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Applied Non-Linear Dynamical Systems

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Applied Non-Linear Dynamical Systems

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Preface

The 12th International Conference "Dynamical Systems-Theory and Applications" (DSTA) was held on December 2–5, 2013, in Łódź, Poland. Organized by the staff of the Department of Automation, Biomechanics and Mechatronics of the Lodz University of Technology, the aim of the conference was to discuss and illustrate the present state and perspective for modeling, simulation, and control of nonlinear dynamical systems—a rapidly developing research front that includes various disciplines in science, bioscience, and high technology.

The Scientific Committee of the conference include the following researchers: Marcilio Alves (Brazil), Igor V. Andrianov (Ukraine), José M. Balthazar (Brazil), Wojciech Blajer (Poland), Tadeusz Burczyński (Poland), Czesław Cempel (Poland), Simona-Mariana Cretu (Romania), Virgil-Florin Duma (Romania), Horst Ecker (Austria), Michal Fečkan (Slovakia), Barry Gallacher (UK), Józef Giergiel (Poland), Peter Hagedorn (Germany), Katica Hedrih (Serbia), Ivana Kovacić (Serbia), Janusz Kowal (Poland), Jan Kozanek (Czech Republic), Vadim A. Krysko (Russia), Lidiya V. Kurpa (Ukraine), Claude-Henri Lamarque (France), Gennady A. Leonov (Russia), Nuno M.M. Maia (Portugal), Leonid I. Manevitch (Russia), Yuriy Mikhlin (Ukraine), Gerard Olivar (Colombia), Carla M.A. Pinto (Portugal), Bogdan Posiadała (Poland), Stanisław Radkowski (Poland), Bogusław Radziszewski (Poland), Giuseppe Rega (Italy), Christos H. Skiadas (Greece), Alexander Seyranian (Russia), Gábor Stépán (Hungary), Jerzy Świder (Poland), Andrzej Tylikowski (Poland), Tadeusz Uhl (Poland), Ferdinand Verhulst (The Netherlands), Jerzy Warmiński (Poland), Edmund Wittbrodt (Poland), Józef Wojnarowski (Poland), Ludmila V. Yakushevich (Russia), Hamad M. Yehia (Egypt), Mikhail V. Zakrzhevsky, and Klaus Zimmermann (Germany).

In the following 38 chapters we present only a small sample of different approaches and understandings of dynamical systems and their applications in physics, mechanics, automation, biomechanics, and applied mathematics.

In Chap. 1 Kovacic and Rand consider nonlinear oscillators with period independent of amplitude and with Duffing-type restoring force. They present Lagrangians, conservation laws, equations of motions, and solutions of the governing equations for oscillators with hardening and softening cubic-type nonlinearities.

Pilipchuk in Chap. 2 presents physical insight and methodologies of asymptotics of "rigid-body" motions for nonlinear dynamics. He shows that by tracking rigid Euclidean transformation nonlinear models can be revealed.

In Chap. 3 Krysko et al. use the Bubnov–Galerkin method and the finite difference method for studies of vibrations of flexible cylindrical and sector shells subjected to the action of uniformly distributed loads. A few novel nonlinear phenomena have been reported.

Bhem and Schwebke analyze in Chap. 4 the wormlike locomotion system. They focused on gear shift patterns and presented procedure for their adjustments to optimize speed and gait/crawling to predefined limits of actuator or spike force load.

Awrejcewicz et al. illustrate and discuss (in Chap. 5) periodic and chaotic dynamics of plates and shells as well as a weak turbulent behavior exhibited by these solid structures, while modeling them as 2D infinite objects. They present also novel approaches to obtain reliable results of nonlinear differential equations, as well as new methods of chaos monitoring.

In Chap. 6 Morcillo et al. propose a methodology of using bifurcation diagrams for computation chaos controllers. They apply this to PWM-controlled power converters method based on an adaptive control, where the sawtooth signal is redefined as a function of the output and reference voltages.

In Chap. 7 Andrianov et al. present studies of anti-plane shear waves. They focus on wave propagation through a cylindrical structured cancellous bone, applying model of a two-dimensional mesh of elastic trabeculae filled by a viscous marrow.

Kizilova et al. focused in Chap. 8 on dynamics of postural sway in human. Analyzing body sway patterns for the group of young healthy individuals and two groups of patients with pathologies of spine and joints, they observed quasi-regular and chaotic dynamics with certain asymmetry of the body acceleration, respectively.

In Chap. 9 Adamiec-Wójcik et al. present modeling of the slender system by means of the rigid finite element method. They considered large deformations and friction using rope as the model. Presented and analyzed are also dynamics of an offshore crane lifting a load from a vessel.

Syta and Litak in Chap. 10 presented results of the investigations of Van der Pol–Duffing system. They focused on the dynamical response of the system with an external harmonic excitation and a memory of a fractional characteristic. The obtained results indicate occurrence of bifurcations of quasiperiodic solutions with short intervals of chaotic solutions.

Analysis of fluid flow around the cylinder is described by Akhmetov and Kutluev in Chap. 11. They apply asymptotic methods for problem of the vortex structure appearance in a stationary viscous incompressible fluid and investigate properties of the flow function in boundary layer.

In Chap. 12 Awrejcewicz et al. present results of application of the asymptotical approach in the form of limiting phase trajectories and multiple timescale methods for analysis of dynamical problem of a two-degree-of-freedom mechanical system. Application of those approaches in the case of spring pendulum allows for determination of critical values of the parameters responsible for change of the character of vibrations.

Kaliński et al. present (Chap. 13) an idea of the workpiece holder with adjustable stiffness to be applied in milling of flexible details. This holder, together with vibration surveillance system, allows for more efficient milling using slender ball-end tools without interference of vibrations.

In Chap. 14 Blajer presents problem of servo-constraint in the inverse simulation problem for underactuated mechanical system. Simulation results are discussed and confronted with computational issues of the governing equations in the form of ordinary differential equations and differential algebraic equations.

Global analysis of nonlinear dynamics of simple hybrid electronic systems through application of the method of complete bifurcation groups is presented by Pikulin in Chap. 15. He also proves that it is possible to design reliable switching power convertor. The applied method of complete bifurcation groups allows to predict and avoid occurrence of undesirable regimes in operation.

Udwadia and Mylapilli in Chap. 16 discuss interrelations and connections between tracking control of nonlinear systems and constrained motion of mechanical systems. By providing diverse examples, they illustrate how ease, simplicity, and efficiency in control can be achieved. They also show that closed-form forces obtained are optimal and minimize the control cost at each instant of time still providing exact trajectory tracking.

In Chap. 17 Ruchkin presents new method of the intellectual investigations of the nonlinear dynamical system. By application of this method the system with a special Hamiltonian structure is studied and its regular and chaotic behavior is analyzed.

Lateral dynamics behavior of the two-axle freight wagon with the UIC double-link suspension in dependence on chosen parameters is shown in Chap. 18 by Matej et al. Based on the Coulomb law regarding friction, applied are the non-smooth mechanics, which allows for derivation of the mathematical models with and without lateral bump-stop.

Zimmerman et al. focus on kinematics and dynamics of a mechanical system with mecanum wheels (Chap. 19). They proceed with comparison of nonholonomic model and approximate model used in robotics, obtaining similar results, which resulted in the production of prototype of a mobile robot with four mecanum wheels.

Chapter 20 by Ritto and Sampaio is devoted to reliability analysis of horizontal drill-string dynamics. It focuses on reliability of the operation, defined as the probability of not achieving a target efficiency and measured by the mean input/output ratio.

Hedrih presents in Chap. 21 changes of elasticity 3D matrix surrounding mammalian oocyte. She considers changes occurring in zona pellucida during maturation and fertilization processes. Applying an oscillatory spherical net model of mouse, eigen circular frequencies of mouse oocyte and mouse embryo have been estimated.

Problem of mass points interacting gravitationally is presented by Szumiński and Przybylska in Chap. 22. They illustrate complicated behavior of trajectories of system with applied certain holonomic constraints using Poincarè cross sections.

Verhulst applies in Chap. 23 slow–fast timescales in the framework of Fenichel geometric singular perturbation theory for analysis of equations with periodic coefficients for singularly perturbed growth.

In Chap. 24 basing on fractional calculus and the concept of fractals Abramov proposes a model of nonlinear fractal oscillator. Investigated are stress field for different structural states of structural dislocation in a nanosystem, as well as its deformation and behavior of energy spectrum.

