Review of Cases of Integrability in Dynamics of Lower- and Multidimensional Rigid Body in a Nonconservative Field of Forces

Maxim V. Shamolin

***Abstract*—Study of the dynamics of a multidimensional solid depends on the force-field structure. As reference results, we consider the equations of motion of low-dimensional solids in the field of a medium-drag force. Then it becomes possible to generalize the dynamic part of equations to the case of the motion of a solid, which is multidimensional in a similarly constructed force field, and to obtain the full list of transcendental first integrals. The obtained results are of importance in the sense that there is a nonconservative moment in the system, whereas it is the potential force field that was used previously.**

***Index Terms*—Case of integrability, dynamic part of motion equations, multidimensional rigid body.**

I. Introduction

THIS activity presents itself the review of either obtained earlier or new cases of integrability in Dynamics of two-dimensional, three-dimensional, and four-dimensional rigid body being in a nonconservative field of forces. The studied problems are described in terms of dynamic systems with so-called variable dissipation with zero mean.

The problem of research of complete choice of transcendental first integrals of the systems with dissipation is rather actual too, and majority of scientific activities was dedicated to it. New class of dynamic systems having the periodic coordinate is introduced in consideration. Due to the presence in such systems the nontrivial groups of symmetries it was shown that the considered systems possess the variable dissipation with zero mean that means the dissipation in the system is equal to zero for the period on available periodical coordinate, although both the sop of energy and its dissipation can be present in the different domains of the phase space of the system. On the base of obtained material the dynamic systems arising in Dynamics of a rigid body were analyzed. As the result the series of the cases of complete integrability of the motion equations in terms of transcendental functions and expressing through the finite combination of elementary functions were discovered. The certain generalizations on the conditions of integrability of more general classes of nonconservative dynamic systems (Dynamics of four-dimensional rigid body) were obtained.

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II. Preliminary

The activity presents the review of both earlier obtained and also new cases of integrability in two-, three-, and four-dimensional rigid body dynamics in a nonconservative force field. The problems studied are described in terms of so-called zero mean variable dissipation dynamic systems.

Therefore, we study nonconservative systems for which the methods for studying, for example, Hamiltonian systems is not applicable in general. Therefore, for such systems, it is necessary, in some sense, to ”directly” integrate the main equation of dynamics. We generalize old cases and also obtain new cases of complete integrability in transcendental functions in two-, three-, and four-dimensional rigid body dynamics in a nonconservative force field.

Of course, in the general case, it is sufficiently difficult to construct some theory of integrating nonconservative systems (even of low dimension). But in a number of cases where the systems considered have additional symmetries, we succeed in finding first integrals through finite combinations of elementary functions [1].

We obtain a whole spectrum of complete integrability cases for nonconservative dynamical systems having nontrivial symmetries. Moreover, in almost all cases, each of the first integrals is expressed through a finite combination of elementary functions, being one transcendental function of its variables. In this case, the transcendence is understood in the complex analysis sense, when after continuation of given functions to the complex domain, they have essentially singular points. The latter fact is stipulated by the existence of attracting and repelling sets in the system (for example, attracting and repelling foci).

We discover new integrable cases of motion of a rigid body, including that in the classical problem of motion of a spherical pendulum in an over-run medium flow.

In [1], [2], [3], [4] we study the general aspects of integrability of so-called variable dissipation dynamic systems. For the beginning we give the visual characteristic of those systems. Therefore, in this case, we will speak of systems with the variable dissipation, where the term ”variable” refers not to the value of the dissipation coefficient, but to the possible alternation of its sign (therefore, it is more reasonable to use the term sign-alternating).

We introduce the class of autonomous dynamic systems having one periodic phase coordinate, and therefore, possessing the certain symmetries which are typical for the pendulum-like systems. We show that offered class of systems are embedded to the class of zero mean variable dissipation systems by natural way, i.e., on the average, for the period of the existing periodic coordinate, the sop and diffusing to energy balance to each other in certain sense. We offer the examples of pendulum-like systems on lower-dimension manifolds from dynamics of a rigid body in a nonconservative field of force [1], [5].

In [1], [6] we study the certain general conditions of integrability in elementary functions for the systems both on two-dimensional planes and the tangent bundle of one-dimensional sphere (i.e., two-dimensional cylinder), and two-dimensional sphere (the four-dimensional manifold). Therefore, we offer the interesting example of three-dimensional phase pattern of pendulum-like system which describes the motion of spherical pendulum, placed in an over-run medium flow [1], [5], [6].

Since we present the cases of complete integrability in spatial rigid body dynamics of the motion in a nonconservative field, we deal with three (at first thought) independent properties:

i) the distinguished class of systems with the symmetries above;

ii) the fact that this class of systems consists of systems with zero mean variable dissipation (in the having periodic variable), which allows us to consider them as "almost" conservative systems;

iii) in certain (although lower-dimensional) cases, these systems have the complete tuple of first integrals, which are transcendental in general (from the viewpoint of complex analysis).

In [1] the obtained results are systematized on study of the dynamic equations of the motion of symmetrical two-dimensional ($2D-$) rigid body which residing in a certain nonconservative field of the forces. Its type is unoriginal from dynamics of the real rigid bodies interacting with a resisting medium on the laws of a jet flow, under which the nonconservative tracing force acts onto the body, and it either forces the value of the velocity of a certain typical point of the rigid body to remain as constant in all time of motion, that means the presence in system of nonintegrable servo-constraint, or forces the center of mass of the body to move rectilinearly and uniformly in all time of motion, that means the presence in system of nonconservative pair of the forces (see also [6], [7]).

Therefore, in [1] the additional transcendental first integral expressing through the finite combination of elementary functions is found to having analytical nonintegrable constraint, and in [8], [9], [10] the same was made to having analytical first integral (the square of the center of mass) only.

New obtained results are systematized and given in invariant form. Herewith, the additional dependence of the moment of the nonconservative force on the angular velocity is introduced. The given dependence can be wide-spread and on the cases of the motions in the spaces of higher dimensions.

In [1], [11] the obtained results are systematized on study of the dynamic equations of the motion of symmetrical three-dimensional ($3D-$) rigid body which residing in a certain nonconservative field of the forces. Its type is also unoriginal from dynamics of the real rigid bodies interacting with a resisting medium on the laws of a jet flow, under which the nonconservative tracing force acts onto the body, and it either forces the value of the velocity of a certain typical point of the rigid body to remain as constant in all time of motion, that means the presence in system of nonintegrable servo-constraint, or forces the center of mass of the body to move rectilinearly and uniformly in all time of motion, that means the presence in system of nonconservative pair of the forces.

Therefore, in [10], [11], [12] three additional transcendental first integrals expressing through the finite combination of elementary functions are found to having analytical invariant relations (nonintegrable constraint and the integral on the equality to zero one of the component of angular velocity), and in [11], [12], [13] the same was made to having analytical first integrals (the square of the center of mass and the integral on the equality to zero one of the component of angular velocity) only.

In [1] we declare the general aspects of dynamics of multi-dimensional rigid body, i.e., the notion of angular velocity tensor, the joint dynamic equations of the motion on direct product $\mathbf{R}^n \times \mathrm{so}(n)$, the Euler and Rivals formulas in multi-dimensional case.

The question on tensor of inertia of four-dimensional ($4D-$) rigid body is considered. In this activity it is proposed to study two possible cases logically on principal moments of inertia, i.e., when there exists *two* relations on the principal moments of inertia:

(i) when there exist *three* equal principal moments of inertia ($I_2 = I_3 = I_4$);

(ii) when there exist *two pairs* of equal moments of inetria ($I_1 = I_2, I_3 = I_4$).

In [12], [13], [14] the obtained results are systematized on study of the dynamic equations of the motion of symmetrical four-dimensional ($4D-$) rigid body which residing in a certain nonconservative field of the forces for the case (i). Its type is also unoriginal from dynamics of lower-dimensional real rigid bodies interacting with a resisting medium on the laws of a jet flow, under which the nonconservative tracing force acts onto the body, and it either forces the value of the velocity of a certain typical point of the rigid body to remain as constant in all time of motion, that means the presence in system of nonintegrable servo-constraint, or forces the center of mass of the body to move rectilinearly and uniformly in all time of motion, that means the presence in system of nonconservative pair of the forces.

Therefore, in [13], [14], [15] four additional transcendental first integrals expressing through the finite combination of elementary functions are found to having four analytical invariant relations (nonintegrable constraint and three integrals on the equalities to zero some of the components of angular velocity tensor), and in [14], [15], [16] the same was made to having four analytical first integrals (the square of the center of mass and three integrals on the equalities to zero some of the components of angular velocity tensor) only.

The results are pertained to the case when all the interaction of a medium with the body is concentrated on that part of the body surface that has the form of three-dimensional disk,

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TABLE I
CLASSIFICATION OF THE CASES OF INTEGRABILITY FROM TWO-DIMENSIONAL SYMMETRIC RIGID BODY DYNAMICS IN $\mathbf{E}^2$ TO FOUR-DIMENSIONAL DYNAMICALLY SYMMETRIC RIGID BODY DYNAMICS IN $\mathbf{E}^4$

Dimension of a Rigid Body	Constraint Condition	
	$v \equiv$ const ($\beta_2 \equiv$ const)	$\mathbf{V}_C \equiv$ **const**
$\mathbf{E}^2$	$h = 0 \oplus$	$h = 0 \oplus$
	$h \neq 0 \oplus$	$h \neq 0 \oplus$
$\mathbf{E}^3$	$h = 0 \oplus$	$h = 0 \oplus$
$(I_2 = I_3)$	$h \neq 0 \oplus$	$h \neq 0 \oplus$
$\mathbf{E}^4$	$h = 0 \oplus$	$h = 0 \oplus$
$(I_2 = I_3 = I_4)$	$h \neq 0 \oplus$	$h \neq 0 \oplus$
$\mathbf{E}^4$	$h = 0 \oplus$	$h = 0 \ominus$
$(I_1 = I_2,\ I_3 = I_4)$	$h \neq 0 \oplus$	$h \neq 0 \ominus$

herewith, the force interaction is concentrated in the direction which is perpendicular to this disk. These results are systematized and given in invariant form. Herewith, the additional dependence of the moment of the nonconservative force on the angular velocity is introduced. The given dependence can be wide-spread and on the cases of the motions in the spaces of higher dimensions.

