they are even vectors. At the same time, there exists an invariant covector (i.e., linear function on vectors), for example, the trace Tr and any proportional covector. The $(n-1)$-dimensional subspace of vectors that are annihilated by this covector is also invariant.

An invariant covector allows one to define the "real" and "imaginary" parts of a vector as follows:

$$\mathrm{Re}(x) = \frac{1}{n}\,\mathrm{Tr}(x), \quad \mathrm{Im}(x) = x - \mathrm{Re}(x) * 1.$$

The conjugate (symmetric) vector for a vector x is defined as $2 * 1\,\mathrm{Re}(x) - x$, where 1 denotes the unit vector.

In algebras that are obtained by several iterations of the doubling procedure starting from some commutative algebra, the transition to the conjugate element is an involution (i.e., an antiautomorphism whose square is the identity automorphism). The doubling is performed so that the extended conjugation remains an involution in the doubled algebra. In physics, the "real" part usually corresponds to the time coordinate and the "imaginary" part to the spatial coordinate.

The polynomial functions $\mathrm{Tr}(x^m)$, $m = 1, \dots, n$, are invariant functions. In the algebras $H_n = \mathbb{R}^n$ (the direct sums of n copies of the set of real numbers), there exist automorphisms that swap n "unities" of each components. This set of n unities is also invariant with respect to automorphisms and their sum—the unity of the algebra—is invariant with respect to automorphisms and antiautomorphisms. Therefore, any invariant function in metrics of Berwals–Moor or Chernov (with symmetric functions) type can be expressed as a function of the above polynomials. Such functions of several vectors can also be expressed in the same form as a function of polynomials of several variables.

In the case of noncommutative multiplication, the difference is only in the use of noncommutative polynomials. However, the trace is independent of the order of factors and hence we obtain the same invariant functions despite the fact that matrices of multiplication are distinct. One can construct homogeneous metrics using homogeneous polynomials of such functions. For example, using three constants, we can construct a metric of rank 3:

$$ds^3 = a\,\mathrm{Tr}(xyz) + b\Big[\mathrm{Tr}(x)\,\mathrm{Tr}(yz) + \mathrm{Tr}(y)\,\mathrm{Tr}(xz) + \mathrm{Tr}(z)\,\mathrm{Tr}(xy) + c\,\mathrm{Tr}(x)\,\mathrm{Tr}(y)\,\mathrm{Tr}(z)\Big]. \qquad (1)$$

2. Polylinear Functions and Metrics

Polylinear functions of rank k can be defined as functions

$$f : V^k \to \mathbb{R}, \quad V = \mathbb{R}^n$$

linear with respect to each of the variables. These functions can also be defined in the form

$$f(x_1, \dots, x_k) = g_{i_1 \dots i_k} x_1^{i_1} \dots x_k^{i_k}.$$

If a functions does not change its value under an arbitrary permutation of arguments, then it is said to be symmetric. As metric tensors, only symmetric polylinear functions can be used. The length of a vector is defined as the power of order $1/k$ of a symmetric polylinear function $f(\cdot, \dots, \cdot)$ (with the same arguments).

Introducing the indices

$$j_m = i_m + m - 1, \quad 1 \le j_1 < \dots < j_k \le n + k - 1,$$

and taking symmetry into account, we can reduce the indices of a tensor to the form

$$i_1 \le \dots \le i_k.$$

Therefore, the number of independent components of a symmetric tensor of rank k is equal to C_{n+k-1}^k. Actually, tensors that are usually used for constructing metrics possess an additional symmetry of numeration of coordinates. In the case of total symmetry with respect to numeration, the number of independent components is expressed by the well-known partition function (see [7]) $p(k, n) \le 2^{k-1}$; it is equal to the number of representations of a number k as the sum of no more than n natural numbers. For $n \ge k$ (this condition holds in all cases) all values are $p(k, n) = p(k)$ and

$p(1) = 1, \quad 2 = 1 + 1,$

$p(2) = 2, \quad 3 = 2 + 1 = 1 + 1 + 1,$

$p(3) = 3, \quad 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1,$

$p(4) = 5, \quad 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1,$

$p(5) = 7,$

$p(6) = 11,$ etc.

For example, the symmetric bilinear function discussed at the end of the previous section are functions of such type. In particular, for $k = 2$ we have two symmetric bilinear functions, one of which corresponds to $g_{ij} = \delta_i^j$ and the other to $g_{ij} = 1 - \delta_i^j$.

All other such functions are linear combinations of these two functions. In the case where an index symmetry occurs for all indices except for one of them (which corresponds to time), the number of independent components is

$$p_1(k,n) = p(k,n-1) + p(k-1,n-1) + \dots + p(1,n-1) + 1.$$

However, in the algebras $H_n = \mathbb{R}^n$ (the direct sum of n copies of the set of real numbers), each component is contained symmetrically (although there is a selected vector, namely, the sum of the components of the unity). Therefore, metrics that are related to the algebra must be completely symmetric.

Polylinear metrics of rank $k \le n$ related to the polynumber structure (i.e., invariant with respect to the action of the automorphism group of polynumbers) are defined by $p(k)$ parameters. Such metrics are said to be polynumber metrics. The number of parameters of a metric in a general-relativity analog is still greater by n^2; it is related to the choice of n generators of the algebra in the n-dimensional space. However, not all n^2 parameters are independent. For example, an analog of the Berwald–Moor metric

$$ds^4 = de_1\, de_2\, de_3\, de_4, \quad e_i = a_i t + b_i x + c_i y + d_i z, \ i = 1,2,3,4,$$

contains 16 parameters related to multiplication by numbers

$$r_i, \ i = 1,2,3,4, \quad r_1 r_2 r_3 r_4 = 1.$$

This leads to 13 parameters of the orientation of the metric for the Berwald–Moor metric; moreover, all these parameters can change from one point to another. In a special-relativity analog, all parameters are constant. The general form of the metric in the SRT is as follows:

$$ds = dt\big(1 - P_2(v) + P_3(v) + \dots\big), \quad P_2(v) = \frac{1}{2}\big(v_1^2 + \dots + v_{n-1}^2\big), \tag{2}$$

where

$$v_i = \frac{dx^i}{dt}$$

and $P_k(v)$ is a symmetric polynomial of degree k of the coordinates of the velocity v.