Dynamics in diagnostic expert systems are addressed by Cholewa in Chap. 25. He introduces dynamic statement network, where statements consist of contents and values. After detailed comparison of networks, he proposes manner of transformation from a static network into dynamic one.

In Chap. 26 Palej proposes method for solving the nonstandard two-point boundary value problem. He considers the case, where the number of boundary conditions is higher than the number of first-order ordinary differential equations containing certain number of unknown parameters and where difference between standard and nonstandard boundary value problem consists in the size of the initial value problem that needs to be solved.

Lainscsek et al. in Chap. 27 propose a differential equation with time delay for automatic sleep scoring from single electrode. Assuming that there is a parameter, where at least one of coefficients varies depending on sleep stages, it is possible to construct hypnogram. In this case brain activity is considered as resulting from a dynamical system.

Chapter 28 by Gyebrószki and Csernák is focused on numerical methods for quick analysis of micro-chaos. They describe and compare different methods used for characterizing chaotic behavior and compare them with simple cell mapping for investigations of chaotic behavior for the case of digitally controlled inverted pendulum.

Studies of a bouncing ball impacting with a periodically moving limiter are presented in Chap. 29 by Okniński and Radziszewski. They analyze this problem within two defined frameworks of the table motion for four cubic polynomial and sinusoidal motion models.

In Chap. 30 Pascal and Stepanov present results of investigations of the behavior of the strongly nonlinear vibrating system excited by dry friction and harmonic force. They consider as the model a system composed of two masses connected by linear springs, where one of the masses is in contact with a driving belt moving at a constant velocity and friction force with Coulomb's characteristics acting between mass and belt.

Selyutskiy in Chap. 31 focuses on studies of aerodynamic pendulum dynamics in low-speed airflow. He applies as an investigation model a phenomenological mode with the internal dynamics of the flow simulated by an oscillator attached to the pendulum and obtained results of simulations are in good correlation with experimental results.

Studies of nonlinear interactions of two coupled oscillators at different timescales are presented by Ture Savadkoohi and Lamarque in Chap. 32. Analytical developments of those investigations are compared with numerical results and discussed is the possibility of the passive control of the main system by means of the time-dependent nonlinear energy sink.

In Chap. 33 Szklarz and Jarzębowska discuss a systematic coordinate-free approach for the formulation of the no-slip condition for wheeled robot models that can be used for derivation of the nonholonomic constraint equations. This approach yields models that are unified, verifiable, comparable, and repeatable.

Original unitary theory of the optical choppers with rotating wheels working with top-hat laser beams is discussed in Chap. 34 by Circa and Duma. They present program developed for designing of the optical choppers with rotating wheels and verified it by application to classical and eclipse choppers.

In Chap. 35 modification of the mathematical model of the HIV dynamics in HIV-specific helper cells is presented by Pinto and Carvalho. Considered are two types of the delay—a latent period for the interval for cells, first one with contact with the virus, to be infected by the virons and released by them and the second one is a viron production period for the virons to be infected by virons and released to the blood stream from the infected cells.

Gallacher et al. discuss in Chap. 36 application of parametric excitation and amplification for MEMS ring gyroscope. Theoretically proved is that the parametric excitation offers suitable excitation method for the rate integrating gyroscope.

In Chap. 37 Piccirillo et al. present analysis of the influence of external parameters on the oscillator dynamics. They investigate two smart materials, i.e., shape memory alloy and magneto rheological damper as well as effect of system dissipation energy related to their hysteretic behavior.

Generalized method of studying plane topographical Poincarè systems to higher dimensions is presented by Shamolin in Chap. 38. He shows also elaborated methods for qualitative study of dissipative systems and systems with anti-dissipation yielding a possibility of obtaining conditions for bifurcation of birth of stable and unstable auto-oscillations.

As can be noticed by the variability of topics of the chapters, dynamical systems analysis can be found in very wide range of scientific disciplines and their constant development is unavoidable and necessary from both theoretical and practical point of view.

I do hope that the readers of this book will be attracted by the chosen topics. I greatly appreciate help of Springer Editor Dr. Eve Mayer in publishing the presented chapters recommended by the Scientific Committee of the DSTA 2013 after the standard review procedure. Finally, I would like to thank all referees for their help in reviewing the manuscripts.

Łódź and Warsaw, Poland Jan Awrejcewicz

Contents

Dynamical Pendulum-Like Nonconservative Systems

Maxim V. Shamolin

Abstract We have elaborated the methods for the qualitative study of dissipative systems and systems with anti-dissipation that allow us, for example, to obtain conditions for bifurcation of birth of stable and unstable auto-oscillations. We succeeded in generalizing the method for studying plane topographical Poincaré systems to higher dimensions. In three-dimensional rigid body dynamics, we have discovered complete lists of first integrals of dissipative systems and systems with anti-dissipation that are transcendental (in the sense of classification of their singularities) functions that are expressed through elementary functions in a number of cases. We have discovered new qualitative analogs between the properties of motion of free bodies in a resisting medium that is fixed at infinity and bodies in an overrun medium flow.

1 Introduction

We study the nonconservative systems for which the methods for studying, for example, Hamiltonian systems, is not applicable in general. Therefore, for such systems, it is necessary, in some sense, to "directly" integrate the main equation of dynamics. Herewith, we offer more universal interpretation of both obtained cases and new ones of complete integrability in transcendental functions in two-, three-, and four-dimensional rigid body dynamics in a nonconservative force field.

The results of the proposed work are a development of the previous studies, including a certain applied problem from rigid body dynamics [4, 9, 11], where complete lists of transcendental first integrals expressed through a finite combination of elementary functions were obtained. Later on, this circumstance allows us to perform a complete analysis of all phase trajectories and show those their properties which have a roughness and are preserved for systems of a more general form. The complete integrability of such systems is related to symmetries of latent type.

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As is known, the concept of integrability is sufficiently fuzzy in general. In its construction, it is necessary to take into account the meaning in which it is understood (we mean a certain criterion with respect to which one makes a conclusion that the structure of trajectories of the dynamical system considered is especially simple and "attractive and simple," in which class of functions we seek for first integrals, etc. (see also [8]).

In this work, we accept the approach that as the class of functions for first integrals, takes transcendental functions, and, moreover, elementary ones. Here, the transcendence is understood not in the sense of elementary function theory (for example, trigonometric functions), but in the sense of existence of essentially singular points for them (according to the classification accepted in the theory of function of one complex variable). In this case, we need to formally continue the function considered in the complex domain (see also [13]).

2 Methods for Analyzing Zero Mean Variable Dissipation Dynamical Systems in Spatial Dynamics of a Rigid Body

In this section, we consider the possibilities of extending the results of the plane dynamics of a rigid body interacting with a medium to the spatial case. We analyze the problems of a spherical pendulum placed in the overrunning medium flow and a spatial body motion under the existence of a certain non-integrable constraint and also show the mechanical and topological analogies of the latter two problems.

2.1 Statement of Problem of Spatial Body Motion in a Resisting Medium Under Streamline Flow Around

The conjectures presented in [4], which concern the medium properties, are reflected in constructing the spatial dynamical model of interaction of a medium with a body. In this connection, there arises the possibility of formalizing the model assumptions and obtaining the complete system of equations.

The whole interaction of the medium with the axe-symmetric body is concentrated on the part of the body surface, which has the form of a circular disk (Fig. 1).

Since the interaction is subjected to the streamline flow around laws, the force $\mathbf{S}$ of this interaction is directed along the normal to the disk, and, moreover, the point N of application of this force is determined by at least one parameter, the angle of attack α, which is made by the velocity vector $\mathbf{v} = \mathbf{v}_D$ of the disk point D and the exterior normal at this point (the line CD). The point D is the intersection point of the middle perpendicular dropped from the center of masses C to the plane of disk. Therefore, $DN = R(\alpha, \ldots)$.