In this activity the obtained results are systematized on study of the dynamic equations of the motion of symmetrical four-dimensional $(4D-)$ rigid body which residing in a certain nonconservative field of the forces for the case (ii). Its type is also unoriginal from dynamics of lower-dimensional real rigid bodies interacting with a resisting medium on the laws of a jet flow, under which the nonconservative tracing force acts onto the body, and it forces both the value of the velocity of a certain typical point of the rigid body and the certain phase variable to remain as constant in all time, that means the presence in system of nonintegrable servo-constraints.

Therefore, in this activity two additional transcendental and three analytical first integrals expressing through the finite combination of elementary functions are found to having four analytical invariant relations (two nonintegrable constraints and two integrals on the equalities to zero some of the components of angular velocity tensor).

The results which are obtained now are pertained to the case when all the interaction of a medium with the body is concentrated on that part of the body surface that has the form of two-dimensional disk, herewith, the force interaction is concentrated in two-dimensional plane which is perpendicular to this disk. These results are systematized and given in invariant form. Herewith, the additional dependence of the moment of the nonconservative force on the angular velocity is introduced. The given dependence can be wide-spread and on the cases of the motions in the spaces of higher dimensions.

And so, in [16], [17], [18] the cases of integrability in lower- and multi-dimensional dynamics of a rigid body placed in a nonconservative force field. To systemize we shall place all of them to the following table.

The notifications $h = 0$ (or $h \neq 0$) mean that the dependence of the force field on the components of angular velocity tensor is present (or is absent) in the system.

The sign $\oplus$ means that the case is placed to this review.

Two signs $\ominus$ in the right below corner of the table mean that these two cases are not placed to this review (indeed, this activity is devoted to the case $I_1 = I_2,\ I_3 = I_4$ only).

Nevertheless, the corresponding results have already obtained for the case $I_2 = \ldots = I_n$ of symmetric n-dimensional rigid body, and those results are not also placed to this review.

Many results of this work were regularly reported at numerous workshops, including the workshop ”Actual Problems of Geometry and Mechanics” named after professor V. V. Trofimov led by D. V. Georgievskii and M. V. Shamolin.

III. CASES OF INTEGRABILITY CORRESPONDING TO A RIGID BODY MOTION IN FOUR-DIMENSIONAL SPACE

In this section the new results are systematized on study of the equations of the motion of dynamically symmetrical four-dimensional $(4D-)$ rigid body which residing in a certain nonconservative field of the forces in the case of special dynamical symmetry. Its type is unoriginal from dynamics of the real lower-dimensional rigid bodies of interacting with a resisting medium on the laws of a jet flow, under which the nonconservative tracing force acts onto the body and forces both the value of the velocity of a certain typical point of the rigid body and the certain phase variable to remain as constant in all time, that means the presence in system of nonintegrable servo-constraints.

Previously, in [1], the author showed the complete integrability of the equations of body planeparallel motion in a resisting medium under the conditions of streamline flow around when the system of dynamical equations has a first integral that is a transcendental (having essentially singular points in the sense of the theory of functions of one complex variable) function of quasi-velocities. At that time, it was assumed that the interaction of the medium with the body is concentrated on the part of the body surface that has the form of a (one-dimensional) plate.

Later on, in [2], [5], [18], the plane problem was generalized to the spatial (three-dimensional) case where the system of dynamical equations has a complete tuple of transcendental first integrals. It was assumed here that the whole interaction of the medium and the body is concentrated on a part of the body surface that has the form of a plane (two-dimensional) disk.

In this section the results which are obtained now are pertained to the case when all the interaction of a medium with the body is concentrated on that part of the body surface that has the form of two-dimensional disk, herewith, the force interaction is concentrated in two-dimensional plane which is perpendicular to this disk. These results are systematized and given in invariant form. Herewith, the additional dependence of the moment of the nonconservative force on the angular velocity is introduced. The given dependence can be wide-spread and on the cases of the motions in the spaces of higher dimensions.

IV. More general problem of the motion with the tracing force

Let consider the motion of a homogeneous dynamically symmetric rigid body with ”the front end-wall” (two-dimensional disk interacting with a medium which filling the four-dimensional space) in the field of force **S** of the resistance under the conditions of quasistationarity.

Let $(v, \alpha, \beta_2, \beta_1)$ are the coordinates of the vector velocity of a certain typical point D of a rigid body (D is the center of two-dimensional disk) such that α is the angle between the vector $\mathbf{v}_D$ and the plane Dx_1x_2, β_2 is the angle measured in the plane Dx_1x_2 up to the projection of the vector $\mathbf{v}_D$ on the plane Dx_1x_2, β_1 is the angle mesured in the plane Dx_3x_4 up to the projection of the vector $\mathbf{v}_D$ on the plane Dx_3x_4,

$$\Omega = \begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

is the angular velocity tensor of the body, $Dx_1x_2x_3x_4$ is the coordinate system related to the body, herewith, the straight line CD lies in the plane Dx_1x_2 (C is the center of mass), and the axes Dx_3, Dx_4 lie in the disk plane, $I_1,\ I_2 = I_1,\ I_3,\ I_4 = I_3,\ m$ are the inertia–mass characteristics.

Let accept the following decompositions in the projections on the axes of the coordinate system $Dx_1x_2x_3x_4$:

$$\mathbf{DC} = \{\sigma \sin\gamma, -\sigma\cos\gamma, 0, 0\},$$

$$\mathbf{v}_D = \{v\cos\alpha\sin\beta_2, v\cos\alpha\cos\beta_2,$$

$$v\sin\alpha\cos\beta_1, v\sin\alpha\sin\beta_1\}. \quad (1)$$

Herewith, in our case the decomposition will be also correct for the function of a medium interaction on four-dimensional body:

$$\mathbf{S} = \{S_1, S_2, 0, 0\}, \quad (2)$$

herewith

$$S_1 = S\sin\gamma,\ S_2 = -S\cos\gamma,\ \gamma = \text{const}, \quad (3)$$

i.e. in this case $\mathbf{F} = \mathbf{S}$, and the angle γ is measured in the plane Dx_1x_2.

Then those part of dynamic equations of the body motion (including and in the case of Chaplygin analytical functions, see below) which describes the center of mass motion and corresponds to the space $\mathbf{R}^4$ under which the tangent forces to three-dimensional disk are absent, has the form:

$$\dot v\cos\alpha\sin\beta_2 - \dot\alpha v\sin\alpha\sin\beta_2 + \dot\beta_2 v\cos\alpha\cos\beta_2 -$$

$$-\omega_6 v\cos\alpha\cos\beta_2 + \omega_5 v\sin\alpha\cos\beta_1 - \omega_3 v\sin\alpha\sin\beta_1 -$$

$$-\sigma(\omega_6^2 + \omega_5^2 + \omega_3^2)\sin\gamma - \sigma(\omega_4\omega_5 + \omega_2\omega_3)\cos\gamma +$$

$$+\sigma\dot\omega_6\cos\gamma = \frac{S_1}{m}, \quad (4)$$

$$\dot v\cos\alpha\cos\beta_2 - \dot\alpha v\sin\alpha\cos\beta_2 - \dot\beta_2 v\cos\alpha\sin\beta_2 +$$

$$+\omega_6 v\cos\alpha\sin\beta_2 - \omega_4 v\sin\alpha\cos\beta_1 + \omega_2 v\sin\alpha\sin\beta_1 +$$

$$+\sigma(\omega_6^2 + \omega_4^2 + \omega_2^2)\cos\gamma + \sigma(\omega_4\omega_5 + \omega_2\omega_3)\sin\gamma +$$

$$+\sigma\dot\omega_6\sin\gamma = \frac{S_2}{m}, \quad (5)$$

$$\dot v\sin\alpha\cos\beta_1 + \dot\alpha v\cos\alpha\cos\beta_1 - \dot\beta_1 v\sin\alpha\sin\beta_1 -$$

$$-\omega_5 v\cos\alpha\sin\beta_2 + \omega_4 v\cos\alpha\cos\beta_2 - \omega_1 v\sin\alpha\sin\beta_1 +$$

$$+\sigma(\omega_4\omega_6 - \omega_1\omega_3)\sin\gamma - \sigma(\omega_5\omega_6 + \omega_1\omega_2)\cos\gamma -$$

$$-\sigma\dot\omega_5\sin\gamma - \sigma\dot\omega_4\cos\gamma = 0, \quad (6)$$

$$\dot v\sin\alpha\sin\beta_1 + \dot\alpha v\cos\alpha\sin\beta_1 + \dot\beta_1 v\sin\alpha\cos\beta_1 +$$

$$+\omega_3 v\cos\alpha\sin\beta_2 - \omega_2 v\cos\alpha\cos\beta_2 + \omega_1 v\sin\alpha\cos\beta_1 -$$

$$-\sigma(\omega_2\omega_6 + \omega_1\omega_5)\sin\gamma + \sigma(\omega_3\omega_6 - \omega_1\omega_4)\cos\gamma +$$

$$+\sigma\dot\omega_3\sin\gamma + \sigma\dot\omega_2\cos\gamma = 0, \quad (7)$$

where

$$S = s(\alpha)v^2,\ \sigma = CD,\ v > 0. \quad (8)$$

Later on, the auxiliary matrix for the calculation of the moment of the resisting force has the form

$$\begin{pmatrix} 0 & 0 & x_{3N} & x_{4N} \\ S_1 & S_2 & 0 & 0 \end{pmatrix}, \quad (9)$$

then those part of the dynamic equations of the body motion which describes the body motion around the center of mass, and corresponds to the Lie algebra so(4), has the form:

$$(\lambda_4 + \lambda_3)\dot\omega_1 + (\lambda_3 - \lambda_4)(\omega_3\omega_5 + \omega_2\omega_4) = 0, \quad (10)$$

$$(\lambda_2 + \lambda_4)\dot\omega_2 + (\lambda_2 - \lambda_4)(\omega_3\omega_6 - \omega_1\omega_4) =$$

$$= -x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2\cos\gamma, \quad (11)$$

$$(\lambda_4 + \lambda_1)\dot\omega_3 + (\lambda_4 - \lambda_1)(\omega_2\omega_6 + \omega_1\omega_5) =$$

$$= -x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2\sin\gamma, \quad (12)$$

$$(\lambda_3 + \lambda_2)\dot\omega_4 + (\lambda_2 - \lambda_3)(\omega_5\omega_6 + \omega_1\omega_2) =$$

$$= x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2\cos\gamma, \quad (13)$$

$$(\lambda_1 + \lambda_3)\dot\omega_5 + (\lambda_3 - \lambda_1)(\omega_4\omega_6 - \omega_1\omega_3) =$$

$$= x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2\sin\gamma, \quad (14)$$

$$(\lambda_1 + \lambda_2)\dot\omega_6 + (\lambda_1 - \lambda_2)(\omega_4\omega_5 + \omega_2\omega_3) = 0. \quad (15)$$