This can be justified as follows: any symmetric function of coordinates (only if they are invariant with respect to all automorphisms) can be represented as a function of symmetric functions. If we factor out dt, we obtain a homogeneous function of degree 0, i.e., a function of the ratios $\dfrac{\Pi_k}{(\Pi_1)^k}$, $k = 2, \dots, n$, where Π_k is the kth symmetric polynomial of generators. We show below that they are symmetric functions of the velocities.

More precisely, there exist metrics invariant with respect to automorphisms whose expansions in (2) starts from terms of order higher than 2. However, they are degenerate critical metrics that can be reduced to a noncritical form by a small deformation and can be written in the form (2) in an appropriate basis.

The coordinates of velocity can be found up to rotations from the expression for $P_2(v)$ (see (2)). Except for degenerate cases, due to the contributions of higher degrees of velocity, the symmetry of coordinates is reduced to permutations of coordinates.

Consider homogeneous metrics starting from the case $k = 2$ (the case where $k = 1$, $ds = dt$, is not of interest). We have

$$ds^2 = a\big(dx_1 + \dots + dx_n\big)^2 + b\big(dx_1^2 + \dots + dx_n^2\big). \tag{3}$$

We pass to the following coordinates:

$$y_0 = \frac{x_1 + \cdots + x_n}{n}, \quad y_i = x_i + qx_n - r\frac{x_1 + \cdots + x_n}{n}, \quad i = 1, \ldots, n-1.$$

The inverse transition is performed by the formulas

$$x_n = \frac{1}{q(n-1)-1}\left[(rn - r - n)y_0 + \sum_{i=1}^{n-1} y_i\right],$$

$$x_i = y_i + ry_0 - \frac{q}{q(n-1)-1}\left[(rn - r - n)y_0 + \sum_{i=1}^{n-1} y_i\right].$$

Taking first differentials

$$dx_i = dy_i - \alpha s_1 + \beta dy_0, \quad i = 1, \ldots, n-1, \quad dx_n = f s_1 + g dy_0,$$

where

$$s_m = \sum_{i=1}^{n-1} (dy_i)^m$$

is a symmetric form of degree m of the variables dy_i, $i = 1, \ldots, n-1$, we obtain

$$f = \alpha(n-1) - 1, \quad g = n - \beta(n-1)$$

and

$$\sum_{i=1}^{n} (dx_i)^2 = s_2 - 2\alpha s_1^2 + 2\beta s_1 dy_0 + (n-1)(\beta dy_0 - \alpha s_1)^2 + (f s_1 + g dy_0)^2.$$

Equating the coefficients of s_1^2 and $s_1 dy_0$ in the last expression to zero, we have

$$\alpha^2 n(n-1) - 2\alpha n + 1 = 0, \quad (\beta - 1)n(1 + \alpha - \alpha n) = 0.$$

Thus, we obtain the following relations:

$$\beta = 1 = g, \quad \alpha = \frac{1}{n \mp \sqrt{n}}, \quad f = \pm\frac{1}{\sqrt{n}}, \quad q = \frac{-1}{n \mp \sqrt{n}} = -r.$$

Substituting these expressions into (3), we have

$$\sum_{i=1}^{n} (dx_i)^2 = s_2 + n dy_0^2,$$

$$ds^2 = (a + bn)(dy_0)^2 + b\sum_{i=1}^{n-1} (dy_i)^2.$$

The metric defined by the last relation is nondegenerate when

$$b \neq 0, \quad a \neq -bn.$$

Finally, we have

$$\begin{gathered} x_i = y_i + y_0 - \frac{1}{n \mp \sqrt{n}}\sum_{j=1}^{n-1} y_j, \quad i < n, \quad x_n = y_0 \pm \frac{1}{\sqrt{n}}\sum_{j=1}^{n-1} y_j, \\ y_0 = \frac{1}{n}\sum_{j=1}^{n} x_j, \quad y_i = x_i - \frac{x_n}{1 \mp \sqrt{n}} + \frac{1}{n(1 \mp \sqrt{n})}\sum_{j=1}^{n} x_j. \end{gathered} \tag{4}$$

Now we show that precisely these coordinates, up to scaling of time and length (spatial measure), correspond to the separation of the time and spatial coordinates for any metric invariant under arbitrary automorphisms of the algebra H_n. More precisely, there is arbitrariness in the elimination of one variable (in our case, x_n; this is similar to the choice of one affine parameter, which is a coordinate in a projective manifold), arbitrariness in the choice of the sign $\pm\sqrt{n}$, and arbitrariness of ordering of the variables y_i (due to symmetry, the metric itself is independent of the last two arbitrary factors), and, finally, arbitrariness of scaling of time and space.