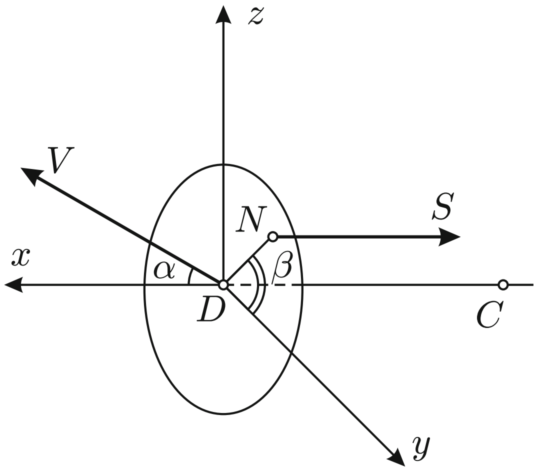

Fig. 1 Spatial model of the dynamical action of the medium on the axe-symmetric body

Assume that the value of the resistance force S has the form $S = s_1 v^2$, where v is the module of the velocity vector of the point D and the resistance coefficient s_1 (as in the plane case; see [9]) is a function of only the angle of attack α: $s_1 = s_1(\alpha)$. Therefore, as previously, we consider such an "extension" of the problem, which was mentioned in [1–4].

As in the plane case, along the line CD, an additional force $\mathbf{T}$ can act on the body; as before, it is called the "following force." The introduction of this force is used for ensuring some given classes of motion (in this case, $\mathbf{T}$ is the reaction of possible imposed constraints). In the case where there is no external force $\mathbf{T}$, the body executes a spatial free drag in the resisting medium.

Denote by $Dxyz$ the coordinate system related to the body (Fig. 1). This coordinate system related to the point D is chosen such that the tensor of inertia in this system has the diagonal form: $\mathrm{diag}\{I_1, I_2, I_3\}$. Assume that the mass distribution is such that the longitudinal principal axis of inertia coincides with the axis CD (this is the axis Dx), whereas the axes Dy and Dz lie in the plane of disk and compose the right coordinate system. Moreover, as it was noted, we consider the case of the dynamically symmetric rigid body, i.e., $I_2 = I_3$.

In this case, to describe the body position in the space, we choose the Cartesian coordinates $(\mathrm{x}_0, \mathrm{y}_0, \mathrm{z}_0)$ of the point D and three angles (θ, ψ, ϕ), which are defined similar to the navigational angles as follows (compare with [5]).

Let us represent the turn from the inertial system $D\mathrm{x}_0\mathrm{y}_0\mathrm{z}_0$ to the system $Dxyz$ as a composition of three turns $T_3(\phi) \circ T_2(\psi) \circ T_1(\theta)$ under which, first, the frame $(\mathbf{e}_{\mathrm{x}_0}, \mathbf{e}_{\mathrm{y}_0}, \mathbf{e}_{\mathrm{z}_0})$ is turned around the vector $\mathbf{e}_{\mathrm{x}_0}$ by the angle θ ($T_1(\theta)$ is executed):

$$(\mathbf{e}_{\mathrm{x}_0}, \mathbf{e}_{\mathrm{y}_0}, \mathbf{e}_{\mathrm{z}_0}) \rightarrow^{(T_1(\theta))} \left(\mathbf{e}_{\mathrm{x}_0}, \mathbf{e}^{(1)}_{\mathrm{y}_0}, \mathbf{e}^{(1)}_{\mathrm{z}_0}\right),$$

then the frame $\left(\mathbf{e}_{x_0}, \mathbf{e}_{y_0}^{(1)}, \mathbf{e}_{z_0}^{(1)}\right)$ is turned around the vector $\mathbf{e}_{y_0}^{(1)}$ by the angle ψ ($T_2(\psi)$ is executed):

$$\left(\mathbf{e}_{x_0}, \mathbf{e}_{y_0}^{(1)}, \mathbf{e}_{z_0}^{(1)}\right) \to^{(T_2(\psi))} \left(\mathbf{e}_{x_0}^{(2)}, \mathbf{e}_{y_0}^{(1)}, \mathbf{e}_z\right),$$

and, finally, the frame $\left(\mathbf{e}_{x_0}^{(2)}, \mathbf{e}_{y_0}^{(1)}, \mathbf{e}_z\right)$ is turned around the vector $\mathbf{e}_z$ by the angle ϕ ($T_3(\phi)$ is executed):

$$\left(\mathbf{e}_{x_0}^{(2)}, \mathbf{e}_{y_0}^{(1)}, \mathbf{e}_z\right) \to^{(T_3(\phi))} (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z).$$

In this case, the vectors having the components in the frame $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ obtain new coordinates in the basis $(\mathbf{e}_{x_0}, \mathbf{e}_{y_0}, \mathbf{e}_{z_0})$. In the basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$, such a transformation is given by certain matrix, and then the phase state of the system is characterized by twelve quantities $(\dot{x}_0, \dot{y}_0, \dot{z}_0, \dot{\theta}; \dot{\psi}; \dot{\phi}; x_0, y_0, z_0, \theta, \psi, \phi)$.

Let us consider the spherical coordinates (v, α, β) of the velocity vector $\mathbf{v} = \mathbf{v}_D$ endpoint of the point D in which the angle β is measured from the axis Dy in the disk plane up to the line (DN) which is the intersection of two planes, one of which contains the vector $\mathbf{v}$ and the axis Dx and the other is the disk plane.

The latter spherical coordinates and also the components of the angular velocity are expressed through the phase variables $(\dot{x}_0, \dot{y}_0, \dot{z}_0, \dot{\theta}; \dot{\psi}; \dot{\phi}; x_0, y_0, z_0, \theta, \psi, \phi)$ via non-integrable relations. Therefore, the phase state of the system is determined by the functions $(v, \alpha, \beta, \Omega_x, \Omega_y, \Omega_z, x_0, y_0, z_0, \theta, \psi, \phi)$, and the first six functions are considered as quasi-velocities of the system. Here, the tuple $(\Omega_x, \Omega_y, \Omega_z)$ is defined as

$$\Omega = \Omega_x \mathbf{e}_x + \Omega_y \mathbf{e}_y + \Omega_z \mathbf{e}_z,$$

where Ω is the absolute angular velocity vector of the rigid body.

Since the generalized forces are independent of the body position in the space, the coordinates $(x_0, y_0, z_0, \theta, \psi, \phi)$ are cyclic. This allows us to consider the dynamical part of the equations of motion as an independent subsystem.

Let us introduce the sign-alternating auxiliary function $s(\alpha) = s_1(\alpha)\mathrm{sign}\cos\alpha$.

By the theorem on the motion of the center of masses in the space in projections to the related axes (x, y, z) and the theorem on the change of the kinematic moment with respect to these axes, we obtain the following complete system of differential equations in the dynamical quasi-velocity space $\mathbf{R}_+^1\{v\} \times \mathbf{S}^2\{\alpha, \beta\} \times \mathbf{R}^3\{\Omega_x, \Omega_y, \Omega_z\}$:

$$\dot{v}\cos\alpha - \dot{\alpha}v\sin\alpha + \Omega_y v \sin\alpha\sin\beta - \Omega_z v\sin\alpha\cos\beta + \sigma(\Omega_y^2 + \Omega_z^2) = F_x/m,$$

$$\dot{v}\sin\alpha\cos\beta + \dot{\alpha}v\cos\alpha\cos\beta - \dot{\beta}v\sin\alpha\sin\beta + \Omega_z v\cos\alpha -$$

$$-\Omega_x v\sin\alpha\sin\beta - \sigma\Omega_x\Omega_y - \sigma\dot{\Omega}_z = 0,$$

$$\dot{v}\sin\alpha\sin\beta + \dot{\alpha}v\cos\alpha\sin\beta + \dot{\beta}v\sin\alpha\cos\beta + \Omega_x v\sin\alpha\cos\beta - \tag{1}$$

$$-\Omega_y v\cos\alpha - \sigma\Omega_x\Omega_z + \sigma\dot{\Omega}_y = 0,$$

$$I_1\dot{\Omega}_x = 0,$$

$$I_2\dot{\Omega}_y + (I_1 - I_2)\Omega_x\Omega_z = -z_N\,(\alpha, \beta, \Omega/v)\,s(\alpha)v^2,$$

$$I_2\dot{\Omega}_z + (I_2 - I_1)\Omega_x\Omega_y = y_N\,(\alpha, \beta, \Omega/v)\,s(\alpha)v^2,$$

where $F_x = T - s(\alpha)v^2$.

The coordinates of the point N in the system $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ take the form

$$\big(0, y_N(\alpha, \beta, \Omega/v), z_N(\alpha, \beta, \Omega/v)\big),$$

$y_N(\alpha, \beta, \Omega/v) = R(\alpha)\cos\beta - h\Omega_z/v$, $z_N(\alpha, \beta, \Omega/v) = R(\alpha)\sin\beta + h\Omega_y/v$.

We can complement the system (1) by the kinematic relations. As before, we use the following notation: σ is the distance DC and m is the mass of the body.