Thus, the following direct product of four-dimensional manifold on the Lie algebra so(4) is the phase space of the tenth order system (4)–(7), (10)–(15):

$$\mathbf{R}^1 \times \mathbf{S}^3 \times \text{so}(4). \quad (16)$$

We notice right now that the system (4)–(7), (10)–(15), by virtue of the having dynamical symmetry

$$I_1 = I_2,\ I_3 = I_4, \quad (17)$$

possesses the cyclic first integrals

$$\omega_1 \equiv \omega_1^0 = \text{const},\ \omega_6 \equiv \omega_6^0 = \text{const}. \quad (18)$$

Herewith, hereinafter we shall consider the dynamics of the system on zero level:

$$\omega_1^0 = \omega_6^0 = 0. \quad (19)$$

And if there exists the more general problem of the body motion with the certain tracing force **T**, which acts on the plane Dx_1x_2 and providing the fulfillment of the following equalities in all time of the motion

$$v \equiv \text{const}, \;\; \beta_2 \equiv \text{const}, \quad (20)$$

that in the system (4)–(7), (10)–(15) the values

$$T_1 + S_1, \;\; T_2 + S_2 \quad (21)$$

will stand instead of F_1 and F_2 accordingly.

As a result of corresponding value choice T of the tracing force it is possible to obtain formally the fulfillment of the equalities (20) in all time of the motion. Really, if we express formally the value T by virtue of the system (4)–(7), (10)–(15) we shall obtain for $\cos\alpha \neq 0$:

$$T_1 = T_{1,v,\beta 2}(\alpha, \beta_1, \Omega) =$$
$$= -m\sigma(\omega_5^2 + \omega_3^2)\sin\gamma - m\sigma(\omega_4\omega_5 + \omega_2\omega_3)\cos\gamma +$$
$$+ m\omega_5 v \sin\alpha\cos\beta_1\cos^2\beta_2 - m\omega_3 v\sin\alpha\sin\beta_1\cos^2\beta_2 +$$
$$+ m\omega_4 v\sin\alpha\cos\beta_1\sin\beta_2\cos\beta_2 -$$
$$- m\omega_2 v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_2 - s(\alpha)v^2 \times$$
$$\times \left[\sin\gamma - \frac{m\sigma}{I_1+I_3}\frac{\sin\alpha}{\cos\alpha}\sin\beta_2 \cdot \Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\right], \quad (22)$$

$$T_2 = T_{2,v,\beta 2}(\alpha, \beta_1, \Omega) =$$
$$= m\sigma(\omega_4^2 + \omega_2^2)\cos\gamma + m\sigma(\omega_4\omega_5 + \omega_2\omega_3)\sin\gamma -$$
$$- m\omega_4 v\sin\alpha\cos\beta_1\sin^2\beta_2 + m\omega_2 v\sin\alpha\sin\beta_1\sin^2\beta_2 -$$
$$- m\omega_5 v\sin\alpha\cos\beta_1\sin\beta_2\cos\beta_2 +$$
$$+ m\omega_3 v\sin\alpha\sin\beta_1\sin\beta_2\cos\beta_2 + s(\alpha)v^2 \times$$
$$\times \left[\cos\gamma - \frac{m\sigma}{I_1+I_3}\frac{\sin\alpha}{\cos\alpha}\cos\beta_2 \cdot \Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right)\right], \quad (23)$$

where

$$\Lambda_{v,\beta_2}\left(\alpha, \beta_1, \frac{\Omega}{v}\right) = x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\cos\beta_1 +$$
$$+ x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)\sin\beta_1. \quad (24)$$

The conditions (18)–(20) are used at the obtaining of the equalities (22) and (23).

It makes possible to look at this procedure from two positions. In first, the transformation of the system has occurred at presence of the tracing (control) force in the system which provides the consideration of interesting classes of the motion (20). In second, it makes possible to look at this like the procedure which allows to deflate the system. Really, the system (4)–(7), (10)–(15) as a result of that action generates an independent system of the sixth order of the following type:

$$\dot\alpha v\cos\alpha\cos\beta_1 - \dot\beta_1 v\sin\alpha\sin\beta_1 - \omega_5 v\cos\alpha\sin\beta_2 +$$
$$+ \omega_4 v\cos\alpha\cos\beta_2 - \sigma\dot\omega_5\sin\gamma - \sigma\dot\omega_4\cos\gamma = 0, \quad (25)$$

$$\dot\alpha v\cos\alpha\sin\beta_1 + \dot\beta_1 v\sin\alpha\cos\beta_1 + \omega_3 v\cos\alpha\sin\beta_2 -$$
$$- \omega_2 v\cos\alpha\cos\beta_2 + \sigma\dot\omega_3\sin\gamma + \sigma\dot\omega_2\cos\gamma = 0, \quad (26)$$

$$(I_1+I_3)\dot\omega_2 = -x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2\cos\gamma, \quad (27)$$

$$(I_1+I_3)\dot\omega_3 = -x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2\sin\gamma, \quad (28)$$

$$(I_1+I_3)\dot\omega_4 = x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2\cos\gamma, \quad (29)$$

$$(I_1+I_3)\dot\omega_5 = x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right)s(\alpha)v^2\sin\gamma, \quad (30)$$

in which the parameters v, β_2 are added to the constant parameters specified above.

A. Two systems of the discourses on integrability

Remark (on analytical first integrals). Obviously that the system (25)–(30) possesses two analytical first integrals which are expressed in terms of the finite combination of the elementary functions:

$$\omega_2\sin\gamma - \omega_3\cos\gamma = W_1' = \text{const}, \quad (31)$$

$$\omega_4\sin\gamma - \omega_5\cos\gamma = W_2' = \text{const}. \quad (32)$$

First of all this means that the system (25)–(30) can be reduced to the fourth order system on its own four-dimensional phase manifold.

Hereafter, it makes possible to develop by the following ways under the study of the system (25)–(30) (i.e. to accept the following systems of the discourses).

I. In first, it makes possible ”not to notice” the existence in the system the first integrals of the forms (31), (32). Then conducting the series of the equivalent transformations it can possible try to reduce the investigated system (25)–(30) to the equivalent system in which the reduction to the systems of lower dimensionality will occur. Herewith, it is sufficient to get the quantity of the independent first integrals smaller then previous one on two units for the complete system integration, by virtue of (31), (32).

II. In second, it makes possible to use the first integrals (31), (32) expressing two interested phase variables from the list $\omega_2, \omega_3, \omega_4, \omega_5$. Herewith, we shall get just the fourth order system as the system which is the reduction of the system (25)–(30) to the certain four-dimensional phase manifold.

In the beginning we shall choose the system of the discourses **I**.

Really, the system (25)–(30) is equivalent to

$$\dot\alpha v\cos\alpha - \omega_5 v\cos\alpha\cos\beta_1\sin\beta_2 + \omega_4 v\cos\alpha\cos\beta_1\cos\beta_2 +$$
$$+ \omega_3 v\cos\alpha\sin\beta_1\sin\beta_2 - \omega_2 v\cos\alpha\sin\beta_1\cos\beta_2 -$$
$$\sigma\dot\omega_5\sin\gamma\cos\beta_1 - \sigma\dot\omega_4\cos\gamma\cos\beta_1 +$$
$$+ \sigma\dot\omega_3\sin\gamma\sin\beta_1 + \sigma\dot\omega_2\cos\gamma\sin\beta_1 = 0, \quad (33)$$

$$\dot\beta_1 v\sin\alpha + \omega_3 v\cos\alpha\cos\beta_1\sin\beta_2 - \omega_2 v\cos\alpha\cos\beta_1\cos\beta_2 +$$

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$$+\omega_5 v\cos\alpha\sin\beta_1\sin\beta_2-\omega_4 v\cos\alpha\sin\beta_1\cos\beta_2+$$
$$+\sigma\dot{\omega_3}\sin\gamma\cos\beta_1+\sigma\dot{\omega_2}\cos\gamma\cos\beta_1+$$
$$+\sigma\dot{\omega_5}\sin\gamma\sin\beta_1+\sigma\dot{\omega_4}\cos\gamma\sin\beta_1=0, \quad (34)$$

$$\dot{\omega}_2=-\frac{v^2}{I_1+I_3}x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)\cos\gamma, \quad (35)$$

$$\dot{\omega}_3=-\frac{v^2}{I_1+I_3}x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)\sin\gamma, \quad (36)$$

$$\dot{\omega}_4=\frac{v^2}{I_1+I_3}x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)\cos\gamma, \quad (37)$$

$$\dot{\omega}_5=\frac{v^2}{I_1+I_3}x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right)s(\alpha)\sin\gamma, \quad (38)$$