We assume that summing of y_i is performed over $i = 1, \dots, n-1$. We denote their symmetric forms (with respect to the variables y) as follows:

$$s_j = \sum dy_i^j, \quad \sigma_m = \sum_{i_1 < \dots < i_m} dy_{i_1} \dots dy_{i_m}.$$

We denote the symmetric form with respect to all variables x by

$$S_j = \sum dx_i^j, \quad \Pi_m = \sum_{i_1 < \dots < i_m} dx_{i_1} \dots dx_{i_m}.$$

The reduction to the form (2) is equivalent to the expression of S_j, $j > 1$, of Π_j, $j > 1$, through the variables

$$z_0 = dy_0 = \frac{1}{n} S_1, \quad s_1, s_2, \dots,$$

as follows:

$$S_j = (z_0 + f s_1)^j + (n-1)(z_0 - \alpha z_1)^j + \sum_{m=1}^{j} s_m C_j^m (z_0 - \alpha z_1)^{j-m}$$
$$= n z_0^j + z_0^{j-2} s_1^2 C_j^2 \big(n(n-1)\alpha^2 - 2n\alpha + 1\big) + \dots,$$

i.e., for

$$f = \frac{1}{\pm\sqrt{n}}, \quad \alpha = \frac{f^2}{1-f} = \frac{1}{n \mp \sqrt{n}}$$

the second term of the expansion considered vanishes for any j.

Therefore, the following relation holds:

$$S_j = n z_0^j + \sum_{m=3}^{j} C_j^m z_0^{j-m} s_1^m \left[f^m + (n-1)(-\alpha)^m + m(-\alpha)^{m-1} + \sum_{m=2}^{j} C_j^m s_m (z_0 - \alpha s_1)^{j-m} \dots \right]. \quad (5)$$

As in to the Newton–Girard formulas, we can obtain metrics expressed through permanents and calculate the corresponding anisotropy coefficients.

The basic metrics used in Finsler geometry are permanent, i.e., they can be expressed in the form

$$ds^k = \Pi_k, \quad \Pi_k = \sum_{i_1 < \dots < i_k} dx_{i_1} \dots dx_{i_k}.$$

We present expressions for the time coordinate for this case.

These formulas can be obtained from the above by using the Newton–Girard formulas. This yields the value $a = \sqrt{3/2}$ for the anisotropy (it depends only on n, which is equal to 4 in our case) of the Chernov metric and two values a and b for the Berwald–Moor metric. Similarly, we obtain

$$\Pi_m = (z_0 + f s_1) \sum_{k=0}^{m-1} C_{m-n+k}^k \sigma_{m-1-k} (z_0 - \alpha s_1)^k + \sum_{k=0}^{m} C_{m-1-n+k}^k \sigma_{m-k} (z_0 - \alpha s_1)^k. \quad (6)$$

This implies that any noncritical metric invariant under automorphisms of polynumbers H_n can be reduced to the form (2) by scaling the time variable y_0 and the space variables y_i ($t = c_1 y_0$, $z_i = c_2 y_i$) under certain inequalities for parameters.

Arbitrariness of the choice of variables leading to the form (2) in the case where $P_2(v)$ is an arbitrary symmetric polynomial of degree 2 is substantially wider and it does not allow one to define spatial relations (length).

We present the final expressions for S_j, Π_j, $j = 1, 2, 3, 4$:

$$S_1 = nz_0, \quad S_2 = nz_0^2 + s_2, \quad S_3 = nz_0^3 + 3s_2 z_0 + A_{31} s_1^3 + A_{32} s_1 s_2 + s_3,$$
$$S_4 = nz_0^4 + 6z_0^2 s_2 + 4z_0 \left(A_{31} s_1^3 + 3\alpha s_2 s_1 + s_3 \right) + P_4,$$
$$P_4 = A_{41} s_1^4 + 6\alpha^2 s_2 s_1^2 + 4\alpha s_3 s_1 + s_4,$$
$$A_{31} = \frac{1}{(n \pm \sqrt{n})^2} \left(2 \pm \frac{1}{\sqrt{n}} \right) \mp \frac{1}{\sqrt{n}} \left(\frac{3}{8} \right), \tag{7}$$
$$A_{32} = -3\alpha = -3 \frac{1}{n \pm \sqrt{n}} \left(\frac{-3}{2} \right),$$
$$A_{41} = \frac{-1}{(n \pm \sqrt{n})^3} \left(3 \pm \frac{1}{\sqrt{n}} \right) + \frac{1}{n^2}.$$

In the case of a metric of degree k, we similarly obtain that

$$ds^k = (dt)^k \left(1 - \frac{k}{2} V_2 + \sum_{i=3}^{k} P_i(V) \right), \quad V_m = \sum_i \left(\frac{dy_i}{dt} \right)^m, \tag{8}$$

where $P_i(V)$ are homogeneous polynomials of degree i of the spatial coordinates of velocities:

$$P_3(V) = a_{31} V_1^3 + a_{32} V_1 V_2 + a_{33} V_3, \quad \ldots, \quad P_k(V) = \sum_{i=1}^{p(k)} a_{ki} T_i(V).$$

Precisely these terms lead to anisotropy. In the Berwald–Moor and Chernov metrics defined by permanents, the coefficients of anisotropic terms cannot be adjusted and are uniquely calculated from (8). Therefore, this does not allow one to introduce a small anisotropy. To remove this shortcoming, one introduces metrics with arbitrary coefficients that preserve their form under arbitrary automorphisms of polynumbers; they are called polynumber metrics.

Using (7), we can reduce a cubic metric to the form

$$ds^3 = a_1 S_1^3 + a_2 S_1 S_2 + a_3 S_3 = b_1 z_0^3 + b_2 z_0 s_2 + a_3 \left(A_{31} s_1^3 + A_{32} s_1 s_2 + s_3 \right),$$
$$b_1 = a_1 n^3 + a_2 n^2 + a_3 n, \quad b_2 = n a_2 + 3a_3.$$

After normalization with respect to the time and space coordinates y, the metric contains only one parameter adjusting anisotropy:

$$ds^3 = dt^3 - \frac{3}{2} dt \left(dx^2 + dy^2 + dz^2 \right) + a P_3(dx, dy, dz), \tag{9}$$

where $P_3(dx, dy, dz)$ is a given symmetric polynomial of degree 3. The last term causes anisotropy.

