On Stability of Certain Key Types of Rigid Body Motion in a Nonconservative Field

Maxim V. Shamolin†

†Institute of Mechanics, Lomonosov Moscow State University
1 Michurinskii Ave, 119192, Moscow, Russian Federation
Email: shamolin@rambler.ru, shamolin@imec.msu.ru

Abstract—In this activity the qualitative analysis of both plane-parallel and spatial problems of the real rigid body motions in a resistant medium is fulfilled. A nonlinear model that describes the interaction of a rigid body with a medium and takes into account (based on experimental data on the motion of circular cylinders in water) the dependence of the arm of the force on the normalized angular velocity of the body and the dependence of the moment of the force on the angle of attack is constructed [1]. An analysis of plane and spatial models (in the presence or absence of an additional tracking force) leads to sufficient stability conditions for translational motion, as one of the key types of motions. Either stable or unstable self-oscillation can be observed under certain conditions.

1. Spatial Motion of an Axial Symmetric Rigid Body in a Resisting Medium

Consider the problem of a spatial motion of *homogeneous* axial symmetric body of mass m. A portion of its surface is a flat circle disk interacting with a medium in a jet flow [1, 2, 3]. The other portion of the body's surface is inside the volume bounded by the jet stalling at the disk edge and is not affected by the medium. Conditions are similar when homogeneous circular cylinders enter water.

If the above conditions are satisfied, the motions of the body include *translational deceleration* similar to the case of plane-parallel (unperturbed) motion: the body can undergo translational motion along its axis of symmetry, i.e., perpendicularly to the disk plane.

We choose the right-hand coordinate system $Dxyz$ with the Dx-axis aligned with the axis of geometrical symmetry of the body and the $Dy-$ and $Dz-$axes fixed to the disk. The components of the angular velocity vector Ω in the system $Dxyz$ are denoted by $\{\Omega_x, \Omega_y, \Omega_z\}$. The inertia tensor of the dynamically symmetric body is diagonalized in the body axes $Dxyz$: $\mathrm{diag}\{I_1, I_2, I_2\}$. We will use the quasitationarity hypothesis and assume for simplicity that $R_1 = DN$ is defined at least by the attack angle α between the velocity vector $\mathbf{v}$ of the center D of the disk and the straight line Dx. Thus, $DN = R_1(\alpha, \ldots)$. Moreover, we assume that $S = |\mathbf{S}| = s_1(\alpha)v^2,\ v = |\mathbf{v}|$. For convenience, we introduce an auxiliary alternating function $s(\alpha)$: $s_1 = s_1(\alpha) = s(\alpha)\mathrm{sgn}\cos\alpha > 0$ instead of the coefficient $s_1(\alpha)$. Thus, the pair of functions $R_1(\alpha, \ldots)$ and $s(\alpha)$ defines the forces and moments exerted by the medium on the disk under such assumptions.

Let us use the spherical coordinates (v, α, β_1) of the tip of the velocity vector $\mathbf{v} = \mathbf{v}_D$ of the point D relative to the flow to measure the angle β_1 in the plane of the disk. Expressing the quantities (v, α, β_1), using nonintegrable relations, in terms of the cyclic kinematic variables and velocities and supplementing them with the projections $(\Omega_x, \Omega_y, \Omega_z)$ of the angular velocity onto the body axes, we consider them as quasivelocities.

Using the theorems on the motion of the center of mass (in the body-fixed frame of reference $Dxyz$) and variation in the angular momentum in the same frame, we obtain the dynamic part of the differential equations of motion in the six-dimensional phase space of quasivelocities ($\sigma = DC$). The first group of equations describes the motion of the center of mass, while the second group the motion around the center of mass