Let introduce new quasivelocities in the system. We shall transform the values $\omega_2,\omega_3,\omega_4,\omega_5$ by means of the composition of the following rotations for this:

$$\begin{pmatrix} z_1 \\ -z_2 \end{pmatrix}=T_*(-\beta_1)\begin{pmatrix} \omega_3 \\ \omega_5 \end{pmatrix},$$
$$\begin{pmatrix} z_3 \\ -z_4 \end{pmatrix}=T_*(-\beta_1)\begin{pmatrix} \omega_2 \\ \omega_4 \end{pmatrix}, \quad (39)$$

where

$$T_*(\beta_1)=\begin{pmatrix} \cos\beta_1 & -\sin\beta_1 \\ \sin\beta_1 & \cos\beta_1 \end{pmatrix}, \quad (40)$$

and also

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}=T_*(\beta_2)\begin{pmatrix} z_3 \\ z_1 \end{pmatrix},$$
$$\begin{pmatrix} w_3 \\ w_4 \end{pmatrix}=T_*(-\beta_2)\begin{pmatrix} -z_4 \\ z_2 \end{pmatrix}. \quad (41)$$

Thus, the following relations are correct:

$$\begin{aligned}
z_1&=\omega_3\cos\beta_1+\omega_5\sin\beta_1,\\
z_2&=\omega_3\sin\beta_1-\omega_5\cos\beta_1,\\
z_3&=\omega_2\cos\beta_1+\omega_4\sin\beta_1,\\
z_4&=\omega_2\sin\beta_1-\omega_4\cos\beta_1,\\
w_1&=-z_1\sin\beta_2+z_3\cos\beta_2,\\
w_2&=z_3\sin\beta_2+z_1\cos\beta_2,\\
w_3&=z_2\sin\beta_2-z_4\cos\beta_2,\\
w_4&=z_4\sin\beta_2+z_2\cos\beta_2.
\end{aligned} \quad (42)$$

As is seen from (33)–(38), on the manifold

$$O_2=\left\{(\alpha,\beta_1,\omega_2,\omega_3,\omega_4,\omega_5)\in\mathbf{R}^6:\ \alpha=\frac{\pi}{2}k,\ k\in\mathbf{Z}\right\} \quad (43)$$

it is impossible to resolve the system uniquely relatively to $\dot{\alpha},\ \dot{\beta_1}$. Thus, the violation of the uniqueness theorem is happened on the manifold (43) formally. Moreover, in first, the indefiniteness is happened for even or odd k by the reason of degeneration of the coordinates $(v,\alpha,\beta_1,\beta_2)$ which are parameterized the three-dimensional sphere (but are not the classical spherical coordinates), and, in second, it is happened the evident violation of the uniqueness theorem for odd k because of the first equation of (33) degenerates for this case.

Really, Jacobian of the transformation $x_1,x_2,x_3,x_4\longrightarrow v,\alpha,\beta_1,\beta_2$

$$\begin{aligned}
x_1&=v\cos\alpha\sin\beta_2,\\
x_2&=v\cos\alpha\cos\beta_2,\\
x_3&=v\sin\alpha\cos\beta_1,\\
x_4&=v\sin\alpha\sin\beta_1
\end{aligned} \quad (44)$$

is equal to

$$v^3\cos\alpha\sin\alpha,$$

in what it differs from Jacobian of the transformation under the transition to the generalized spherical coordinates v,α,β_1,β_2, which, in turn, is equal to

$$v^3\sin\alpha\sin\beta_1.$$

It follows that the system (33)–(38) outside of and only outside of the manifold (43) is equivalent to the system

$$\dot{\alpha}=-w_3+\frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\cos\alpha}\cdot\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right), \quad (45)$$

$$\dot{z_4}=-\frac{v^2}{I_1+I_3}s(\alpha)\cos\gamma\cdot\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)+$$
$$+z_3\left[w_1\frac{\cos\alpha}{\sin\alpha}-\frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha}\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\right], \quad (46)$$

$$\dot{z_3}=\frac{v^2}{I_1+I_3}s(\alpha)\cos\gamma\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)-$$
$$-z_4\left[w_1\frac{\cos\alpha}{\sin\alpha}-\frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha}\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\right], \quad (47)$$

$$\dot{z_2}=-\frac{v^2}{I_1+I_3}s(\alpha)\sin\gamma\cdot\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)+$$
$$+z_1\left[w_1\frac{\cos\alpha}{\sin\alpha}-\frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha}\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\right], \quad (48)$$

$$\dot{z_1}=\frac{v^2}{I_1+I_3}s(\alpha)\sin\gamma\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)-$$
$$-z_2\left[w_1\frac{\cos\alpha}{\sin\alpha}-\frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha}\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\right], \quad (49)$$

$$\dot{\beta_1}=w_1\frac{\cos\alpha}{\sin\alpha}-\frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha}\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right), \quad (50)$$

or finally

$$\dot{\alpha}=-w_3+\frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\cos\alpha}\cdot\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right), \quad (51)$$

$$\dot{w_4}=-\frac{v^2}{I_1+I_3}s(\alpha)\sin(\beta_2+\gamma)\cdot\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)+$$
$$+w_2\left[w_1\frac{\cos\alpha}{\sin\alpha}-\frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha}\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\right], \quad (52)$$

$$\dot{w_3}=\frac{v^2}{I_1+I_3}s(\alpha)\cos(\beta_2+\gamma)\cdot\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)-$$
$$-w_1\left[w_1\frac{\cos\alpha}{\sin\alpha}-\frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha}\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\right], \quad (53)$$

$$\dot{w}_2 = \frac{v^2}{I_1+I_3}s(\alpha)\sin(\beta_2+\gamma)\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)-$$
$$-w_4\left[w_1\frac{\cos\alpha}{\sin\alpha}-\frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha}\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\right], \quad (54)$$
$$\dot{w}_1 = \frac{v^2}{I_1+I_3}s(\alpha)\cos(\beta_2+\gamma)\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)+$$
$$+w_3\left[w_1\frac{\cos\alpha}{\sin\alpha}-\frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha}\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\right], \quad (55)$$
$$\dot{\beta}_1 = w_1\frac{\cos\alpha}{\sin\alpha}-\frac{\sigma v}{I_1+I_3}\frac{s(\alpha)}{\sin\alpha}\cdot\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right), \quad (56)$$

where

$$\Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right) = -x_{4N}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1+$$
$$+x_{3N}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1, \quad (57)$$

and the function $\Lambda_{v,\beta_2}(\alpha,\beta_1,\Omega/v)$ is represented in the form (24).

Hereafter, the dependence on the groups of the variables $(\alpha,\beta_1,\beta_2,\Omega/v)$ is understood like the complicated dependence on $(\alpha,\beta_1,\beta_2,z_1/v,z_2/v,z_3/v,z_4/v)$ (or $(\alpha,\beta_1,\beta_2,w_1/v,w_2/v,w_3/v,w_4/v)$) by virtue of (42).

The violation of the uniqueness theorem is happened for the system (33)–(38) on the manifold (43) for odd k in following sense: the regular phase trajectory of the system (33)–(38) passes through nearly any point from the manifold (43) for odd k intersecting the manifold (43) under right angle, and also there exist the phase trajectory which coincides completely with the specified point in all moments of time. But those are the different trajectories physically since the different values of the tracing force correspond them. Let show this.

As it is shown above, it is necessary to choose the values T_1 and T_2 for $\cos\alpha\neq 0$ in the form of (22) and (23) to fulfill the constraints (20).

Let

$$\lim_{\alpha\to\pi/2}\frac{s(\alpha)}{\cos\alpha}\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right) = L\left(\beta_1,\beta_2,\frac{\Omega}{v}\right). \quad (58)$$

Let note that $|L|<+\infty$ iff, when

$$\lim_{\alpha\to\pi/2}\left|\frac{\partial}{\partial\alpha}\left(\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha)\right)\right| < +\infty. \quad (59)$$

The necessary values of the tracing force for $\alpha=\pi/2$ should be found from the equalities

$$T_1 = T_{1,v,\beta 2}\left(\frac{\pi}{2},\beta_1,\Omega\right) =$$
$$= -m\sigma(\omega_5^2+\omega_3^2)\sin\gamma - m\sigma(\omega_4\omega_5+\omega_2\omega_3)\cos\gamma+$$
$$+m\omega_5 v\cos\beta_1\cos^2\beta_2 - m\omega_3 v\sin\beta_1\cos^2\beta_2+$$
$$+m\omega_4 v\cos\beta_1\sin\beta_2\cos\beta_2 - m\omega_2 v\sin\beta_1\sin\beta_2\cos\beta_2+$$
$$+v^2\frac{m\sigma}{I_1+I_3}\sin\beta_2\cdot L, \quad (60)$$
$$T_2 = T_{2,v,\beta 2}\left(\frac{\pi}{2},\beta_1,\Omega\right) =$$
$$= m\sigma(\omega_4^2+\omega_2^2)\cos\gamma + m\sigma(\omega_4\omega_5+\omega_2\omega_3)\sin\gamma-$$
$$-m\omega_4 v\cos\beta_1\sin^2\beta_2 + m\omega_2 v\sin\beta_1\sin^2\beta_2-$$
$$-m\omega_5 v\cos\beta_1\sin\beta_2\cos\beta_2 + m\omega_3 v\sin\beta_1\sin\beta_2\cos\beta_2-$$
$$-v^2\frac{m\sigma}{I_1+I_3}\cos\beta_2\cdot L, \quad (61)$$

where the values of $\omega_2,\omega_3,\omega_4,\omega_5$ are arbitrary.

On the other hand, if we make the rotation around a certain point W by means of the tracing force it will be necessary to choose the projections of the tracing force in the form of

$$T = T_1\left(\frac{\pi}{2},\beta_1,\beta_2,\Omega\right) = \frac{mv^2}{R_{01}}, \quad (62)$$
$$T = T_2\left(\frac{\pi}{2},\beta_1,\beta_2,\Omega\right) = \frac{mv^2}{R_{02}}, \quad (63)$$

where R_{01}, R_{02} are the projections of the cut CW onto the corresponding axes of the coordinates.

The equalities (22), (23) and (62) (63) define, generally speaking, the different values of the tracing force T for almost all the points of the manifold (43), and that is proved the suitable remark.