In particular, under reversal of space orientation, this term changes its sign. It is hard to imagine that the reversal of the orientation of a ruler changes its length. That is why the authors hold the view that the spatial metric (without time) is Euclidean.

The speed of light and the so-called Lorentz transforms (transforms of one inertial reference system into another) depend on direction (near singularities). In the latter case, anisotropy appears only in terms of the third order with respect to velocities (the speed of light has order 1 but depends on direction). In the case $k = 4$, the metric has two anisotropy parameters:

$$ds^4 = dt^4 - 2\, dt^2 \left(dx^2 + dy^2 + dz^2 \right) + a\, dt\, P_3(dx, dy, dz) + b\, P_4(dx, dy, dz). \tag{10}$$

For $a = 0$, the anisotropy is even and properties do not change under the reversal of spatial orientation.

By a rank-k metric, one can obtain a metric of lower rank by a convolution with fixed vectors.

Let $m > k$ and vectors A^i_l, $l = 1, \dots, m-k$, be given. Then a metric of rank m is obtained as follows:

$$g'_{j_1 \dots j_m} = g_{i_1 \dots i_{m-k} j_1 \dots j_m} A^{i_1}_1 \dots A^{i_{m-k}}_{m-k}$$

(here summation over repeated indices is assumed).

Metrics obtained in this way with positive (exponentially representable) vectors are called accompanying metrics. Accompanying metrics are invariant with respect to automorphisms of the algebra if all vectors A^i_l are proportional to the unit vector $1^i = 1$. Such accompanying metrics are called subordinated metrics. For example, the Chernov metric is a subordinated metric for the Berwald–Moor metric.

A symmetric metric whose accompanying metrics of rank 2 have the signature $(+, -, -, \dots, -)$ and which is negative definite on the hyperplane $\mathrm{Tr}(x) = 0$ and positive on the unit vector 1 is called a Lorentz-type metric. This notion is similar to the notion of the hyperbolicity of a polynomial in the sense of Gårding, which defines a metric with respect to a unit vector [3].

Metrics that are subordinated to the Berwald–Moor metric are called metrics of permanent type. They are conformally equivalent to Π_k.

In the following section, we present remarkable properties of such metrics. Here we mention only the following obvious property (which follows directly from the definition): an accompanying metric for a Lorentz-type metric is also a Lorentz-type metric.

In view of the importance of the case $n = 4$ for physics (in this case, in addition, irrationalities in transition expressions disappear), we rewrite for this case all types of anisotropy of metrics of third and fourth orders, which depend on n. The case $n = 4$ is remarkable not only by the disappearance of irrationalities, but also by the fact that, in the case of positive sign, transitions are realized supersymmetrically (this holds only for $n = 4$) and anisotropy polynomials are of the simplest form. Therefore, the case $n = 4$ is remarkable.

We rescale the spatial variables as follows:

$$t = y_0, \quad x = \frac{y_1}{2}, \quad y = \frac{y_2}{2}, \quad z = \frac{z_2}{2}.$$

Then we have

$$x_1 = t + x - y - z, \quad x_2 = t + y - x - z, \quad x_3 = t + z - x - y, \quad x_4 = t + x + y + z. \tag{11}$$

Therefore,

$$\begin{aligned}
\Pi_1 &= 4dt, \\
\Pi_2 &= 2\left(3dt^2 - dx^2 - dy^2 - dz^2\right), \\
\Pi_3 &= 4dt\left(dt^2 - dx^2 - dy^2 - dz^2\right) + 12 dx\, dy\, dz, \\
\Pi_4 &= dt^4 - 2dt^2\left(dx^2 + dy^2 + dz^2\right) + 8dt\, dx\, dy\, dz + dx^4 + dy^4 + dz^4 \\
&\quad - 2\left(dx^2dy^2 + dy^2dz^2 + dz^2dx^2\right).
\end{aligned} \tag{12}$$

Here Π_2 corresponds to the Minkowski metric (we still have freedom of choice of the spatial and time scale), Π_3 corresponds to the Chernov metric (with the simplest anisotropy for $n = 4$), and Π_4 corresponds to the Berwald–Moor metric. All these metrics are permanent.

If there is a chosen vector 1, from a metric of rank k we can obtain metrics of lower rank by replacing some vectors (missing in scalar products of lower rank) by the vector 1 in the scalar product of rank k. For example, if we have a trilinear scalar product (A, B, C), then it is possible to obtain the associated bilinear scalar product $(A, B) = (A, B, 1)$ by using the chosen vector.

In polynumber metrics, there always exists a chosen vector 1 corresponding to the identity matrix. By means of the unit vector one can obtain metrics of lower rank and define angles of lower rank in such a subordinated metric. Moreover, the case $k = n$ corresponds to the Berwald–Moor metric:

$$\frac{1}{n!}\det\begin{pmatrix} x_1^1 & x_2^1 & \dots & x_n^1 \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \dots & \dots & \dots & \dots \\ x_1^n & x_2^n & \dots & x_n^n \end{pmatrix}$$

(the permanent of the coordinates of vectors in a specific basis). It also is a norm of a vector as in number theory.