The dynamical system (1) contains the functions $R(\alpha)$ and $s(\alpha)$. To qualitatively describe them, we use the existing experimental information about the properties of the streamline flow around (see [3, 5]).

2.2 *Case of Body Motion in a Medium Under Existence of a Certain Non-integrable Constraint and Beginning of Qualitative Analysis*

As in the plane case, we consider the class of motions under which identity

$$v = |\mathbf{v}| \equiv \text{const}, \tag{2}$$

being a non-integrable constraint holds.

2.2.1 On Analytical Integral

In the case of the dynamically symmetric rigid body, system (1) has the analytic first integral

$$\Omega_x \equiv \Omega_{x0} = \text{const}, \tag{3}$$

i.e., the generalized forces admit a body self-rotation around the longitudinal dynamical symmetry axis.

2.2.2 On Appearance of an Independent Subsystem

Let us use the methodological tool of reducing the system order, i.e., from the first equation of (1), let us express the function T so that the total derivative of v by the system (1) vanishes. As a result of this order reduction, system (1) has an independent subsystem of the form

$$\dot{\alpha} v \cos\alpha \cos\beta - \dot{\beta} v \sin\alpha \sin\beta + \Omega_z v \cos\alpha - \Omega_{x0} v \sin\alpha \sin\beta 4-$$

$$-\sigma \Omega_{x0} \Omega_y - \sigma \dot{\Omega}_z = 0,$$

$$\dot{\alpha} v \cos\alpha \sin\beta + \dot{\beta} v \sin\alpha \cos\beta - \Omega_y v \cos\alpha + \Omega_{x0} v \sin\alpha \cos\beta + \tag{4}$$

$$+\sigma \Omega_{x0} \Omega_z + \sigma \dot{\Omega}_y = 0,$$

$$I_2 \dot{\Omega}_y = -z_N (\alpha, \beta, \Omega/v)\, s(\alpha) v^2,$$

$$I_2 \dot{\Omega}_z = y_N (\alpha, \beta, \Omega/v)\, s(\alpha) v^2,$$

in which the parameter v is added to constant parameters.

2.2.3 Case of Zero Projection of Angular Velocity on Longitudinal Axis and Case of Analytic System

Let us consider the trajectories of motion of system (4) on the level of integral (3) for

$$\Omega_{x0} = 0. \tag{5}$$

In this case, it takes the form

$$\dot{\alpha} v \cos\alpha \cos\beta - \dot{\beta} v \sin\alpha \sin\beta + \Omega_z v \cos\alpha - \sigma \dot{\Omega}_z = 0,$$

$$\dot{\alpha} v \cos\alpha \sin\beta + \dot{\beta} v \sin\alpha \cos\beta - \Omega_y v \cos\alpha + \tag{6}$$

$$+\sigma \dot{\Omega}_y = 0,$$

$$I_2 \dot{\Omega}_y = -z_N (\alpha, \beta, \Omega/v)\, s(\alpha) v^2, \; I_2 \dot{\Omega}_z = y_N (\alpha, \beta, \Omega/v)\, s(\alpha) v^2. \tag{7}$$

The first two equations (i.e., (6)) reduce to the form

$$\dot{\alpha} v \cos\alpha + v \cos\alpha \left[\Omega_z \cos\beta - \Omega_y \sin\beta\right] +$$

$$+\sigma\left[-\dot{\Omega}_z\cos\beta+\dot{\Omega}_y\sin\beta\right]=0, \tag{8}$$

$$\dot{\beta}v\sin\alpha-v\cos\alpha\left[\Omega_y\cos\beta+\Omega_z\sin\beta\right]+$$

$$+\sigma\left[\dot{\Omega}_y\cos\beta+\dot{\Omega}_z\sin\beta\right]=0, \tag{9}$$

If, for example, S. A. Chaplygin conditions [1]

$$R(\alpha)=A\sin\alpha,\ s(\alpha)=B\cos\alpha;\ A,B>0, \tag{10}$$

hold, then, introducing the notations

$$n_0^2=\frac{AB}{I_2},\ b=\sigma n_0,\ H_1=\frac{Bh}{I_2 n_0},\ [b]=[H_1]=1,$$

we have the following analytic system:

$$\dot{\alpha}-bn_0v\sin\alpha+(1+bH_1)\left[\Omega_z\cos\beta-\Omega_y\sin\beta\right]=0, \tag{11}$$

$$\dot{\beta}\sin\alpha-(1+bH_1)\cos\alpha[\Omega_y\cos\beta+\Omega_z\sin\beta]=0. \tag{12}$$

Let us complement it by the equations

$$\dot{\Omega}_y=-n_0^2v^2\sin\alpha\cos\alpha\sin\beta-H_1n_0v\Omega_y\cos\alpha, \tag{13}$$

$$\dot{\Omega}_z=n_0^2v^2\sin\alpha\cos\alpha\cos\beta-H_1n_0v\Omega_z\cos\alpha. \tag{14}$$

The system (11)–(14) is closed. Note that system (11), (12) is equivalent to (8), (9) only outside the manifold

$$\mathscr{O}=\{(\alpha,\beta,\Omega_y,\Omega_z):\ \cos\alpha=0\}. \tag{15}$$

Let us project the angular velocity to the movable coordinate system Dz_1z_2 (turning the system Dyz by the angle $-\beta$) such that

$$z_1=\Omega_y\cos\beta+\Omega_z\sin\beta,\quad z_2=\Omega_z\cos\beta-\Omega_y\sin\beta. \tag{16}$$

In this case, since $\dot{z_1}=\dot{\beta}z_2$, system (6), (7) is equivalent to the following system outside the manifold $\mathscr{O}'$:

$$\dot{\alpha}=-\left(1+b\frac{h}{I_2n_0}\frac{s(\alpha)}{\cos\alpha}\right)z_2+b\frac{v}{I_2n_0}\frac{\mathscr{F}(\alpha)}{\cos\alpha}, \tag{17}$$

$$\dot{z_2}=\frac{v^2}{I_2}\mathscr{F}(\alpha)-\left(1+b\frac{h}{I_2n_0}\frac{s(\alpha)}{\cos\alpha}\right)z_1^2\frac{\cos\alpha}{\sin\alpha}-\frac{hv}{I_2}\frac{s(\alpha)}{\cos\alpha}z_2\cos\alpha, \tag{18}$$

$$\dot{z}_1 = \left(1 + b\frac{h}{I_2 n_0}\frac{s(\alpha)}{\cos\alpha}\right) z_1 z_2 \frac{\cos\alpha}{\sin\alpha} - \frac{hv}{I_2}\frac{s(\alpha)}{\cos\alpha} z_1 \cos\alpha, \tag{19}$$

$$\dot{\beta} = \left(1 + b\frac{h}{I_2 n_0}\frac{s(\alpha)}{\cos\alpha}\right) z_1 \frac{\cos\alpha}{\sin\alpha}, \tag{20}$$

where

$$\mathcal{O}' = \{(\alpha, \beta, \Omega_y, \Omega_z) : \ \sin\alpha\cos\alpha = 0\}, \ \mathcal{F}(\alpha) = R(\alpha)s(\alpha). \tag{21}$$

Under S. A. Chaplygin condition (10) and the substitutions

$$< \cdot > \mapsto n_0 v <'>, \ z_k \mapsto n_0 v z_k, \ k = 1, 2,$$

the system (17)–(20) takes the form of the analytic system

$$\dot{\alpha} = -(1 + bH_1)\, z_2 + b\sin\alpha, \tag{22}$$

$$\dot{z}_2 = \sin\alpha\cos\alpha - (1 + bH_1)\, z_1^2 \frac{\cos\alpha}{\sin\alpha} - H_1 z_2 \cos\alpha, \tag{23}$$

$$\dot{z}_1 = (1 + bH_1)\, z_1 z_2 \frac{\cos\alpha}{\sin\alpha} - H_1 z_1 \cos\alpha, \tag{24}$$

$$\dot{\beta} = (1 + bH_1)\, z_1 \frac{\cos\alpha}{\sin\alpha}, \tag{25}$$

Outside the manifold $\mathcal{O}'$, from the dynamical system (17)–(20), the independent third-order system (17)–(19) is separated. The system (17)–(20) is considered on the tangle bundle to the two-dimensional sphere (see [13]). Since we have the "separation" of the third-order system, the phase space of our system has a number of bundles.