V. Case of the absence of the dependence of the moment of the nonconservative forces on the angular velocity

A. Reduced system

Similarly to the choice of the Chaplygin analytical functions, we shall accept the dynamic functions s, x_{3N} and x_{4N} as the following form:

$$s(\alpha) = B\cos\alpha,\ A,B>0,\ v\neq 0,$$
$$x_{3N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = x_{3N0}(\alpha,\beta_1) = A\sin\alpha\cos\beta_1, \quad (64)$$
$$x_{4N}\left(\alpha,\beta_1,\beta_2,\frac{\Omega}{v}\right) = x_{4N0}(\alpha,\beta_1) = A\sin\alpha\sin\beta_1,$$

which convinces us that the dependence of the moment of the nonconservative forces on the angular velocity is absent in considered system (and there exist the dependences on the angles α,β_1,β_2 only).

Herewith, the functions $\Lambda_{v,\beta_2},\Pi_{v,\beta_2}$, appearing in the system (51)–(56), have the following forms:

$$\Lambda_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right) = A\sin\alpha,\ \Pi_{v,\beta_2}\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\equiv 0. \quad (65)$$

Then the dynamic part of the motion equations (the system (51)–(56)) will have the form as the following analytical system by means of the nonintegrable constraints (20) outside of and only outside of the manifold (43)

$$\dot{\alpha} = -w_3 + \frac{\sigma ABv}{I_1+I_3}\sin\alpha, \quad (66)$$
$$\dot{w}_4 = -\frac{ABv^2}{I_1+I_3}\sin(\beta_2+\gamma)\sin\alpha\cos\alpha + w_1w_2\frac{\cos\alpha}{\sin\alpha}, \quad (67)$$
$$\dot{w}_3 = \frac{ABv^2}{I_1+I_3}\cos(\beta_2+\gamma)\sin\alpha\cos\alpha - w_1^2\frac{\cos\alpha}{\sin\alpha}, \quad (68)$$

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$$\dot{w}_2 = -w_1 w_4 \frac{\cos\alpha}{\sin\alpha}, \qquad (69)$$

$$\dot{w}_1 = w_1 w_3 \frac{\cos\alpha}{\sin\alpha}, \qquad (70)$$

$$\dot{\beta}_1 = w_1 \frac{\cos\alpha}{\sin\alpha}, \qquad (71)$$

If we introduce the dimensionless variables, parameters and differentiability as follows:

$$w_k \mapsto n_0 v w_k,\ k = 1,2,3,4,\ n_0^2 = \frac{AB}{I_1 + I_3},$$
$$b = \sigma n_0, \quad <\cdot> = n_0 v <'>, \qquad (72)$$

we shall reduce the system (66)–(71) to the form

$$\alpha' = -w_3 + b\sin\alpha, \qquad (73)$$

$$w_4' = -\sin(\beta_2 + \gamma)\sin\alpha\cos\alpha + w_1 w_2 \frac{\cos\alpha}{\sin\alpha}, \qquad (74)$$

$$w_3' = \cos(\beta_2 + \gamma)\sin\alpha\cos\alpha - w_1^2 \frac{\cos\alpha}{\sin\alpha}, \qquad (75)$$

$$w_2' = -w_1 w_4 \frac{\cos\alpha}{\sin\alpha}, \qquad (76)$$

$$w_1' = w_1 w_3 \frac{\cos\alpha}{\sin\alpha}, \qquad (77)$$

$$\beta_1' = w_1 \frac{\cos\alpha}{\sin\alpha}, \qquad (78)$$

As is seen, the independent fifth order system (73)–(77) on its own five-dimensional manifold was formed in the sixth order system (73)–(78) which can be considered on its own six-dimensional manifold

$$T\mathbf{S}^2 \times \mathbf{R}^2 \qquad (79)$$

— the direct product of the tangent stratification $T\mathbf{S}^2$ of two-dimensional sphere $\mathbf{S}^2$ on two-dimensional plane.

Furthermore, the independent third order subsystem

$$\alpha' = -w_3 + b\sin\alpha, \qquad (80)$$

$$w_3' = \cos(\beta_2 + \gamma)\sin\alpha\cos\alpha - w_1^2 \frac{\cos\alpha}{\sin\alpha}, \qquad (81)$$

$$w_1' = w_1 w_3 \frac{\cos\alpha}{\sin\alpha}, \qquad (82)$$

was formed from the sixth order system (73)–(78), and also (while dependent) second order system

$$w_4' = -\sin(\beta_2 + \gamma)\sin\alpha\cos\alpha + w_1 w_2 \frac{\cos\alpha}{\sin\alpha}, \qquad (83)$$

$$w_2' = -w_1 w_4 \frac{\cos\alpha}{\sin\alpha}, \qquad (84)$$

and the equation

$$\beta_1' = w_1 \frac{\cos\alpha}{\sin\alpha} \qquad (85)$$

can be chosen.

In general, for the complete integrability of the system (73)–(78) it is sufficient to know five independent first integrals. However, after the partition of the system on three parts (the system (80)–(82), the system (83), (84) and the equation (85)) for the complete integrability it is sufficient to know two independent first integrals of the system (80)–(82), one — of the system (83), (84) (after the reduction of the latter system to the independent subsystem) and one more first integral which "joining" the equation (85).

Immediately we shall notice that the latter discourses are typical for the choice of the system of discourses **I** (see above). Really, while we "do not notice" the existence of two analytical first integrals (31), (32). Therefore, when we get two independent first integrals of the independent third order system (80)–(82), and also the first integral "joining" the equation (85), we shall have the complete tuple of the independent first integrals of the fourth order system (80)–(82), (85). The obtained assigned complete tuple (three integrals) and together with the analytical first integrals (31), (32) forms the complete tuple of five first integrals of the sixth order system (80)–(85).

Hereafter, in particular, it will is seen that the composition of the analytical first integrals (31), (32) gives the first integral of the (potentially separated) system (83), (84).

B. Complete list of invariant relations

At the beginning we compare the third order system (80)–(82) to the nonautonomous second order system

$$\begin{aligned} \frac{dw_3}{d\alpha} &= \frac{\cos(\beta_2+\gamma)\sin\alpha\cos\alpha - w_1^2\cos\alpha/\sin\alpha}{-w_3 + b\sin\alpha}, \\ \frac{dw_1}{d\alpha} &= \frac{w_1 w_3 \cos\alpha/\sin\alpha}{-w_3 + b\sin\alpha}. \end{aligned} \qquad (86)$$

Let rewrite the system (86) on algebraic form using the substitution $\tau = \sin\alpha$

$$\begin{aligned} \frac{dw_3}{d\tau} &= \frac{\cos(\beta_2+\gamma)\tau - w_1^2/\tau}{-w_3 + b\tau}, \\ \frac{dw_1}{d\tau} &= \frac{w_1 w_3/\tau}{-w_3 + b\tau}. \end{aligned} \qquad (87)$$

Later on, if we introduce the uniform variables by the formulas

$$w_1 = u_1\tau,\ w_3 = u_2\tau, \qquad (88)$$

we shall reduce the system (87) to the following form:

$$\begin{aligned} \tau\frac{du_2}{d\tau} + u_2 &= \frac{\cos(\beta_2+\gamma) - u_1^2}{-u_2 + b}, \\ \tau\frac{du_1}{d\tau} + u_1 &= \frac{u_1 u_2}{-u_2 + b}, \end{aligned} \qquad (89)$$

that is equivalent to

$$\begin{aligned} \tau\frac{du_2}{d\tau} &= \frac{\cos(\beta_2+\gamma) - u_1^2 + u_2^2 - bu_2}{-u_2 + b}, \\ \tau\frac{du_1}{d\tau} &= \frac{2u_1 u_2 - bu_1}{-u_2 + b}. \end{aligned} \qquad (90)$$

Let compare the second order system (90) to the nonautonomous first order equation

$$\frac{du_2}{du_1} = \frac{\cos(\beta_2+\gamma) - u_1^2 + u_2^2 - bu_2}{2u_1 u_2 - bu_1}, \qquad (91)$$

which is reduced uncomplicated to the complete differential:

$$d\left(\frac{u_2^2 + u_1^2 - bu_2 + \cos(\beta_2+\gamma)}{u_1}\right) = 0. \qquad (92)$$

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And so, the equation (91) has the following first integral:

$$\frac{u_2^2 + u_1^2 - bu_2 + \cos(\beta_2 + \gamma)}{u_1} = C_1 = \text{const}, \tag{93}$$

which in former variables is looked like

$$\frac{w_3^2 + w_1^2 - bw_3 \sin\alpha + \cos(\beta_2 + \gamma)\sin^2\alpha}{w_1 \sin\alpha} = C_1 = \text{const}. \tag{94}$$

Remark 8.1. Let consider the system (80)–(82) with zero mean variable dissipation which becomes the conservative for $b = 0$:

$$\begin{aligned} \alpha' &= -w_3, \\ w_3' &= \cos(\beta_2 + \gamma)\sin\alpha\cos\alpha - w_1^2 \frac{\cos\alpha}{\sin\alpha}, \\ w_1' &= w_1 w_3 \frac{\cos\alpha}{\sin\alpha}. \end{aligned} \tag{95}$$

It has two the analytical first integrals of the forms

$$w_3^2 + w_1^2 + \cos(\beta_2 + \gamma)\sin^2\alpha = C_1^* = \text{const}, \tag{96}$$

$$w_1 \sin\alpha = C_2^* = \text{const}. \tag{97}$$

It is obviously that the ratio of two the first integrals (96), (97) is also the first integral of the system (95). But for $b \neq 0$ each of functions

$$w_3^2 + w_1^2 - bw_3 \sin\alpha + \cos(\beta_2 + \gamma)\sin^2\alpha \tag{98}$$

and (97) are not the first integrals of the system (80)–(82) separately. However, the ratio of the functions (98), (97) is the first integral of the system (80)–(82) for any b.