Thus, the norm of the product of polynumbers is equal to the product of their norms. Therefore, polynumber metrics of lower rank can be obtained by filling $n - k$ columns with 1's. In particular, the Chernov metric is obtained in this way from the Berwald–Moor metric by replacing one column by a column consisting of 1's. This procedure, up to a conformal factor, is equivalent to the rejection of the highest anisotropic term in the metric, reduction by z_0, and decreasing the degree of ds by 1.

A general metric of rank 3 in the case $n = 4$ can be reduced by scaling (when the signs of a, $2a + b$, and c in the expression $a\Pi_1\Pi_2 + b\Pi_3 + c\Pi_1^3$ coincide) to the form

$$ds^3 = dt\left(dt^2 - dx^2 - dy^2 - dz^2\right) + a\,dx\,dy\,dz. \tag{13}$$

Therefore, in the general case a metric of rank 4 (under appropriate conditions) has the form

$$ds^4 = dt^4 - 2dt^2\left(dx^2 + dy^2 + dz^2\right) + a\,dt\,dx\,dy\,dz + b\left(dx^4 + dy^4 + dz^4\right) + c\left(dx^2dy^2 + dy^2dz^2 + dz^2dx^2\right). \tag{14}$$

The number of parameters is equal to $p(4) - 2 = 3$, where $p(4)$ is the total number of parameters minus two parameters owing to spatial and time scaling.

A spatial metric (a definition of lengths) can be introduced as follows:

$$dr^2 = \lim_{dt\to\infty}\left(dt^2 - ds^2\right).$$

The degree 2 in this formula corresponds to all nondegenerate (noncritical) metrics and can be replaced by a higher degree only in the degenerate case. Moreover, in all nondegenerate cases, this metric is the usual Euclidian metric.

Using this metric, it is possible to calculate the speed of light in various directions by (12). For the Chernov and Berwald–Moor metrics, the speed of light in the directions of the spatial axes x, y, and z is equal to 1. The minimal speed corresponds to the direction $(-1,-1,-1)$; for the Chernov metric it is equal to 0.823, and for the Berwald–Moor metric to $1/\sqrt{3}$. In the last case, the speed of light reaches the maximal value ($\sqrt{3}$) in the direction $(1, 1, 1)$; however, for the Chernov metric, the inequality $ds \neq 0$ holds for this direction (it seems that light does not propagate). This shows that one need examine metrics whose anisotropy is less than the anisotropy of the metrics considered above.

3. Polyingles

We discuss several approaches to the notion of polyingles between k vectors in a space with a metric of rank k, which generalizes the notion of an angle (bingle) between two vectors for quadratic metrics. A polyingle must be defined as a function of k vectors $\{A_1, \dots, A_k\}$ in a space with a metric of rank k satisfying the following conditions:

(1) the function is symmetric, i.e., invariant under any permutation of arguments:

$$\{A_1, A_2, A_3, \dots, A_k\} = \{A_2, A_1, A_3, \dots, A_k\},$$
$$\{A_1, A_2, \dots, A_{k-1}, A_k\} = \{A_k, A_1, A_2, \dots, A_{k-1}\};$$

(2) the k-ingle is independent of the lengths of vectors and depends only on their directions, i.e., the function is homogeneous of degree 0 with respect to all arguments:

$$\{\lambda A_1, \dots, A_k\} = \lambda\{A_1, \dots, A_k\}, \quad \lambda \in \mathbb{R}_+;$$

(3) if c is an isometry of the space, then

$$\{c(A_1), \dots, c(A_k)\} = \{A_1, \dots, A_k\};$$

(4) for positive (i.e., exponentially representable) vectors, polyingles are always defined and continuously depend on variations of vectors in this domain.

Actually, these conditions imply that property (3) is also valid for conformal transforms. Therefore, if conformal transforms transitively act on the set of vectors (this is valid for polynumber metrics), then

$$\{A, \dots, A\} = \{B, \dots, B\} = \text{const}.$$

By analogy to bingles, this constant can be made equal to 1 by normalization.

(5) A polyingle is not a constant function for positive vectors.

However, these requirements do not suffice to define k-ingles. Assume that there exist a transform u preserving a k-ingle and changing only the first two vectors so that it is possible to decrease the angle between them to zero. Using such transforms repeatedly, we superpose all the vectors into a single vector. This can be easily proved for the transform

$$(A_1, A_2) \to (B, B), \quad B = \frac{A_1 + A_2}{2}.$$

In this case, the sum of the vectors is preserved and, therefore, in the limit, all the vectors become equal to their arithmetic mean value and, bearing in mind the remark after condition (3), this leads to a constant value.

In the general case, one can similarly reduce a polyingle to a constant value.

The following lemma can be proved similarly.

Lemma 1. *If the symmetry group of a k-ingle is the general linear group* $\mathrm{GL}(n)$*, then this k-ingle is constant, i.e., contradicts condition* (5).

Proof. Let k vectors $\{A_1, \dots, A_k\}$ be linearly independent. Then we take any vector from this set, say A_1, and apply the transforms

$$A_i \to \frac{A_1 + A_i}{2}, \quad i \neq 1.$$

Further, applying such transforms repeatedly (the vectors still remain independent and hence such transforms belong to the group $\mathrm{Gl}(n)$), we obtain a contradiction with condition (5).

The case where the initial vectors $\{A_1, \dots, A_k\}$ are linearly dependent can be reduced to the above case by taking independent vectors arbitrarily close to given vectors. □

The mapping, which associates with any set of k vectors $V^k \in \mathbb{R}^n$ their k-ingle, stratifies the given set into levels, and the transformation group G preserving k-ingles acts so that each orbit belongs to a certain level set. Thus, we find that the group G cannot contain elements that change only two vectors and preserves the other vectors.

Some orbits of this group may not contain a "standard" type that annihilates bingles. For example, if all k vectors are equal, then it is unlikely that the group G enables their "disconnection" so that they become pairwise orthogonal.