2.2.4 Symmetries of System Vector Field in Quasi-velocity Phase Space

The vector field of system (17)–(19) has the following three types of symmetries:

1. The central symmetry. Near $(\pi k, 0, 0)$, $k \in \mathbf{Z}$, such a symmetry arises owing to that the field in the coordinates (α, z_2, z_1) alternates its sign under the change

$$\begin{pmatrix} \pi k - \alpha \\ -z_2 \\ -z_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \pi k + \alpha \\ z_2 \\ z_1 \end{pmatrix}$$

2. A certain mirror symmetry (CMS). With respect to the planes Λ_i, $i \in \mathbf{Z}$, where

$$\Lambda_i = \left\{(\alpha, z_2, z_1) : \alpha = \frac{\pi}{2} + \pi i\right\},$$

such a symmetry arises owing to that the α-component of the system vector fields in the coordinates (α, z_2, z_1) is preserved under the change

$$\begin{pmatrix} \pi/2 + \pi i - \alpha \\ z_2 \\ z_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \pi/2 + \pi i + \alpha \\ z_2 \\ z_1 \end{pmatrix}$$

whereas the z_2- and z_1-components alternate their signs.

3. The symmetry with respect to the plane

$$\{(\alpha, z_2, z_1) : z_1 = 0\};$$

precisely, the z_2- and α-components of the system vector field are preserved under the change

$$\begin{pmatrix} \alpha \\ z_2 \\ -z_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha \\ z_2 \\ z_1 \end{pmatrix}$$

whereas the z_1-component alternates its sign.

2.3 On Transcendental Integrability of System

This subsection is devoted to studying the possibilities of complete integration of the dynamical system considered. Here, we present first integrals of system (22)–(25) expressed through elementary functions and also discuss the way of integrating the general system (17)–(20).

2.3.1 Complete List of Invariant Relations

At the beginning we compare the third-order system (22)–(24) to the nonautonomous second-order system

$$\frac{dz_2}{d\alpha} = \frac{\sin\alpha\cos\alpha - (1 + bH_1)z_1^2\cos\alpha/\sin\alpha - H_1 z_2\cos\alpha}{-(1 + bH_1)z_2 + b\sin\alpha}, \tag{26}$$

$$\frac{dz_1}{d\alpha} = \frac{(1 + bH_1)z_1 z_2\cos\alpha/\sin\alpha - H_1 z_1\cos\alpha}{-(1 + bH_1)z_2 + b\sin\alpha}. \tag{27}$$

Let us rewrite the system (26), (26) on algebraic form using the substitution $\tau = \sin\alpha$

$$\frac{dz_2}{d\tau} = \frac{\tau - (1 + bH_1)z_1^2/\tau - H_1 z_2}{-(1 + bH_1)z_2 + b\tau}, \ \frac{dz_1}{d\tau} = \frac{(1 + bH_1)z_1 z_2/\tau - H_1 z_1}{-(1 + bH_1)z_2 + b\tau}. \quad (28)$$

Later on, if we introduce the uniform variables by the formulas $z_k = u_k\tau, \ k = 1, 2$, we shall reduce the system (28) to the following form:

$$\tau\frac{du_2}{d\tau} = \frac{(1 + bH_1)(u_2^2 - u_1^2) - (b + H_1)u_2 + 1}{-(1 + bH_1)u_2 + b}, \quad (29)$$

$$\tau\frac{du_1}{d\tau} = \frac{2(1 + bH_1)u_1 u_2 - (b + H_1)u_1}{-(1 + bH_1)u_2 + b}. \quad (30)$$

Let us compare the second-order system (29), (30) to the nonautonomous first-order

$$\frac{du_2}{du_1} = \frac{1 - (1 + bH_1)(u_1^2 - u_2^2) - (b + H_1)u_2}{2(1 + bH_1)u_1 u_2 - (b + H_1)u_1}, \quad (31)$$

which is reduced uncomplicated to the complete differential:

$$d\left(\frac{(1 + bH_1)(u_2^2 + u_1^2) - (b + H_1)u_2 + 1}{u_1}\right) = 0. \quad (32)$$

And so, equation (31) has the following first integral:

$$\frac{(1 + bH_1)(u_2^2 + u_1^2) - (b + H_1)u_2 + 1}{u_1} = C_1 = \text{const}, \quad (33)$$

which in former variables is looked like

$$\frac{(1 + bH_1)(z_2^2 + z_1^2) - (b + H_1)z_2 \sin\alpha + \sin^2\alpha}{z_1 \sin\alpha} = C_1 = \text{const}. \quad (34)$$

Later on, let us find the evident form of the additional first integral of the third-order system (22)–(24). At the beginning for this we shall transform the invariant relation (33) for $u_1 \neq 0$ as follows:

$$\left(u_2 - \frac{b + H_1}{2(1 + bH_1)}\right)^2 + \left(u_1 - \frac{C_1}{2(1 + bH_1)}\right)^2 = \frac{(b - H_1)^2 + C_1^2 - 4}{4(1 + bH_1)^2}. \quad (35)$$

As is seen, the parameters of the given invariant relation should satisfy the condition

$$(b - H_1)^2 + C_1^2 - 4 \geq 0, \quad (36)$$

and the phase space of the system (22)–(24) is stratified on the family of the surfaces which is assigned by the equality (35).

Thus, by virtue of the relation (33), equation (29) has the form

$$\tau \frac{du_2}{d\tau} = \frac{2(1+bH_1)u_2^2 - 2(b+H_1)u_2 + 2 - C_1 U_1(C_1, u_2)}{b - (1+bH_1)u_2}, \tag{37}$$

where

$$U_1(C_1, u_2) = \frac{1}{2(1+bH_1)}\{C_1 \pm U_2(C_1, u_2)\}, \tag{38}$$

$$U_2(C_1, u_2) = \sqrt{C_1^2 - 4(1+bH_1)(1-(b+H_1)u_2 + (1+bH_1)u_2^2)},$$

herewith, the constant of the integration C_1 is chosen from the condition (36).

Therefore, the quadrature for the search of the additional first integral of the system (22)–(24) has the form

$$\int \frac{d\tau}{\tau} =$$

$$= \int \frac{(b-(1+bH_1)u_2)du_2}{2(1-(b+H_1)u_2 + (1+bH_1)u_2^2) - C_1\{C_1 \pm U_2(C_1, u_2)\}/(2(1+bH_1))}. \tag{39}$$

The left-hand side (accurate to the additive constant), obviously, is equal to $\ln|\sin\alpha|$. If

$$u_2 - \frac{b+H_1}{2(1+bH_1)} = w_1, \; b_1^2 = (b-H_1)^2 + C_1^2 - 4, \tag{40}$$

then the right-hand side of the equality (39) has the form

$$-\frac{1}{4}\int \frac{d(b_1^2 - 4(1+bH_1)w_1^2)}{(b_1^2 - 4(1+bH_1)w_1^2) \pm C_1\sqrt{b_1^2 - 4(1+bH_1)w_1^2}} -$$

$$-(b-H_1)(1+bH_1)\int \frac{dw_1}{(b_1^2 - 4(1+bH_1)w_1^2) \pm C_1\sqrt{b_1^2 - 4(1+bH_1)w_1^2}} =$$

$$= -\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2 - 4(1+bH_1)w_1^2}}{C_1} \pm 1\right| \pm \frac{b-H_1}{2} I_1, \tag{41}$$

where

$$I_1 = \int \frac{dw_3}{\sqrt{b_1^2 - w_3^2}(w_3 \pm C_1)}, \; w_3 = \sqrt{b_1^2 - 4(1 + bH_1)w_1^2}. \tag{42}$$

Three cases are possible for the calculation of the integral (42):

I. $|b - H_1| > 2$.

$$I_1 = -\frac{1}{2\sqrt{(b-H_1)^2-4}} \ln \left| \frac{\sqrt{(b-H_1)^2-4} + \sqrt{b_1^2 - w_3^2}}{w_3 \pm C_1} \pm \frac{C_1}{\sqrt{(b-H_1)^2-4}} \right| +$$

$$+ \frac{1}{2\sqrt{(b-H_1)^2-4}} \ln \left| \frac{\sqrt{(b-H_1)^2-4} - \sqrt{b_1^2 - w_3^2}}{w_3 \pm C_1} \mp \frac{C_1}{\sqrt{(b-H_1)^2-4}} \right| + \text{const.} \tag{43}$$