Later on, let find the evident form of the additional first integral of the third order system (80)–(82). At the beginning for this we shall transform the invariant relation (93) for $u_1 \neq 0$ as follows:

$$\left(u_2 - \frac{b}{2}\right)^2 + \left(u_1 - \frac{C_1}{2}\right)^2 = \frac{b^2 + C_1^2}{4} - \cos(\beta_2 + \gamma). \tag{99}$$

As is seen, the parameters of given invariant relation should satisfy the condition

$$b^2 + C_1^2 - 4\cos(\beta_2 + \gamma) \geq 0, \tag{100}$$

and the phase space of the system (80)–(82) is stratified on the family of the surfaces which is assigned by the equality (99).

Thus, by virtue of the relation (93) the first equation of the system (90) has the form

$$\tau \frac{du_2}{d\tau} = \frac{2(\cos(\beta_2 + \gamma) - bu_2 + u_2^2) - C_1 U_1(C_1, u_2)}{-u_2 + b}, \tag{101}$$

where

$$U_1(C_1, u_2) = \frac{1}{2}\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + \cos(\beta_2 + \gamma))}\}, \tag{102}$$

herewith, the constant of the integration C_1 is chosen from the condition (100).

Therefore, the quadrature for the search of the additional first integral of the system (80)–(82) has the form ($q_1 = \cos(\beta_2 + \gamma) - bu_2 + u_2^2$)

$$\int \frac{d\tau}{\tau} = \int \frac{(b - u_2)du_2}{2q_1 - C_1\{C_1 \pm \sqrt{C_1^2 - 4q_1}\}/2}. \tag{103}$$

The left-hand side (accurate to the additive constant), obviously, is equal to

$$\ln|\sin\alpha|. \tag{104}$$

If

$$u_2 - \frac{b}{2} = p_1, \; b_1^2 = b^2 + C_1^2 - 4\cos(\beta_2 + \gamma), \tag{105}$$

then the right-hand side of the equality (103) has the form

$$\begin{aligned} &-\frac{1}{4}\int \frac{d(b_1^2 - 4p_1^2)}{(b_1^2 - 4p_1^2) \pm C_1\sqrt{b_1^2 - 4p_1^2}} - \\ &-b\int \frac{dp_1}{(b_1^2 - 4p_1^2) \pm C_1\sqrt{b_1^2 - 4p_1^2}} = \\ &= -\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2 - 4p_1^2}}{C_1} \pm 1\right| \pm \frac{b}{2} I_1, \end{aligned} \tag{106}$$

where

$$I_1 = \int \frac{dp_3}{\sqrt{b_1^2 - p_3^2}(p_3 \pm C_1)}, \; p_3 = \sqrt{b_1^2 - 4p_1^2}. \tag{107}$$

Three cases are possible for the calculation of the integral (107).

I. $b > 2$.

$$I_1 = -\frac{1}{2\sqrt{b^2 - 4}}\ln\left|\frac{\sqrt{b^2 - 4} + \sqrt{b_1^2 - p_3^2}}{p_3 \pm C_1} \pm \frac{C_1}{\sqrt{b^2 - 4}}\right| + \\ + \frac{1}{2\sqrt{b^2 - 4}}\ln\left|\frac{\sqrt{b^2 - 4} - \sqrt{b_1^2 - p_3^2}}{p_3 \pm C_1} \mp \frac{C_1}{\sqrt{b^2 - 4}}\right| + \text{const}. \tag{108}$$

II. $b < 2$.

$$I_1 = \frac{1}{\sqrt{4 - b^2}} \arcsin \frac{\pm C_1 p_3 + b_1^2}{b_1(p_3 \pm C_1)} + \text{const}. \tag{109}$$

III. $b = 2$.

$$I_1 = \mp \frac{\sqrt{b_1^2 - p_3^2}}{C_1(p_3 \pm C_1)} + \text{const}. \tag{110}$$

When we return to the variable

$$p_1 = \frac{w_3}{\sin\alpha} - \frac{b}{2}, \tag{111}$$

we shall have the final form for the value I_1:

I. $b > 2$.

$$I_1 = -\frac{1}{2\sqrt{b^2 - 4}}\ln\left|\frac{\sqrt{b^2 - 4} \pm 2p_1}{\sqrt{b_1^2 - 4p_1^2} \pm C_1} \pm \frac{C_1}{\sqrt{b^2 - 4}}\right| + \\ + \frac{1}{2\sqrt{b^2 - 4}}\ln\left|\frac{\sqrt{b^2 - 4} \mp 2p_1}{\sqrt{b_1^2 - 4p_1^2} \pm C_1} \mp \frac{C_1}{\sqrt{b^2 - 4}}\right| + \text{const}. \tag{112}$$

II. $b < 2$.

$$I_1 = \frac{1}{\sqrt{4 - b^2}} \arcsin \frac{\pm C_1\sqrt{b_1^2 - 4p_1^2} + b_1^2}{b_1(\sqrt{b_1^2 - 4p_1^2} \pm C_1)} + \text{const}. \tag{113}$$

III. $b = 2$.

$$I_1 = \mp \frac{2p_1}{C_1(\sqrt{b_1^2 - 4p_1^2} \pm C_1)} + \text{const}. \tag{114}$$

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So, the additional first integral was found right before for the third order system (80)–(82) i.e. it was presented the complete tuple of the first integrals which are the transcendental functions of its own phase variables.

Remark 8.2. It is necessary to substitute formally the left-hand side of the first integral (93) instead of C_1 in the expression of the found first integral.

Then the obtained additional first integral has the following structural form (which is similar to the transcendental first integral from the planeparallel dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha, \frac{w_3}{\sin\alpha}, \frac{w_1}{\sin\alpha}\right) = C_2 = \text{const}. \quad (115)$$

Thus, there are already found two the independent first integrals for the integration of the sixth order system (80)–(85). And now, under the acceptance of the discourses type system **I** (see above, when as we were ”do not notice” the existence of two analytical first integrals (31), (32)), and for its complete integrability it is sufficient to find one first integral for (separated potentially) system (83), (84), and also the additional first integral which ”connects” the equation (85).

After the change of the variables

$$\begin{aligned} w_* &= w_3\sin(\gamma+\beta_2) + w_4\cos(\gamma+\beta_2),\\ w_{**} &= w_1\sin(\gamma+\beta_2) - w_2\cos(\gamma+\beta_2) \end{aligned} \quad (116)$$

the system (83), (84) can be reduced to the form

$$\begin{aligned} \frac{dw_*}{d\beta_1} &= -w_{**},\\ \frac{dw_{**}}{d\beta_1} &= w_*, \end{aligned} \quad (117)$$

which expects the existence of the analytical first integral:

$$w_*^2 + w_{**}^2 = C_3 = \text{const}. \quad (118)$$

Let ask the question: how is related the obtained right now first integral (118) with the analytical first integrals of the forms (31), (32)?

Really, two discourses types (**I** and **II**, see above) correspond to two following alternatives. For the complete integration of the sixth order system (25)–(30):

1) Either we find five the independent first integrals of the sixth order system (25)–(30);

2) Or we transform the sixth order system (25)–(30) such as there are stand out the independent subsystems else more low order.

So, for instance, since after observation of such coordinates as w_*, w_{**} the stratification of the system vector field is occur such as the independent second order subsystem is formed (117), it needs to find four the independent first integrals instead of five ones (three — for the integration of the fourth order system (80)–(82), (85) and one — for the integration of the separated second order system (117)).

And now, finally, let rewrite the forms of the analytical first integrals (31), (32) in new variables as follows:

$$w_{**}\cos\beta_1 - w_*\sin\beta_1 = W_1'' = \text{const}, \quad (119)$$

$$w_{**}\sin\beta_1 + w_*\cos\beta_1 = W_2'' = \text{const}. \quad (120)$$

Obviously, that the analytical first integrals (119), (120) involve the founded analytical first integral (118) (it is sufficient for this to add the squares of the left-hand side of the equalities (119), (120)).

Later on, finally, for the integration of the fourth order system (80)–(82), (85) two independent the first integrals have already founded. And for the complete its integrability it is sufficient to find one more (additional) the first integral which ”joining” the equation (85).

Since

$$\frac{du_1}{d\tau} = \frac{u_1(2u_2-b)}{(b-u_2)\tau}, \ \frac{d\beta_1}{d\tau} = \frac{u_1}{(b-u_2)\tau}, \quad (121)$$

then

$$\frac{du_1}{d\beta_1} = 2u_2 - b. \quad (122)$$

Obviously, for $u_1 \neq 0$ the equality

$$u_2 = \frac{1}{2}\left(b \pm \sqrt{b_1^2 - 4\left(u_1 - \frac{C_1}{2}\right)^2}\right),$$

$$b_1^2 = b^2 + C_1^2 - 4\cos(\beta_2+\gamma), \quad (123)$$

is fulfilled, then the integration of the following quadrature:

$$\beta_1 + \text{const} = \pm\int \frac{du_1}{\sqrt{b_1^2 - 4\left(u_1 - \frac{C_1}{2}\right)^2}} \quad (124)$$

will bring to the invariant relation

$$2(\beta_1 + C_4) = \pm\arcsin\frac{2u_1 - C_1}{\sqrt{b^2 + C_1^2 - 4\cos(\beta_2+\gamma)}}, \quad (125)$$

$$C_4 = \text{const}.$$

In other words, the equation

$$\sin[2(\beta_1 + C_4)] = \pm\frac{2u_1 - C_1}{\sqrt{b^2 + C_1^2 - 4\cos(\beta_2+\gamma)}} \quad (126)$$

is fulfilled, or, under the transition to the old variables

$$\sin[2(\beta_1+C_4)] = \pm\frac{2w_1 - C_1\sin\alpha}{\sqrt{b^2 + C_1^2 - 4\cos(\beta_2+\gamma)}\sin\alpha}. \quad (127)$$

In principle, it makes possible to stop on the latter equality to achieve the additional invariant relation ”connecting” the equation (85), if we add to this equality that it is necessary to substitute formally the left-hand side of the first integral (93) instead of C_1 in the latter expression.