Most likely, such an orbit is degenerate and contains only all sets of identical k vectors. Moreover, for some metrics, the fact that a set of k vectors contains a vector orthogonal to all other vectors probably implies that the corresponding k-ingle vanishes. This means that counting of the number of conditions and the reduction of k vectors to the standard form, provided that the number of these conditions is less than the dimension of the group, does not imply the possibility of such a reduction.

One way of defining k-ingles is as follows. First, we define this notion for coinciding vectors and then introduce formulas that describe the change of k-ingles under the change of one of the vectors in a direction perpendicular to the other vectors.

The analysis of various definitions of k-ingles allows one to reject most of them, whereby only definitions based on homogeneous functions are adopted. Taking into account the fact that not all spaces contain a selected vector, we conclude that the definition of k-ingles should be rather general and independent of this property. Therefore, the general form of a k-ingle is a combination of the scalar products of k vectors taken in various combination.

It is possible to construct 2^{k-1} types of combinations of k vectors (see the previous section). However, many types of combinations related to permutations give the same results; they correspond to various partitions of the number k into the sums $p(k)$. For example, we obtain two types of partitions for $k = 2$:

$$1 + 1, \quad 2;$$

therefore,

$$(A, B)$$

is the first type and

$$(A, A), \quad (B, B)$$

is the second type. For $k = 3$, we have

$$1 + 1 + 1, \quad 2 + 1, \quad 3;$$

therefore,

$$(A, B, C)$$

is the first type,

$$(A, A, B), \quad (A, A, C), \quad (B, B, A), \quad (B, B, C), \quad (C, C, A), \quad (C, C, B)$$

is the second type, and

$$(A, A, A), \quad (B, B, B), \quad (C, C, C)$$

is the third type.

As the first type, we take the representation of k as k unities. In this type, we have a unique scalar product. For all types we introduce the following quotient:

$$\rho_i = \frac{(\Pi(A_{j(1)}, A_{j(2)}, \ldots, A_{j(k)}))^{1/m_i}}{|A_1| \cdot |A_2| \cdot \cdots \cdot |A_k|};$$

the denominator contains the norms of the vectors:

$$|A| = (A, A, \ldots, A)^{1/k}$$

(the case $i = p(k)$ yields the unity (1)). All these values satisfy the necessary requirements for the definition of k-ingles.

If for a k-ingle there exist several functions, then any function of such k-ingles can also be taken as a k-ingle. Therefore, any function of $p(k) - 1$ arguments

$$\rho_i, \quad i > 1, \quad \rho_{p(k)} \equiv 1,$$

defines a k-ingle.

Note that if all vectors are the same ($A_i = A$), then $\rho_i \equiv 1$ and, therefore, we can assume that the k-ingle is equal to 1. In the case where one vector is B and all the other vectors are the same and equal to A, then

$$\rho_1 = \frac{(A, \dots, A, B)}{|A|^{k-1}|B|}, \quad \rho_2 = \frac{(A, \dots, A, B)^{\frac{k-2}{k}}(A, \dots, A, B, B)^{\frac{1}{k}}}{|A|^{k-1}|B|}, \quad \dots$$

Since the function $p(k)$ rapidly increases as k grows, the number of functionally independent quantities can be substantially less than their total number. Therefore, it is advisable to operate with the smallest quantity of these numbers.

For $m_1 = 1$, we can find ρ_1 from the definition of bingles. For an even number $m_2 = k(k-1)$ of changes of the composition of vectors, we can calculate the value ρ_2 corresponding to the partition $2 + 1 + 1 + \cdots + 1$ ($k-2$ unities and one deuce).

If a k-ingle can be expressed through only one of these numbers, there is no need to take a function of it. For bingles, we use a trigonometric function (or, in the case of a pseudo-Euclidean metric, a hyperbolic function), since the rotation group that changes the value of a bingle is known. This can be done by various methods and hence one cannot uniquely recover the method used in the construction of a k-ingle of k equal vectors. Moreover, in the case of tringles, there are two numbers ρ_1 and ρ_2, but even two numbers cannot determine a unique way of construction of an initial triple of vectors from a triple of equal vectors.

The form of ρ_1 resembles the form of a bingle:

$$\rho_1 = \frac{(A_1, \dots, A_k)}{|A_1| \dots |A_k|}, \quad |A_i| = (A_i, \dots, A_i)^{1/k}. \tag{15}$$

This is because for $k = 2$ we have $\theta(k) = 2$ and the two formulas coincide.

Actually, if we take all functions satisfying the required conditions (1)–(5), then we can supplement this by all k-ingles with other symmetric metrics and l-ingles ($l \neq k$) with symmetric metrics of rank l.

We will henceforth consider arbitrary functions of such polyingles.

The Berwald–Moor metric possesses a continuous isometry group I, whereas the isometry groups of the other symmetric metrics do not contain this isometry group isometry I. Therefore, in this case, other polyingles become unusable. For other metrics (e.g., for the Chernov metric without a continuous isometry group) we could significantly expand the class of possible polyingles. In this case, only the restriction not to use other symmetric metrics except for the initial metric enables us to restrict ourselves to only these polyingles.

Now we prove some properties of these polyingles.