II. $|b - H_1| < 2$.

$$I_1 = \frac{1}{\sqrt{4-(b-H_1)^2}} \arcsin \frac{\pm C_1 w_3 + b_1^2}{b_1(w_3 \pm C_1)} + \text{const.} \tag{44}$$

III. $|b - H_1| = 2$.

$$I_1 = \mp \frac{\sqrt{b_1^2 - w_3^2}}{C_1(w_3 \pm C_1)} + \text{const.} \tag{45}$$

When we return to the variable

$$w_1 = \frac{z_2}{\sin \alpha} - \frac{b + H_1}{2(1 + bH_1)}, \tag{46}$$

we shall have the final form for the value I_1:

I. $|b - H_1| > 2$.

$$I_1 = -\frac{1}{2\sqrt{(b-H_1)^2-4}} \ln \left| \frac{\sqrt{(b-H_1)^2-4} \pm 2(1+bH_1)w_1}{\sqrt{b_1^2 - 4(1+bH_1)^2 w_1^2} \pm C_1} \pm \frac{C_1}{\sqrt{(b-H_1)^2-4}} \right| +$$

$$+\frac{1}{2\sqrt{(b-H_1)^2-4}}\ln\left|\frac{\sqrt{(b-H_1)^2-4}\mp 2(1+bH_1)w_1}{\sqrt{b_1^2-4(1+bH_1)^2w_1^2}\pm C_1}\mp\frac{C_1}{\sqrt{(b-H_1)^2-4}}\right|+\text{const.} \tag{47}$$

II. $|b-H_1|<2$.

$$I_1=\frac{1}{\sqrt{4-(b-H_1)^2}}\arcsin\frac{\pm C_1\sqrt{b_1^2-4(1+bH_1)^2w_1^2}+b_1^2}{b_1(\sqrt{b_1^2-4(1+bH_1)^2w_1^2}\pm C_1)}+\text{const.} \tag{48}$$

III. $|b-H_1|=2$.

$$I_1=\mp\frac{2(1+bH_1)w_1}{C_1(\sqrt{b_1^2-4(1+bH_1)^2w_1^2}\pm C_1)}+\text{const.} \tag{49}$$

So, the additional first integral was found right before for the third-order system (22)–(24), i.e., it was presented the complete tuple of the first integrals which are the transcendental functions of its own phase variables.

It is necessary to substitute formally the left-hand side of the first integral (33) instead of C_1 in the expression of the found first integral.

Then the obtained additional first integral has the following structural form (which is similar to the transcendental first integral from the plane-parallel dynamics; see [4]):

$$\ln|\sin\alpha|+G_2\left(\sin\alpha,\frac{z_2}{\sin\alpha},\frac{z_1}{\sin\alpha}\right)=C_2=\text{const.} \tag{50}$$

Thus, there are already found two the independent first integrals for the integration of the fourth-order system (22)–(25). And to complete its integrability it is sufficient to find one more (additional) first integral which "joins" equation (25).

Since

$$\frac{du_1}{d\tau}=\frac{u_1(2(1+bH_1)u_2-(b+H_1))}{(b-(1+bH_1)u_2)\tau},\ \frac{d\beta_1}{d\tau}=\frac{(1+bH_1)u_1}{(b-(1+bH_1)u_2)\tau}, \tag{51}$$

then

$$\frac{du_1}{d\beta_1}=2u_2-\frac{b+H_1}{1+bH_1}. \tag{52}$$

It is obvious that for $u_1\neq 0$ the following equality is fulfilled:

$$u_2=\frac{1}{2(1+bH_1)}\left((b+H_1)\pm\sqrt{b_1^2-(2(1+bH_1)u_1-C_1)^2}\right), \tag{53}$$

$$b_1^2 = (b - H_1)^2 + C_1^2 - 4,$$

and then the integration of the following quadrature

$$\beta_1 + \text{const} = \pm(1 + bH_1) \int \frac{du_1}{\sqrt{b_1^2 - (2(1 + bH_1)u_1 - C_1)^2}} \tag{54}$$

will bring to the invariant relation

$$2(\beta_1 + C_3) = \pm \arcsin \frac{2(1 + bH_1)u_1 - C_1}{\sqrt{(b - H_1)^2 + C_1^2 - 4}}, \; C_3 = \text{const}. \tag{55}$$

In other words, the equality

$$\sin[2(\beta_1 + C_3)] = \pm \frac{2(1 + bH_1)u_1 - C_1}{\sqrt{(b - H_1)^2 + C_1^2 - 4}} \tag{56}$$

is fulfilled and under the transition to the old variables

$$\sin[2(\beta_1 + C_3)] = \pm \frac{2(1 + bH_1)z_1 - C_1 \sin\alpha}{\sqrt{(b - H_1)^2 + C_1^2 - 4} \sin\alpha}. \tag{57}$$

In principle, it makes possible to stop on the latter equality to achieve the additional invariant relation "joining" equation (25), if we add to this equality that it is necessary to substitute formally the left-hand side of the first integral (33) instead of C_1 in the latter expression.

But we shall make the certain transformations which is reduced to obtaining the following evident form of the additional first integral (herewith, the equality (33) is used):

$$\text{tg}^2[2(\beta_1 + C_3)] =$$

$$= \frac{((1 + bH_1)u_1^2 - (1 + bH_1)u_2^2 + (b + H_1)u_2 - 1)^2}{u_1^2(2(1 + bH_1)u_2 - (b + H_1))^2}. \tag{58}$$

Returning to the old coordinates, we shall obtain the additional invariant relation as the form

$$\text{tg}^2[2(\beta_1 + C_3)] =$$

$$= \frac{((1 + bH_1)z_1^2 - (1 + bH_1)z_2^2 + (b + H_1)z_2 \sin\alpha - \sin^2\alpha)^2}{z_1^2(2(1 + bH_1)z_2 - (b + H_1)\sin\alpha)^2}, \tag{59}$$

or finally

$$-\beta_1 \pm \frac{1}{2}\times$$

$$\times\mathrm{arctg}\frac{(1+bH_1)z_1^2-(1+bH_1)z_2^2+(b+H_1)z_2\sin\alpha-\sin^2\alpha}{z_1(2(1+bH_1)z_2-(b+H_1)\sin\alpha)} = C_3 = \mathrm{const.} \tag{60}$$

And so, the system of dynamic equations (1) under S. A. Chaplygin conditions (10) has five invariant relations in considered case: there exist the analytical non-integrable constraint (2), the cyclic first integral (3), (5), and the first integral (34), and also there exists the first integral expressed by the relations (43)–(50) which is the transcendental function of its phase variables (in sense of complex analysis also) and expresses in terms of finite combination of the elementary functions, and finally, there exists the transcendent first integral (60).

Theorem 1. *The system (1) under the conditions (10), (2), (5) possesses five invariant relations (the complete tuple), three of which are the transcendental functions from the complex analysis view of point. Herewith, all the relations express in terms of the finite combination of the elementary functions.*

2.3.2 On Pendulum with Variable Dissipation

The system (17)–(20) is also a pendulum zero mean variable dissipation system. The motion is executed under the action of the following two forces: the potential force

$$\frac{v^2}{I_2}\mathscr{F}(\alpha)$$

and the linear-in-velocity force

$$b\frac{v}{I_2 n_0}\alpha'\frac{d}{d\alpha}\frac{\mathscr{F}(\alpha)}{\cos\alpha}$$

with variable coefficient. In a certain subspace, this coefficient has the strictly positive sign, and, therefore, the energy pumping from the is occurred here. In the other subspace, this coefficient has the strictly negative sign. Therefore the force scatters the energy forcing the body to damp its motion.

The latter remarks show that we deal with dissipative *zero mean variable dissipation system*. The phase space is decomposed into a union of alternating domain, in each of which there is a dissipation of only one sign.

Remark. In the case of the system (17)–(20) the search for integrals reduces to integrating the Riccati equations whose solutions are not expressed in elementary functions in the most general case.

2.4 Problem on Spatial Pendulum in Overrunning Medium Flow

Analogously to the plane case [4], let us consider the problem of the spatial pendulum placed in the overrunning medium flow.

Let a convex plane domain (circular disk, for simplicity) be clamped perpendicularly to the holder OD by a spherical hinge, and let it be in the overrunning medium flow, which moves with a constant velocity $\mathbf{v}_\infty \neq \mathbf{0}$. Assume that the holder does not create any resistance (Fig. 2).