$$
\begin{gathered}
\dot v\cos\alpha - \dot\alpha v\sin\alpha + \Omega_y v\sin\alpha\sin\beta_1 - \\
-\Omega_z v\sin\alpha\cos\beta_1 + \sigma(\Omega_y^2 + \Omega_z^2) = -s(\alpha)v^2/m, \\
\dot v\cos\alpha - \dot\alpha v\sin\alpha + \Omega_y v\sin\alpha\sin\beta_1 - \\
-\Omega_z v\sin\alpha\cos\beta_1 + \sigma(\Omega_y^2 + \Omega_z^2) = 0, \\
\dot v\sin\alpha\cos\beta_1 + \dot\alpha v\cos\alpha\cos\beta_1 - \\
-\dot\beta_1 v\sin\alpha\sin\beta_1 + \Omega_z v\cos\alpha - \\
-\Omega_x v\sin\alpha\sin\beta_1 - \sigma\Omega_x\Omega_y - \sigma\dot\Omega_z = 0, \\
\dot v\sin\alpha\sin\beta_1 + \dot\alpha v\cos\alpha\sin\beta_1 + \\
+\dot\beta_1 v\sin\alpha\cos\beta_1 + \Omega_x v\sin\alpha\cos\beta_1 - \\
-\Omega_y v\cos\alpha - \sigma\Omega_x\Omega_z + \sigma\dot\Omega_y = 0, \\
I_1\dot\Omega_x = 0, \\
I_2\dot\Omega_y + (I_1 - I_2)\Omega_x\Omega_z = -z_N s(\alpha)v^2, \\
I_2\dot\Omega_z + (I_2 - I_1)\Omega_x\Omega_y = y_N s(\alpha)v^2,
\end{gathered}
\tag{1}
$$

where $(0, y_N, z_N)$ are the coordinates of the point N in the system $Dxyz$.

2. Motion of a Symmetric Body Subject to Force of Resistance and Tracking Force

Let us consider the class of problems where a rigid body moves through a medium under a tracking force acting along the axis of geometrical symmetry of the body and producing (under some conditions) classes of motions (imposed constraints) of interest, this force being the reaction of the constraints imposed. Here, the tracking force is such that $v \equiv \text{const}$.

The cyclic invariant relation $\Omega_x \equiv \Omega_{x0} = \text{const}$ holds at all instants of time. In what follows, we will examine the case where the rigid body does not rotate about its longitudinal axis, i.e., $\Omega_{x0} = 0$.

Then the independent dynamic part of the equations of motion in the four-dimensional phase space is given by

$$\begin{gathered}\dot\alpha \cos\alpha \cos\beta_1 - \dot\beta_1 v \sin\alpha \sin\beta_1 + \\ + \Omega_z v \cos\alpha - \sigma\dot\Omega_z = 0,\end{gathered} \tag{2}$$

$$\begin{gathered}\dot\alpha \cos\alpha \sin\beta_1 + \dot\beta_1 v \sin\alpha \cos\beta_1 - \\ - \Omega_y v \cos\alpha + \sigma\dot\Omega_y = 0,\end{gathered} \tag{3}$$

$$I_2 \dot\Omega_y = -z_N s(\alpha) v^2, \tag{4}$$

$$I_2 \dot\Omega_z = y_N s(\alpha) v^2. \tag{5}$$

Here y_N, z_N are Cartesian coordinates of the point N of resisting force application.

System (2)–(5) includes the influence functions y_N, z_N, and s. To determine them qualitatively, we will use experimental data on the properties of jet flow.

For beginning, we will analyze the system (2)–(5) for the following influence functions (of S. A. Chaplygin [1]); such an analysis can be performed for an arbitrary pair of functions y_N, z_N, and s, see below:

$$\begin{gathered} y_N = A\sin\alpha\cos\beta_1 - h\Omega_z/v, \\ z_N = A\sin\alpha\sin\beta_1 + h\Omega_y/v, \\ s(\alpha) = B\cos\alpha, \\ A = \tfrac{\partial y_N}{\partial\alpha}|_{\alpha=0,\ \beta_1=0} = \tfrac{\partial z_N}{\partial\alpha}|_{\alpha=0,\ \beta_1=\pi/2}, \\ B = s(0), h > 0. \end{gathered} \tag{6}$$

The resultant system will be called a *reference* one.