Lemma 2. *Let g_{ij} be a symmetric metric of rank 2 all of whose off-diagonal elements are zero. If all diagonal elements are nonnegative, then the following inequality holds:*

$$(A, B) \leq (A, A)(B, B),$$

where

$$(A, B) = \sum g_{ij} A^i B^j.$$

If

$$g_{11} > 0, \quad g_{ii} \leq 0, \quad i > 1,$$

then for time-like vectors

$$A^1 > 0, \quad B^1 > 0, \quad (A, A) > 0, \quad (B, B) > 0,$$

and also

$$(A, B) \geq (A, A)(B, B).$$

Proof. Consider the quadratic expression

$$f(x) = (xA' + B', xA' + B') = (A', A')x^2 + 2(A', B')x + (B', B'),$$

where

$$A'^i = |A^i|, \quad B'^i = |B^i|.$$

In the first case, the quadratic form is positive-definite and hence the discriminant $D = (A', B')^2 - (A', A')(B', B')$ of the form is nonpositive. Therefore, we have the estimates

$$(A, B)^2 \leq (A', B')^2 \leq (A', A')(B', B') = (A, A)(B, B).$$

In the second case, the quadratic form is not positive-definite and hence the discriminant is nonnegative. Therefore,

$$(A, B)^2 \geq (A', B')^2 \geq (A', A')(B', B') = (A, A)(B, B).$$

The lemma is proved. □

In fact, in this proof we have not used the diagonality of g_{ij}. The main thing here is not the positive definiteness of the corresponding quadratic form, but the existence of x such that $(Ax-B, Ax-B) \leq 0$.

Lemma 3. *If a metric of rank k is of Lorentz type, then for any positive vectors A_l^i, $l = 1, \ldots, k$, the following inequality holds*:

$$(A_1, \ldots, A_k)^2 \geq (A_1, \ldots, A_k)(A_1, \ldots, A_k). \tag{16}$$

Proof. This lemma immediately follows from the second case of Lemma 2 with using accompanying metrics. □

Lemma 4. *Any linear combination of metrics of Lorentz type with positive coefficients is also a metric of Lorentz type.*

This lemma follows directly from the definition.

Lemma 5. *Permanent metrics (i.e., metrics subordinated to the Berwald–Moor metric and conformally equivalent to the metric $ds^k = \Pi_k$) are metrics of Lorentz type.*

Proof. This lemma is implied by the fact that all accompanying metrics of rank 2 can be represented as linear combinations with positive coefficients of metrics of Lorentz type. □

Lemma 3 implies the following assertion.

Proposition. *All k-ingles take values not less than 1 on positive vectors.*

Proof. Note that on positive vectors, metrics of Lorentz type satisfy property (16). Therefore, successively amalgamating the partition of k into summands, we decrease the product in the geometric mean.

For simplicity, we illustrate this by examples of tringles and "quadringles." For tringles, we write the following property:

$$(A, B, C)^6 \geq (A, A, C)(B, B, C)(A, B, A)(C, B, C)(A, B, B)(A, C, C).$$

Raising to the power 1/6 and dividing by $|A||B||C|$, we obtain the following inequality:

$$\rho_1 \geq \rho_2.$$

Similarly, we write inequality (16) for six terms:

$$(A, A, C)^2 \geq (A, C, C)(A, A, A), \quad (B, B, C)^2 \geq (B, C, C)(B, B, B), \quad \ldots.$$

Multiplying them and dividing the left-hand side by the right-hand side (all terms are positive for positive vectors), we obtain the inequality

$$\rho_1 \geq \rho_2 \geq 1. \tag{17}$$

Further, for $k = 4$ we enumerate the partitions of 4:

(1) $1 + 1 + 1 + 1$;
(2) $2 + 1 + 1$;
(3) $2 + 2$;
(4) $3 + 1$;
(5) 4.

For any of the six variants we write an inequality of the form (16)

$$(A, B, C, D)^2 \geq (A, A, C, D)(B, B, C, D), \ldots$$

and multiply them; we obtain

$$\rho_1 \geq \rho_2.$$

Multiplying the twelve inequalities of the form

$$(A, A, C, D)^2 \geq (A, A, C, C)(A, A, D, D),$$

we have

$$\rho_2 \geq \rho_3.$$

Further, multiplying the six inequalities of the form

$$(A, A, B, B)^2 \geq (A, A, A, B)(A, B, B, B),$$

we have

$$\rho_3 \geq \rho_4.$$

Finally, multiplying the six inequalities of the form

$$(A, A, A, B)^4 \geq (A, A, A, A)^2(A, A, B, B)^2 \geq (A, A, A, A)(A, A, A, B)(A, B, B, B),$$

we obtain the final inequalities for quadringles:

$$\rho_1 \geq \rho_2 \geq \rho_3 \geq \rho_4 \geq 1. \tag{18}$$

Inequalities for higher k-ingles can be obtained similarly. □

Now we consider calculations of k-ingles up to second order of smallness in the case where the vectors do not greatly differ from the unit vector:

$$A^i_l = 1 + x^i_l, \quad |x^i_l| \ll 1.$$

This corresponds to the case of vectors whose spatial coordinates satisfy the inequality

$$\max(|dx|, |dy|, |dz|) \ll dt,$$

i.e., vectors corresponding to small velocities (compared to the speed of light):

$$v \ll 1.$$

We denote the subordinated second-order metric by g_{ij}, the first-order metric by

$$g_i = \sum_j g_{ij},$$

and the zero-order metric by

$$g = \sum_i g_i.$$

The following relations hold:

$$(A_1, \ldots, A_k) = g + \sum_{l=1}^{k} \sum_{i=1}^{n} g_i x_l^i + \sum_{1 \le l < m \le k} \sum_{i,j} g_{ij} x_l^i x_m^j + O\left(\varepsilon^3\right), \tag{19}$$

and

$$\begin{aligned} |A_l| &= \left(g + k \sum_{i=1}^{n} g_i x_l^i + \frac{k(k-1)}{2} \sum_{i,j} g_{ij} x_l^i x_l^j \right)^{1/k} + O\left(\varepsilon^3\right) \\ &= g^{1/k} \left(1 + \frac{1}{g} \sum_{i=1}^{n} g_i x_l^i + \frac{k-1}{2g} \sum_{i,j} g_{ij} x_l^i x_l^j - \frac{k-1}{2g^2} \left(\sum_i g_i x^i \right)^2 \right) + O\left(\varepsilon^3\right). \end{aligned}$$