But we shall make the certain transformations which reduce to the obtaining of the following evident form of the additional first integral (herewith, the equality (93) is used):

$$\text{tg}^2[2(\beta_1 + C_4)] = \frac{(u_1^2 - u_2^2 + bu_2 - \cos(\beta_2+\gamma))^2}{u_1^2(4u_2^2 - 4bu_2 + b^2)}. \quad (128)$$

Returning to the old coordinates, we shall obtain the additional invariant relation as the form

$$\text{tg}^2[2(\beta_1 + C_4)] =$$

$$= \frac{(w_1^2 - w_3^2 + bw_3\sin\alpha - \cos(\beta_2+\gamma)\sin^2\alpha)^2}{w_1^2(4w_3^2 - 4bw_3\sin\alpha + b^2\sin^2\alpha)}, \quad (129)$$

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or finally

$$-\beta_1 \pm \frac{1}{2}\text{arctg}\frac{w_1^2 - w_3^2 + bw_3 \sin\alpha - \cos(\beta_2+\gamma)\sin^2\alpha}{w_1(2w_3 - b\sin\alpha)} = \tag{130}$$
$$= C_4 = \text{const}.$$

And so, the system of dynamic equations (4)–(7), (10)–(15) under the condition (64) has eight invariant relations in considered case: there exist the analytical nonintegrable constraints (20), the cyclic first integrals (18), (19), the first integral (94) and also there exists the first integral expressed by the relations (108)–(115) which is the transcendental function of its phase variables (in sense of complex analysis also) and expresses in terms of finite combination of the elementary functions, and finally the transcendent first integral (130) ((129)) and analytical first integral (118).

Theorem 8.1. *The system (4)–(7), (10)–(15) under the conditions (20), (64), (19) possesses eight invariant relations (the complete tuple), three of which are the transcendental functions from the complex analysis view of point. Herewith, all the relations express in terms of the finite combination of the elementary functions.*

C. Topological analogies

Let consider the following third order system of the equations:

$$\begin{aligned} &\ddot{\xi} + b_*\dot{\xi}\cos\xi + R_3\sin\xi\cos\xi - \dot{\eta_1}^2\frac{\sin\xi}{\cos\xi} = 0, \\ &\ddot{\eta_1} + b_*\dot{\eta_1}\cos\xi + \dot{\xi}\dot{\eta_1}\frac{1+\cos^2\xi}{\cos\xi\sin\xi} = 0,\ b_* > 0, \end{aligned} \tag{131}$$

describing the fixed spherical pendulum which is placed in a flow of the filling medium under the absence of the dependence of the moment of the forces on the angular velocity, i.e. the mechanical system in the nonconservative field of the forces. In general, the order of such system should be equal to 4, but the phase variable η_1 is the cyclic, that reduces to the stratification of the phase space and the deflation.

Its phase space is the tangent stratification

$$T\mathbf{S}^2\{\dot{\xi}, \dot{\eta_1}, \xi, \eta_1\} \tag{132}$$

to two-dimensional sphere $\mathbf{S}^2\{\xi, \eta_1\}$, herewith, the equation of the big circles

$$\dot{\eta_1} \equiv 0 \tag{133}$$

assigns the family of the integral manifolds.

It is not difficult to make sure that the system (131) is equivalent to the dynamic system with the zero mean variable dissipation on the tangent stratification (132) to two-dimensional sphere. Moreover, the following theorem is equitable.

Theorem 8.2. *The system (4)–(7), (10)–(15) under the conditions (20), (138), (19) is equivalent to the dynamic system (131).*

Really, it is sufficient to accept $\alpha = \xi,\ \beta_1 = \eta_1,\ b = -b_*,\ R_3 = \cos(\gamma+\beta_2)$.

VI. Case of the dependence of the moment of the nonconservative forces on the angular velocity

A. Introduction on the dependence on the angular velocity

This section is devoted to dynamics of four-dimensional rigid body on the four-dimensional space. But since this section is devoted to the study of the case of the motion under the presence of the dependence of the moment of forces on the angular velocity tensor, we introduce such dependence from more general positions. Additionally, the given point of view helps us to introduce this dependence and for many-dimensional ones.

Let $x = (x_{1N}, x_{2N}, x_{3N}, x_{4N})$ are the coordinates of the point N of the action of the nonconservative force (of a medium interaction) to two-dimensional disk, $Q = (Q_1, Q_2, Q_3, Q_4)$ are the components not depending on the angular velocity tensor. We shall introduce the dependence of the functions $(x_{1N}, x_{2N}, x_{3N}, x_{4N})$ on the angular velocity tensor by the linear form only since given introduction itself is not obvious a priori.

And so, let accept the following dependence:

$$x = Q + R, \tag{134}$$

where $R = (R_1, R_2, R_3, R_4)$ is the vector-function containing the components of angular velocity tensor. Herewith, the dependence of the function R on the angular velocity tensor is gyroscopic:

$$R = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{pmatrix} =$$
$$= -\frac{1}{v}\begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}. \tag{135}$$

Here (h_1, h_2, h_3, h_4) are the certain positive parameters.

And now, with the reference to our problem, since $x_{1N} \equiv x_{2N} \equiv 0$, then

$$x_{3N} = Q_3 - \frac{h_1}{v}(\omega_4 - \omega_5),\ x_{4N} = Q_4 - \frac{h_1}{v}(\omega_3 - \omega_2). \tag{136}$$

B. Reduced system

Similarly to the choice of the Chaplygin analytical functions

$$Q_3 = A\sin\alpha\cos\beta_1,\ Q_4 = A\sin\alpha\sin\beta_1,\ A > 0, \tag{137}$$

we shall accept the dynamic functions s, x_{3N} and x_{4N} as the following form:

$$\begin{aligned} &s(\alpha) = B\cos\alpha,\ B > 0,\ h = h_1 > 0,\ v \neq 0,\ h = h_2 > 0, \\ &x_{3N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = A\sin\alpha\cos\beta_1 - \frac{h}{v}(\omega_4 - \omega_5), \\ &x_{4N}\left(\alpha, \beta_1, \beta_2, \frac{\Omega}{v}\right) = A\sin\alpha\sin\beta_1 - \frac{h}{v}(\omega_3 - \omega_2), \end{aligned} \tag{138}$$

which convinces us that the additional dependence of the damping moment of the nonconservative forces (and the dispersing one in some domains of the phase space) is also present in considered system (i.e. the dependence of the moment on the angular velocity tensor is present). Moreover, $h_1 = h_2,\ h_3 = h_4$ by virtue of the dynamical symmetry (17) of the body.

Later on, let accept the system of discourses **I** which takes into account and the system of discourses **II** (see above).

We shall arouse to introduce the following variables in this section:

$$\begin{aligned} u_1 &= \omega_2 - \omega_3, \\ u_2 &= \omega_4 - \omega_5, \\ u_3 &= \omega_2 \cos\beta_2 - \omega_3 \sin\beta_2, \\ u_4 &= \omega_4 \cos\beta_2 - \omega_5 \sin\beta_2. \end{aligned} \tag{139}$$

Really, the assigned coordinates are defined correctly for

$$\cos\beta_2 \neq \sin\beta_2, \tag{140}$$

and Jacobian of the mapping is equal to

$$-\frac{1}{(\cos\beta_2 - \sin\beta_2)^2}, \tag{141}$$

herewith, the inverse transformation is assigned as follows:

$$\begin{aligned} \omega_2 &= \frac{u_3 - u_1 \sin\beta_2}{\cos\beta_2 - \sin\beta_2}, \\ \omega_3 &= \frac{u_3 - u_1 \cos\beta_2}{\cos\beta_2 - \sin\beta_2}, \\ \omega_4 &= \frac{u_4 - u_2 \sin\beta_2}{\cos\beta_2 - \sin\beta_2}, \\ \omega_5 &= \frac{u_4 - u_2 \cos\beta_2}{\cos\beta_2 - \sin\beta_2}, \end{aligned} \tag{142}$$

and the particular case

$$\cos\beta_2 = \sin\beta_2, \tag{143}$$

which simplifies the dynamic equations can be considered separately.

Then the equations (33)–(38) under the condition (138) outside of and only outside of the manifold

$$O_3 = \left\{(\alpha, \beta_1, \omega_2, \omega_3, \omega_4, \omega_5) \in \mathbf{R}^6 :\ \alpha = \frac{\pi}{2} + \pi k,\ k \in \mathbf{Z}\right\} \tag{144}$$

transform to the following equations:

$$\dot\alpha - u_3 \sin\beta_1 + u_4 \cos\beta_1 - \\ -\sigma n_0^2 v \sin\alpha + \sigma H_1'[-u_1 \sin\beta_1 + u_2 \cos\beta_1] = 0, \tag{145}$$

$$\dot\beta_1 \sin\alpha - \cos\alpha[u_3 \cos\beta_1 + u_4 \sin\beta_1] - \\ -\sigma H_1' \cos\alpha[u_1 \cos\beta_1 + u_2 \sin\beta_1] = 0, \tag{146}$$

$$\dot u_1 = -n_0^2 v^2 r_1 \sin\alpha \cos\alpha \sin\beta_1 - \frac{Bvh}{I_1 + I_3} r_1 u_1 \cos\alpha, \tag{147}$$

$$\dot u_2 = n_0^2 v^2 r_1 \sin\alpha \cos\alpha \cos\beta_1 - \frac{Bvh}{I_1 + I_3} r_1 u_2 \cos\alpha, \tag{148}$$

$$\dot u_3 = -n_0^2 v^2 \sin\alpha \cos\alpha \sin\beta_1 \cos(\gamma + \beta_2) - \\ -\frac{Bvh}{I_1 + I_3} u_1 \cos\alpha \cos(\gamma + \beta_2), \tag{149}$$

$$\dot u_4 = n_0^2 v^2 \sin\alpha \cos\alpha \cos\beta_1 \cos(\gamma + \beta_2) - \\ -\frac{Bvh}{I_1 + I_3} u_2 \cos\alpha \cos(\gamma + \beta_2), \tag{150}$$

where

$$r_1 = \cos\gamma - \sin\gamma \neq 0,\ n_0^2 = \frac{AB}{I_1 + I_3},\ H_1' = \frac{Bh}{I_1 + I_3}. \tag{151}$$

We note that the particular case

$$\cos\gamma = \sin\gamma\ (r_1 = 0), \tag{152}$$

which simplifies the dynamic equations can also be considered separately (similarly the case (143)).