The coefficient h in (6) appears in the terms proportional to the rotary derivatives of the moment of hydroaerodynamic forces (drag) with respect to the components of the angular velocity of the body.

Projecting the angular velocities onto the moving axes not fixed to the body so that $z_1 = \Omega_y \cos\beta_1 + \Omega_z \sin\beta_1$, $z_2 = -\Omega_y \sin\beta_1 + \Omega_z \cos\beta_1$ and introducing dimensionless variables $w_k,\ k = 1, 2$, and parameters by the formulas $b = \sigma n_0,\ n_0^2 = AB/I_2,\ H_1 = Bh/I_2 n_0,\ z_k = n_0 v w_k,\ k = 1, 2$, (herewith $< \cdot >= n_0 v <'>$), we obtain the following analytic dynamic system (*reference system*) of the fourth order:

$$\alpha' = -(1 + bH_1) w_2 + b\sin\alpha, \tag{7}$$

$$\begin{gathered} w_2' = \sin\alpha\cos\alpha - (1 + bH_1) w_1^2 \tfrac{\cos\alpha}{\sin\alpha} - \\ - H_1 w_2 \cos\alpha, \end{gathered} \tag{8}$$

$$w_1' = (1 + bH_1) w_1 w_2 \frac{\cos\alpha}{\sin\alpha} - H_1 w_1 \cos\alpha, \tag{9}$$

$$\beta_1' = (1 + bH_1) w_1 \frac{\cos\alpha}{\sin\alpha}, \tag{10}$$

which includes the independent third-order subsystem (7)–(9).

If $b = H_1$ then after the change of variables $w^* = \ln|w_1|$, the divergence of the right-hand side of (7)–(9) ((7)–(10)) will become identically equal to zero, which allows considering the system(s) to be conservative.

2.1. On Stability of Translational Motion

Let research the stability of key type (unperturbed motion) with respect to the perturbations of angle of attack and angular velocity, i.e. to the variables α, w_1, w_2. In other words, we research the stability of trivial solution of independent third order system (7)–(9).

Consider the following positive definite function in the phase space of the third-order system (7)–(9):

$$\begin{gathered} V(\alpha, w_1, w_2) = (1 + b^2)(w_2^2 + w_1^2) - \\ - 2bw_2\sin\alpha + \sin^2\alpha. \end{gathered} \tag{11}$$

Theorem 1. Function (11) is a Lyapunov (Chetaev) function for system (7)–(9), i.e., its derivative by virtue of the system is negative definite for $b < H_1$ and positive definite for $b > H_1$.

Corollary. The origin of coordinates of system (7)–(9) (after the right-hand side is redefined at it) is an attracting singular point for $b < H_1$ and a repulsing singular point for $b > H_1$.

Note once again that a similar theorem is also valid for the general system with arbitrary influence functions y_N, z_N, and s. The asymptotic stability condition for the origin of coordinates of the system of reduced dynamic equations with respect to the variables (α, w_1, w_2) remains the same, $b < H_1$.

Indeed, in the more general case where the influence functions are represented as

$$\begin{gathered} y_N = R(\alpha)\cos\beta_1 - h_1\Omega_z/v, \\ z_N = R(\alpha)\sin\beta_1 + h_1\Omega_y/v, \end{gathered} \tag{12}$$

and the functions R, s satisfy the typical conditions (function R corresponds to the function y in such case), the dynamic equations of motion become:

$$\begin{gathered} \alpha' = -w_2 + \tfrac{\sigma}{I_2 n_0} \tfrac{F(\alpha)}{\cos\alpha} - \tfrac{\sigma h_1}{I_2} w_2 \tfrac{s(\alpha)}{\cos\alpha}, \\ w_2' = \tfrac{F(\alpha)}{I_2 n_0^2} - w_1^2 \tfrac{\cos\alpha}{\sin\alpha} - \\ - \tfrac{\sigma h_1}{I_2} w_1^2 \tfrac{s(\alpha)}{\sin\alpha} - \tfrac{h_1}{I_2 n_0} w_2 s(\alpha), \\ w_1' = w_1 w_2 \tfrac{\cos\alpha}{\sin\alpha} + \tfrac{\sigma h_1}{I_2} w_1 w_2 \tfrac{s(\alpha)}{\sin\alpha} - \\ - \tfrac{h_1}{I_2 n_0} w_1 s(\alpha), \\ \beta_1' = w_1 \tfrac{\cos\alpha}{\sin\alpha} + \tfrac{\sigma h_1}{I_2} w_1 \tfrac{s(\alpha)}{\sin\alpha}, \end{gathered} \tag{13}$$

here $F(\alpha) = R(\alpha)s(\alpha)$.

Consider the function (similar (11))

$$V(\alpha, w_1, w_2) = w_2^2 + (1+b^2)w_1^2 + [bw_2 - \sin\alpha]^2\,, \quad (14)$$

that is positive definite in the neighborhood of the origin.

Theorem 2. Function (14) is a Lyapunov (Chetaev) function for system (13), i.e., its derivative by virtue of the system is negative definite for $\sigma R'(0) < h_1$ and positive definite for $\sigma R'(0) > h_1$.

Corollary. The origin of coordinates of system (13) (after the right-hand side is redefined at it) is an attracting singular point for $\sigma R'(0) < h_1$ and a repulsing singular point for $\sigma R'(0) > h_1$.

The asymptotic-stability condition for moving homogeneous circular cylinders will be satisfied if $\sigma k < hD$, where D is the diameter of the cylinder, k and h are dimensionless influence parameters, and σ is the distance DC.

3. Motion of a Symmetric Body Subject to Force of Resistance and Tracking Force. II

In given case, the tracking force is such that the following condition is satisfied all the time, i.e. $\mathbf{V}_C \equiv \mathbf{const}$.

The cyclic invariant relation $\Omega_x \equiv \Omega_{x0} = \text{const}$ holds at all instants of time. In what follows, we will examine the case where the rigid body does not rotate about its longitudinal axis, i.e., $\Omega_{x0} = 0$.

Then the following value has to stand in the right-hand side of the first equation of the system (1) instead of $-s(\alpha)v^2/m$ and to be equal to zero identically, since the nonconservative pair of forces will act onto the body: $T - s(\alpha)v^2 \equiv 0$. Obviously, that it needs to choose the value of tracking force T in the type $T = T(v, \alpha, \Omega) = s(\alpha)v^2$, $\mathbf{T} \equiv -\mathbf{S}$.

Similar to the choose of influence functions, we define the dynamic functions s, y_N, and z_N in the types of (12). Herewith, the additional damping (but in the certain domains of the phase space and dispersing) moment of a nonconservative force is present in considered system as before.

Projecting the angular velocities onto the moving axes not fixed to the body so that $z_1 = \Omega_y \cos\beta_1 + \Omega_z \sin\beta_1$, $z_2 = -\Omega_y \sin\beta_1 + \Omega_z \cos\beta_1$ and introducing new dimensionless phase variables and differentiation by the formulas $z_k = n_0 v Z_k$, $k = 1, 2$, $<\cdot> = n_0 v <'>$, system (1) will lead to the following type:

$$v' = v\Psi_1(\alpha, Z_1, Z_2), \quad (15)$$

$$\begin{gathered}\alpha' = -Z_2 + \mu_2(Z_1^2 + Z_2^2)\sin\alpha + \\ + \tfrac{\sigma}{I_2 n_0}F(\alpha)\cos\alpha - \tfrac{\sigma h_1}{I_2}Z_2 s(\alpha)\cos\alpha,\end{gathered} \quad (16)$$