Dividing the k-scalar product by the product of norms, we see that k-ingles are identically equal to 1 up to the first order of smallness. This implies the following:

$$\begin{aligned} \rho_1 = 1 &+ \frac{1}{g} \sum_{1 \le l < m \le k} \sum_{i,j} g_{ij} x_l^i x_m^j - \frac{k-1}{2g} \sum_{l,i,j} g_{ij} x_l^i x_l^j \\ &+ \frac{k-1}{2g^2} \sum_l \left(\sum_i g_i x_l^i \right)^2 - \frac{1}{g^2} \sum_{1 \le l < m \le k} \sum_i g_i x_l^i \sum_i g_i x_m^i + O\left(\varepsilon^3\right). \end{aligned} \tag{20}$$

Other k-ingles have a similar form. However, only k-ingles calculated up to the second order of smallness are invariant under Lorentz transforms that preserve subordinated metrics up to the second order of smallness.

Clearly, the symmetry group of a function calculated with less accuracy is not less than the exact symmetry group (the exact symmetry preserves approximations of all orders). Therefore, the question arises, what is the exact symmetry group. To this end, we can calculate k-ingles up to third order and recognize which symmetries of second order are also symmetries of third order.

However, this approach is too cumbersome. The following approach is simpler. Assume that the symmetry group G of a certain function is broader than the group K of conformal transformations. If G contains a certain rotation g (circular or hyperbolic), then it also contains all rotations aga^{-1} for any automorphism a of the algebra of polynumbers.

It is easy to show that the minimum group containing the group of Bervald–Moor conformal transformations and elements g and aga^{-1} is the complete group $\mathrm{Gl}(n)$. As was noted above (see Lemma 1), such a function must be a constant, which contradicts the necessary condition (5) and the definition of a k-ingle. Therefore, this remark completely solves the problem of polyingles.

It remains to verify that the tringles $\rho_1, \rho_2, \ldots$ introduced above are functionally independent. This question is intuitively clear, and to prove independence of the tringles mentioned, we restrict ourselves to subordinated metrics for the Bervald–Moor metrics. In such polynumerical metrics, the scalar products are calculated as permanents and represent some average values. The k-ingles themselves represent a certain type of correlation (more exactly, the value reciprocal to the correlation) between the values of the coordinates. In the case where all coordinates are positive, they are bounded from below by 1 (the strong correlation) and they are always greater than 1 if the set contains miscellaneous vectors. Moreover, for positive vectors we can order these k-ingles as follows:

$$1 \le \rho_{p(k)-1} \le \cdots \le \rho_2 \le \rho_1.$$

First, we consider the case of H_2. As was mentioned above, we obtain in this case the Minkowski metric

$$ds^2 = dt^2 - dx^2, \quad dt = \frac{x_1 + x_2}{2}, \quad dx = \frac{x_1 - x_2}{2}.$$

Therefore, bingles characterize hyperbolic rotations; they are not defined for two arbitrary vectors. The same situations occurs also for the case $k > 2$.

Calculating tringles, we can combine one vector C with the unit vector by a conformal transformation; then the vectors A and B become A' and B', respectively, and the tringle can be calculated by the subordinated scalar product of rank 2:

$$\rho_1 = \frac{(A', B', 1)}{|A'||B'||1|}.$$

However, the norms in the denominator are calculated by a metric of rank 3, which need not coincide with the norms in the subordinated metric of rank 2.

Consider the case H_3, $k = n = 3$. We have

$$\rho_1 = \frac{\operatorname{per}(x, y, z)}{[\operatorname{per}(x, x, x)\operatorname{per}(y, y, y)\operatorname{per}(z, z, z)]^{1/3}},$$

$$\rho_2 = \frac{[\operatorname{per}(x, x, y)\operatorname{per}(x, x, z)\operatorname{per}(x, y, y)\operatorname{per}(y, y, z)\operatorname{per}(x, z, z)\operatorname{per}(y, z, z)]^{1/6}}{[\operatorname{per}(x, x, x)\operatorname{per}(y, y, y)\operatorname{per}(z, z, z)]^{1/3}}.$$

To prove the functional independence of these quantities, it suffices to present examples of sets of vectors for which one quantity is the same whereas the other is not. Consider an arbitrary set of three vectors x, y, and z one of whose coordinates is equal to 0 and the other two coordinates are positive numbers. Then $1/\rho_1 = 0$ for any such sets whereas the quantities of ρ_2 can differ, which proves their functional independence.

The case of the Chernov metric ($k = 3$, $n = 4$) is reduced to the previous case if we consider sets of vectors in which one fixed coordinate is equal to 0.

The case H_4, $k = n = 4$, is reduced to the case H_3 if one of the four vectors is $E = (1, 1, 1, 1)$. In this case, we obtain four types of 4-ingles. One can prove their functional independence and that they are ordered for sets with positive coordinates.

The question also arises, aren't elliptic functions used for calculation of tringles for the Berwald–Moor metric of rank 3? An analysis shows that in this case elliptic functions do not play a key role. Elliptic functions and the theory of Abelian varieties (in the case where a variety is a commutative group whose operation is represented by a rational function of coordinates of the summands) are applicable for the so-called Penrose isotropic cones (see [1, 2, 4, 6, 8–12, 14–27]).

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