The total force $\mathbf{S}$ of the medium flow action on the body is directed parallel to the holder, and the point N of application of this force is determined by only one parameter, the angle of attack α measured between the velocity vector $\mathbf{v}_D$ of the point D with respect to the flow and the holder. Therefore, the force $\mathbf{S}$ is directed along the normal to the side opposite to the direction of the velocity $\mathbf{v}_D$ and passes through a certain point N of the plane domain displaced from the point D forward with respect to the direction of $\mathbf{v}_D$. Such conditions arise in using the model of streamline flow around spatial bodies [4].

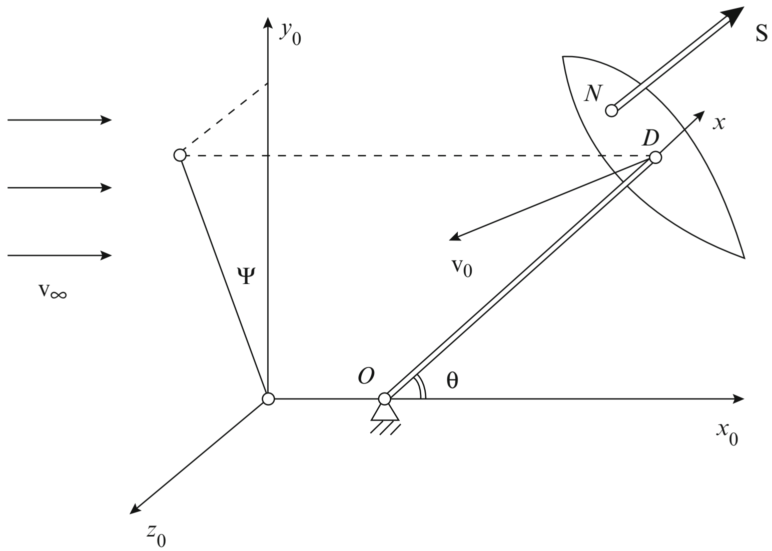

Fig. 2 Spherical pendulum in the homogeneous overrunning medium flow

The vector **e** determines the orientation of the holder. Then $\mathbf{S} = s_1(\alpha)v_D^2\mathbf{e}$, where the resistance coefficient has the form $s_1 = s_1(\alpha) = s(\alpha)\operatorname{sign}\cos\alpha$. Let $O\mathrm{x}_0\mathrm{y}_0\mathrm{z}_0$ be the immovable coordinate system. The direction of the overrunning flow coincides with the direction of the axis x_0. Let us relate the coordinate system $Dxyz$ with the body, where the axis Dx is directed along the holder and axes Dy and Dz are rigidly related to the plane domain.

The coordinates of the point N in the system $Dxyz$ have the form $(0, y_N, z_N)$. Analogously to the problem of the free body motion, we introduce the function $R(\alpha)$ and also the angle β measured in the plane Dyz. In thus case for simplicity, let S. A. Chaplygin properties (10) hold. For any admissible function $R(\alpha)$, the analysis is performed analogously.

If the body is dynamically symmetric (I_1 and $I_2 = I_3$ are principal moments of inertia in the system $Dxyz$) and (Ω_x, Ω_y, and Ω_z) are projections of the angular velocity in the system $Dxyz$, then the equations of motion take the following form, which is analogous to (13), (14) ($H_1 = 0$, for simplicity):

$$\Omega_y' = -n_0^2v_D^2\sin\alpha\cos\alpha\sin\beta,\ \Omega_z' = n_0^2v_D^2\sin\alpha\cos\alpha\cos\beta. \tag{61}$$

The resistance force admits the existence of the first integral (3), and, in this case, the condition $\Omega_{x0} = 0$ is taken into account in equations (61).

Let us introduce the angles (θ, ψ) determining the orientation of the pendulum (Fig. 2). The angle θ is measured from the axis x_0 to the holder, whereas ψ is measured from the projection of the holder on the plane $O\mathrm{y}_0\mathrm{z}_0$ to the axis y_0. Then

$$\cos\theta = \cos\psi\cos\phi, \quad \sin\theta\cos\psi = \cos\psi\sin\phi, \quad \sin\theta\sin\psi = \sin\psi. \tag{62}$$

2.4.1 Complete System of Equations

The relations connecting (v_D, α, β) and $(\theta, \psi, \Omega_y, \Omega_z)$ (l is the holder length) have the form

$$\begin{aligned} v_D\cos\alpha &= -v_\infty\cos\theta, \\ v_D\sin\alpha\cos\beta &= l\Omega_z + v_\infty\sin\theta\cos\psi, \\ v_D\sin\alpha\sin\beta &= -l\Omega_y - v_\infty\sin\theta\sin\psi. \end{aligned} \tag{63}$$

Later on, we have

$$\begin{aligned} \dot\theta &= -\Omega_y\tfrac{\sin\phi}{\cos\psi}, \\ \dot\phi &= \Omega_z + \Omega_y\sin\phi\tfrac{\sin\phi}{\cos\psi}, \\ \dot\psi &= \Omega_y\cos\phi, \end{aligned} \tag{64}$$

whence we easily deduce that

$$\Omega_y = \frac{\dot{\psi}}{\cos\phi}, \ \Omega_z = \dot{\phi} - \dot{\psi}\frac{\sin\phi}{\cos\phi}\,\frac{\sin\psi}{\cos\psi}. \tag{65}$$

Using properties (62) and (65), we have the identities

$$\Omega_y = \dot{\theta}\sin\psi + \dot{\psi}\frac{\sin\theta}{\cos\theta}\cos\psi, \ \Omega_z = \dot{\theta}\cos\psi - \dot{\psi}\frac{\sin\theta}{\cos\theta}\sin\psi. \tag{66}$$

Equations from (61), (63), and (66) compose a complete system for determining the pendulum motion on the level of the integral $\Omega_{x0} = 0$.

2.4.2 Systems of Differential Equations and Topological Analogy

Starting from three groups of equations two of which are differential and the third is algebraic, it is easy to prove the following proposition.

Theorem 2. *The complete system of the pendulum motion on the tangent bundle to the two-dimensional sphere has the following form:*

$$\ddot{\theta} + ln_0^2 v_\infty \dot{\theta}\cos\theta + n_0^2 v_\infty^2 \sin\theta\cos\theta - \dot{\psi}^2 \tfrac{\sin\theta}{\cos\theta} = 0, \tag{67}$$

$$\ddot{\psi} + \dot{\theta}\dot{\psi}\left(\tfrac{1+\cos^2\theta}{\cos\theta\sin\theta}\right) + ln_0^2 v_\infty \dot{\psi}\cos\theta = 0. \tag{68}$$

As in the case of a free body, system (67), (68) has symmetries. It also has a complete tuple of first integrals, and the angle ψ is a cyclic coordinate.

Theorem 3. *System (67), (68) is topologically equivalent to the system (17)–(20). Therefore, as in the plane case, there is a mechanical analogy between the pendulum in the medium flow and the free body under the presence of a certain non-integrable constraint.*

Remark. The angle α for a free body is equivalent to the angle θ, whereas the angle β is equivalent to the angle ψ. Moreover, for systems (22)–(25) and (67), (68) to be identical, it is necessary to set $l = -\sigma$ and $v_\infty = v$. The constant velocity of the characteristic point of the circular disk for a free body corresponds to the constant velocity of the overrunning flow on the pendulum. The relation $l = -\sigma$ tells us that for a free body, the stationary motion $\alpha \equiv 0$ is exponentially unstable, and for the pendulum, the stationary motion $\theta \equiv 0$ is exponentially stable.

2.5 Topological Structure of the Phase Portrait of the System Studied

In this subsection, we present a scheme of global qualitative analysis of the dynamical system (17)–(19) on the whole phase space $\{\alpha, z_2, z_1\}$. For any function $\mathscr{F}$, the phase portrait of system (17)–(19) has the same topological type.

System (17)–(19) has no trajectories having infinitely distant points of the phase space as its α- and ω-limit sets. Moreover, the system has no simple and complicated limit cycles (see also [10]).