$$\begin{gathered}Z_2' = \tfrac{F(\alpha)}{I_2 n_0^2} - Z_2\Psi_1(\alpha, Z_1, Z_2) - \\ -Z_1^2\tfrac{\cos\alpha}{\sin\alpha} - \tfrac{\sigma h_1}{I_2}Z_1^2\tfrac{s(\alpha)}{\sin\alpha} - \tfrac{h_1}{I_2 n_0}Z_2 s(\alpha),\end{gathered} \quad (17)$$

$$\begin{gathered}Z_1' = -Z_1\Psi_1(\alpha, Z_1, Z_2) + Z_1 Z_2\tfrac{\cos\alpha}{\sin\alpha} + \\ + \tfrac{\sigma h_1}{I_2}Z_1 Z_2\tfrac{s(\alpha)}{\sin\alpha} - \tfrac{h_1}{I_2 n_0}Z_1 s(\alpha),\end{gathered} \quad (18)$$

$$\beta_1' = Z_1\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma h_1}{I_2}Z_1\frac{s(\alpha)}{\sin\alpha}, \quad (19)$$

where

$$\Psi_1(\alpha, Z_1, Z_2) = -\mu_2(Z_1^2 + Z_2^2)\cos\alpha +$$

$$+\frac{\sigma}{I_2 n_0}F(\alpha)\sin\alpha - \frac{\sigma h_1}{I_2}Z_2 s(\alpha)\sin\alpha,$$

and analytical system of equations in the case of Chaplygin influence functions (6):

$$v' = v\Psi_1(\alpha, Z_1, Z_2), \quad (20)$$

$$\alpha' = -Z_2 + \mu_2(Z_1^2 + Z_2^2)\sin\alpha + \mu_2\sin\alpha\cos^2\alpha -$$

$$-\mu_2\mu_3 Z_2\cos^2\alpha, \quad (21)$$

$$Z_2' = \sin\alpha\cos\alpha - Z_2\Psi_1(\alpha, Z_1, Z_2) - (1+\mu_2\mu_3)Z_1^2\frac{\cos\alpha}{\sin\alpha} -$$

$$-\mu_3 Z_2\cos\alpha, \quad (22)$$

$$Z_1' = -Z_1\Psi_1(\alpha, Z_1, Z_2) + (1+\mu_2\mu_3)Z_1 Z_2\frac{\cos\alpha}{\sin\alpha} -$$

$$-\mu_3 Z_1\cos\alpha, \quad (23)$$

$$\beta_1' = (1+\mu_2\mu_3)Z_1\frac{\cos\alpha}{\sin\alpha}, \quad (24)$$

where

$$\Psi_1(\alpha, Z_1, Z_2) = -\mu_2(Z_1^2 + Z_2^2)\cos\alpha + \mu_2\sin^2\alpha\cos\alpha -$$

$$-\mu_2\mu_3 Z_2\sin\alpha\cos\alpha,$$

herewith, we will choose the dimensionless parameters $b = \mu_2$, $H_1 = \mu_3$ as follows: $b = \sigma n_0$, $n_0^2 = AB/I_2$, $H_1 = Bh_1/I_2 n_0$ as above.

The equations (16)–(19) of the system (15)–(19) form the independent fourth order subsystem, and the equations (16)–(18) the independent third order one.

3.1. On Stability of Translational Motion

Let research the stability of key type (unperturbed motion) with respect to the perturbations of angle of attack and angular velocity, i.e. to the variables α, Z_1, Z_2. In other words, we research the stability of trivial solution of independent third order system (16)–(18).

Proposition 1. The plane

$$\{(\alpha, Z_1, Z_2) \in \mathbf{R^3}:\ \mathbf{Z_1 = 0}\} \quad (25)$$

is integral one for the system (16)–(18).

Furthermore, two remaining equations on α, Z_2 form the system describing plane-parallel rigid body dynamics (see above) after formal substitution $Z_1 = 0$ in the system (16)–(18).

Thus, the phase pattern from the plane dynamics "packs" into the plane (25). Furthermore, the plane (25) separates three-dimensional phase space on two parts:

$$\{(\alpha, Z_1, Z_2) \in \mathbf{R^3} : \ \mathbf{0} < \alpha < \pi, \ \mathbf{Z_1} > \mathbf{0}\} \quad (26)$$

and

$$\{(\alpha, Z_1, Z_2) \in \mathbf{R^3} : \ \mathbf{0} < \alpha < \pi, \ \mathbf{Z_1} < \mathbf{0}\}, \quad (27)$$

in each of which the motion occurs by itself. But it is not arbitrarily from each other, since the following symmetry is present in the system:

i) α- and Z_2-components of vector field of the system (16)–(18) *do not change the signs* under the symmetry

$$\begin{pmatrix} \alpha \\ Z_1 \\ Z_2 \end{pmatrix} \rightarrow \begin{pmatrix} \alpha \\ -Z_1 \\ Z_2 \end{pmatrix} \quad (28)$$

relatively the plane (25);

ii) Z_1-component of vector field of the system (16)–(18) *changes the sign* under the symmetry (28) relatively the plane (25).

The latter facts state that it is sufficiently to research the system (16)–(18) in semibounded shell (26), although it is not the valid phase space.

The possibility of use of the function

$$V_1(\alpha, Z_1) = Z_1 \sin\alpha \quad (29)$$

as Lyapunov (Chetaev) function in semibounded shell (26) is the important effect of latter remarks, since given function is positive defined in it.

Theorem 3. Function (29) is a Lyapunov (Chetaev) function for system (16)–(18), i.e., its derivative by virtue of the system is negative definite for $\mu_2 < \mu_3$ and positive definite for $\mu_2 > \mu_3$.

Corollary. The origin of coordinates of system (16)–(18) (after the right-hand side is redefined at it) is an attracting singular point for $\mu_2 < \mu_3$ and a repulsing singular point for $\mu_2 > \mu_3$.

In particular, the similar theorem is valid and for the systems (21)–(23), considered for influence functions (6) of Chaplygin.

Let consider the function (similar (11))

$$V(\alpha, Z_1, Z_2) = Z_2^2 + (1+b^2)Z_1^2 + [bZ_2 - \sin\alpha]^2, \quad (30)$$

which is positive defined in a vicinity of the origin.

Theorem 4. Function (30) is a Lyapunov (Chetaev) function for system (16)–(18), i.e., its derivative by virtue of the system (16)–(18) is negative definite for $\mu_2 < \mu_3$ and positive definite for $\mu_2 > \mu_3$.

Corollary. The origin of coordinates of system (16)–(18) (after the right-hand side is redefined at it) is an attracting singular point for $\mu_2 < \mu_3$ and a repulsing singular point for $\mu_2 > \mu_3$.

The asymptotic-stability condition for moving homogeneous circular cylinders will be satisfied if $\sigma k < hD$, where D is the diameter of the cylinder, k and h are dimensionless influence parameters, and σ is the distance DC.

4. Conclusions

The experiment on the motion of homogeneous circular cylinders in water conducted at the Research Institute of Mechanics of the Lomonosov Moscow State University confirmed that in modeling the influence of a medium on a rigid body, it is necessary to introduce an additional parameter to account for dissipation in the system.

In studying the deceleration of a body with finite angles of attack, a key task is to establish the conditions under which self-oscillations occur in a finite neighborhood of translational deceleration. Thus, a comprehensive nonlinear analysis is of necessity.

The initial stage of such an analysis is neglecting the damping effect of the medium. This corresponds to the assumption that the pair of dynamic functions describing the influence of the medium depends on a unique parameter (angle of attack). The dynamic systems resulting from such a nonlinear description behave as systems with variable dissipation. Therefore, it is necessary to develop a procedure for analyzing such systems. Since experimental data on the properties of jet flow is used, there is some scatter in forces and moments characteristics in the qualitative description of the body-medium interaction. This makes it natural to define relative robustness and to prove such robustness for the systems under study.

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