2.5.1 Reducing System to the Form Studied

Let us consider the case of the absence of an additional dependence of the functions y_N, z_N on angular velocity (i.e., for simplicity, $H_1 = 0$). For convenience of drawing the three-dimensional phase portrait and preserving the right-oriented coordinate system $\{\alpha, z1_2, z_1\}$, let us make the formal change $\alpha \to -\alpha$. Moreover, because of the existence of symmetries, we study the domain

$$\{(\alpha, z_2, z_1) : -\pi < \alpha < 0, z_1 > 0\}. \tag{69}$$

In this case, the system takes the form

$$\begin{aligned} \dot{\alpha} &= z_2 + b\tfrac{v}{I_2 n_0}\tfrac{\mathscr{F}(\alpha)}{\cos\alpha}, \\ \dot{z_2} &= -\tfrac{v^2}{I_2}\mathscr{F}(\alpha) + z_1^2\tfrac{\cos\alpha}{\sin\alpha}, \\ \dot{z_1} &= -z_1 z_2\tfrac{\cos\alpha}{\sin\alpha}, \end{aligned} \tag{70}$$

and under S. A. Chaplygin condition (10), the analytic system has the form

$$\begin{aligned} \dot{\alpha} &= z_2 + b\sin\alpha, \\ \dot{z_2} &= -\sin\alpha\cos\alpha + z_1^2\tfrac{\cos\alpha}{\sin\alpha}, \\ \dot{z_1} &= -z_1 z_2\tfrac{\cos\alpha}{\sin\alpha}. \end{aligned} \tag{71}$$

For simplicity, let us study the system (71), and let us “rectify” the field along the cylinders $\{(\alpha, z_2, z_1) : z_2 + b\sin\alpha = 0\}$; precisely, making the change of the phase variables $u = z_2 + b\sin\alpha$, we pass from system (71) to the system

$$\begin{aligned} \dot{\alpha} &= u, \\ \dot{u} &= -\sin\alpha\cos\alpha + z_1^2\tfrac{\cos\alpha}{\sin\alpha} + bu\cos\alpha, \\ \dot{z_1} &= -z_1[u - b\sin\alpha]\tfrac{\cos\alpha}{\sin\alpha}. \end{aligned} \tag{72}$$

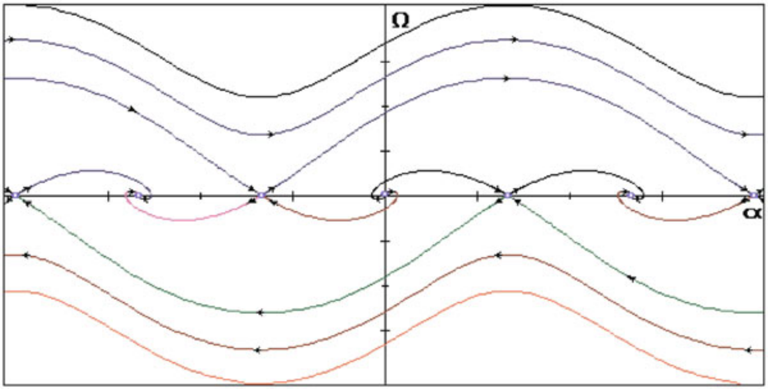

Fig. 3 Phase portrait of the system (72) for $z_1 \equiv 0$ and $\Omega \to u$ formally

For $b = 0$, system (72) (denoted by (72′)) has two analytical integrals. The following but important proposition is obvious.

Theorem 4. *The plane*

$$\{(\alpha, u, z_1) : z_1 = 0\} \tag{73}$$

is integral for the system (70).

Theorem 5. *Plane (73) "contains" the portrait of the system from the plane dynamics (see Fig. 3 if we extend the system field to the lines* $\{(\alpha, u) : si\,n\alpha = 0\}$ *by continuity).*

Let us introduce the family of (three-dimensional) layers

$$\Pi_{(\alpha_1,\alpha_2)} = \{(\alpha, x_1, x_2) \in \mathbf{R}^3 : \alpha_1 < \alpha < \alpha_2\}. \tag{74}$$

2.5.2 Conservative Third-Order Comparison System

We have already presented many assertions concerning Poincaré topographical systems (PTS) and more general comparison systems on two-dimensional manifolds [6,7]. To study third-order systems, we need PTS and comparison systems of higher order. We do not dwell on the general theory of PTS and comparison systems of higher dimension and restrict ourselves to its application to the system studied.

Since system (72′) has two analytical integrals

$$\Phi_1 = z_1^2 + u^2 + \sin^2\alpha = C_1^0,\ \Phi_2 = z_1 \sin\alpha = C_2^0,\ C_1^0, C_2^0 = \text{const}; \tag{75}$$

the latter fiber the phase space at each point of which we can draw two surfaces given by relations (75) that intersect along the phase characteristic of system (72′). For each point of the phase space of system (72), let us define two pairs of subspaces in each of which the characteristic of system (72) enters or emanates from. The first integrals (75) "help us" to study the behavior of phase trajectories of system (72) (compare with [14, 15]).

If $g_1 = \{\sin\alpha\cos\alpha, u, z_1\}$, $g_2 = \{z_1\cos\alpha, 0, \sin\alpha\}$ are the gradients of surfaces Φ_1, Φ_2, then the inner products

$$\chi_k = (g_k, \bar{v}),\ k = 1, 2$$

($\bar{v}$ is the vector field of system (72)) have all the properties of the characteristic functions (see also [12]).

Theorem 6. *The characteristic functions χ_k have the form*

$$\chi_1(\alpha, u, z_1) = b\cos\alpha[u^2 + z_1^2], \quad \chi_2(\alpha, z_1) = bz_1\sin\alpha\cos\alpha.$$

2.5.3 Equilibrium Points of System Studied

The system (72) under study has the following equilibrium states:

1. The repelling point $(0, 0, 0)$
2. The attracting point $(-\pi, 0, 0)$
3. The saddles $(-\pi/2, 0, Q_1)$ and $Q_1 \in [0, n_0)$ in each small area parallel to the plane $O\alpha z_2$
4. The centers $(-\pi/2, 0, Q_2)$ and $Q_2 \in (n_0, +\infty)$ in each small area parallel to the plane $O\alpha z_2$

In cases 3 and 4, the singular points are *not isolated.*

Through the whole section, we have performed the qualitative analysis in sufficient detail. The next subsection is a consequence of the previous material.

2.5.4 Phase Portrait Structure

Theorem 6 implies the following assertions:

1. Point 1 is the α-limit set of the separatrices entering points 3 in the layer $\Pi_{(-\pi/2,0)}$.
2. Point 2 is the ω-limit set of the separatrices emanating from points 3 in the strip $\Pi_{(-\pi,-\pi/2)}$.
3. The ω-limit (α-limit) sets of the separatrices emanating from (entering) points 3 in the layer $\Pi_{(-\pi/2,0)}$ (in the layer $\Pi_{(-\pi,-\pi/2)}$) are the same points.
4. The part of the phase space containing points 4 entirely filled with closed trajectories.

2.6 *Trajectories of Spherical Pendulum Motion and Case of Its Nonzero Twist Near Longitudinal Axis*

2.6.1 Pendulum Trajectories on Sphere

In accordance with the properties of the phase space partition into trajectories, the typical trajectories of the point D of the plane domain fall into classes:

1. The trajectories corresponding to the oscillatory domain. Such trajectories are curves on the sphere that unboundedly approach the poles of the sphere (along the flow) as $t \to \pm\infty$.
2. The trajectories corresponding to the rotational domain. Such trajectories are curves on the sphere that almost always fill annulus-like domains on the sphere symmetric with respect to the equator.

2.6.2 Spherical Pendulum Under Nonzero Proper Twist

We immediately present the equations of the pendulum motion under the condition $\Omega_{x0} \neq 0$. These equations have the following form:

$$\ddot{\theta} + l n_0^2 v_\infty \dot{\theta} \cos\theta + n_0^2 v_\infty^2 \sin\theta \cos\theta - \dot{\psi}^2 \frac{\sin\theta}{\cos\theta} - \frac{I_1}{I_2} \Omega_{x0} \dot{\psi} \frac{\sin\theta}{\cos\theta} = 0, \quad (76)$$

$$\ddot{\psi} + \dot{\theta}\dot{\psi} \left(\frac{1+\cos^2\theta}{\cos\theta \sin\theta} \right) + l n_0^2 v_\infty \dot{\psi} \cos\theta + \frac{I_1}{I_2} \Omega_{x0} \dot{\theta} \frac{\cos\theta}{\sin\theta} = 0. \quad (77)$$

Now let us immediately pass to the classification of possible pendulum trajectories on the sphere:

1. Trajectories analogous to trajectories 1 for the case $\Omega_{x0} = 0$. The asymptotics of behavior of such trajectories is the same as above.
2. The trajectories analogous to trajectories 2 for the case $\Omega_{x0} = 0$. Such trajectories are everywhere dense on the whole sphere [15].

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