Let introduce the following phase variables by the formulas:

$$\begin{aligned} v_1 &= -u_1 \sin\beta_1 + u_2 \cos\beta_1, \\ v_2 &= u_1 \cos\beta_1 + u_2 \sin\beta_1, \\ v_3 &= -u_3 \sin\beta_1 + u_4 \cos\beta_1, \\ v_4 &= u_3 \cos\beta_1 + u_4 \sin\beta_1. \end{aligned} \tag{153}$$

then outside of and only outside of the manifold

$$O_4 = \left\{(\alpha, \beta_1, u_1, u_2, u_3, u_4) \in \mathbf{R}^6 :\ \beta_1 = \pi k,\ k \in \mathbf{Z}\right\} \tag{154}$$

the system (145)–(150) has the form

$$\dot\alpha = -v_3 - bH_1 v_1 + b \sin\alpha, \tag{155}$$

$$\dot\beta_1 = [v_4 + bH_1 v_2] \frac{\cos\alpha}{\sin\alpha}, \tag{156}$$

$$\dot v_1 = n_0^2 v^2 r_1 \sin\alpha \cos\alpha - \\ -H_1' v r_1 v_1 \cos\alpha - v_2 \cdot [v_4 + bH_1 v_2] \frac{\cos\alpha}{\sin\alpha}, \tag{157}$$

$$\dot v_2 = -H_1' v r_1 v_2 \cos\alpha + v_1 \cdot [v_4 + bH_1 v_2] \frac{\cos\alpha}{\sin\alpha}, \tag{158}$$

$$\dot v_3 = n_0^2 v^2 \sin\alpha \cos\alpha \cos(\gamma + \beta_2) - \\ -H_1' v v_1 \cos\alpha \cos(\gamma + \beta_2) - v_4 \cdot [v_4 + bH_1 v_2] \frac{\cos\alpha}{\sin\alpha}, \tag{159}$$

$$\dot v_4 = -H_1' v v_2 \cos\alpha \cos(\gamma + \beta_2) + \\ +v_3 \cdot [v_4 + bH_1 v_2] \frac{\cos\alpha}{\sin\alpha}, \tag{160}$$

where we introduce as before the dimensionless parameters as follows:

$$n_0^2 = \frac{AB}{I_1 + I_3},\ b = \sigma n_0,\ [b] = 1, \\ H_1 = \frac{H_1'}{n_0} = \frac{Bh}{(I_1 + I_3) n_0},\ [H_1] = 1. \tag{161}$$

Let also introduce one more auxiliary change of the part of the phase variables, as follows:

$$s_1 = v_3 + bH_1 v_1,\ s_2 = v_4 + bH_1 v_2. \tag{162}$$

Then the investigated system (155)–(160) after the introduction of dimensionless variables and differentiability

$$v_k \mapsto n_0 v v_k,\ k = 1, \ldots, 4,\ \ <\cdot> = n_0 v <'>, \tag{163}$$

will rewrite as the form

$$\alpha' = -s_1 + b \sin\alpha, \tag{164}$$

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$$\beta_1' = s_2 \frac{\cos\alpha}{\sin\alpha}, \tag{165}$$

$$s_1' = R_1 \sin\alpha\cos\alpha - s_2^2 \frac{\cos\alpha}{\sin\alpha} - R_1 H_1 v_1 \cos\alpha, \tag{166}$$

$$s_2' = s_1 s_2 \frac{\cos\alpha}{\sin\alpha} - R_1 H_1 v_2 \cos\alpha, \tag{167}$$

$$v_1' = R_2 \sin\alpha\cos\alpha - s_2 v_2 \frac{\cos\alpha}{\sin\alpha} - H_1 R_2 v_1 \cos\alpha, \tag{168}$$

$$v_2' = s_2 v_1 \frac{\cos\alpha}{\sin\alpha} - H_1 R_2 v_2 \cos\alpha, \tag{169}$$

where

$$R_1 = bH_1(\cos\gamma - \sin\gamma) + \cos(\gamma + \beta_2),$$
$$R_2 = r_1 = \cos\gamma - \sin\gamma. \tag{170}$$

Obviously, that for $H_1 = 0$ formally the independent fourth order subsystem (164)–(167) stands out in the system (164)–(169) on the tangent stratification $T\mathbf{S}^2$ to two-dimensional sphere $\mathbf{S}^2\{0 < \alpha < \pi,\ 0 \leq \beta_1 < 2\pi\}$, in which, in turn, it can be stand out the independent third order subsystem (164), (166), (167) on its own three-dimensional phase manifold.

But, in principle, it is just understood, since for $H_1 = 0$ we are under the conditions of absence of moment of the forces on the angular velocity tensor (see previous section and the system (80)–(82), (85)). The latter fact allows to integrate completely similarly the considered fourth order system (164)–(167), but signifies, and the considered sixth order system (164)–(169), since there exist two independent analytical first integrals (31), (32) or (119), (120) (see above on two systems of discourses **I** and **II**).

And in the given case it is great for us that $H_1 \neq 0$. Therefore, we transform the having analytical first integrals (31), (32) or (119), (120). We have the evident type of its in the different variables:

$$\frac{u_3 - u_1 \sin\beta_2}{\cos\beta - 2 - \sin\beta_2}\sin\gamma - \frac{u_3 - u_1\cos\beta_2}{\cos\beta - 2 - \sin\beta_2}\cos\gamma =$$
$$= W_1' = \text{const}, \tag{171}$$

$$\frac{u_4 - u_2 \sin\beta_2}{\cos\beta - 2 - \sin\beta_2}\sin\gamma - \frac{u_4 - u_2\cos\beta_2}{\cos\beta - 2 - \sin\beta_2}\cos\gamma =$$
$$= W_2' = \text{const}. \tag{172}$$

If we consider the case (20) (i.e., in particular, when the value β_2 is the identical constant along the phase trajectories), then the following analytical functions are constant on the phase trajectories of the considered system:

$$u_3(\sin\gamma - \cos\gamma) + u_1\cos(\gamma + \beta_2) = W_1^0 = \text{const}, \tag{173}$$

$$u_4(\sin\gamma - \cos\gamma) + u_2\cos(\gamma + \beta_2) = W_2^0 = \text{const}. \tag{174}$$

In another variables the latter two invariant relations have the forms

$$(v_2\cos\beta_1 - v_1\sin\beta_1)\cos(\gamma + \beta_2) +$$
$$+ (v_4\cos\beta_1 - v_3\sin\beta_1)(\sin\gamma - \cos\gamma) = W_1^0 = \text{const}, \tag{175}$$

$$(v_2\sin\beta_1 + v_1\cos\beta_1)\cos(\gamma + \beta_2) +$$
$$+ (v_4\sin\beta_1 + v_3\cos\beta_1)(\sin\gamma - \cos\gamma) = W_2^0 = \text{const}, \tag{176}$$

or

$$R_1 v_2 \cos\beta_1 - R_1 v_1 \sin\beta_1 +$$
$$+ R_2[s_1\sin\beta_1 - s_2\cos\beta_1] = W_1^0 = \text{const}, \tag{177}$$

$$R_1 v_2 \sin\beta_1 + R_1 v_1 \cos\beta_1 -$$
$$- R_2[s_1\cos\beta_1 + s_2\sin\beta_1] = W_2^0 = \text{const}, \tag{178}$$

where

$$R_1 = \cos(\gamma + \beta_2) + bH_1(\cos\gamma - \sin\gamma),$$
$$R_2 = \cos\gamma - \sin\gamma \tag{179}$$

as before.

Later on, let express from the relations (177), (178) the values v_1, v_2. We have:

$$v_2 R_1 = R_2 s_2 + \psi_1(\beta_1, W_1^0, W_2^0), \tag{180}$$

$$v_1 R_1 = R_2 s_1 + \psi_2(\beta_1, W_1^0, W_2^0), \tag{181}$$

where

$$\begin{aligned} \psi_1(\beta_1, W_1^0, W_2^0) &= W_1^0\cos\beta_1 + W_2^0\sin\beta_1, \\ \psi_2(\beta_1, W_1^0, W_2^0) &= W_2^0\cos\beta_1 - W_1^0\sin\beta_1. \end{aligned} \tag{182}$$

Then the system (164)–(167) has the form of the independent fourth order system:

$$\alpha' = -s_1 + b\sin\alpha, \tag{183}$$

$$s_1' = R_1\sin\alpha\cos\alpha - s_2^2\frac{\cos\alpha}{\sin\alpha} -$$
$$- R_2 H_1 s_1 \cos\alpha - H_1\psi_2(\beta_1, W_1^0, W_2^0)\cos\alpha, \tag{184}$$

$$s_2' = s_1 s_2 \frac{\cos\alpha}{\sin\alpha} -$$
$$- R_2 H_1 s_2 \cos\alpha - H_1\psi_1(\beta_1, W_1^0, W_2^0)\cos\alpha, \tag{185}$$

$$\beta_1' = s_2\frac{\cos\alpha}{\sin\alpha}. \tag{186}$$

The system (183)–(186) can be considered as the system (164)–(167) which is reduced to the levels (W_1^0, W_2^0) of the analytical first integrals (177), (178).

Obviously, that

$$\psi_1(\beta_1, 0, 0) \equiv \psi_2(\beta_1, 0, 0) \equiv 0. \tag{187}$$

Therefore, we shall consider the system (183)–(186) on the zero levels of the analytical first integrals (177), (178):

$$W_1^0 = W_2^0 = 0, \tag{188}$$

which has the form

$$\alpha' = -s_1 + b\sin\alpha, \tag{189}$$

$$s_1' = R_1\sin\alpha\cos\alpha - s_2^2\frac{\cos\alpha}{\sin\alpha} - R_2 H_1 s_1\cos\alpha, \tag{190}$$

$$s_2' = s_1 s_2\frac{\cos\alpha}{\sin\alpha} - R_2 H_1 s_2 \cos\alpha, \tag{191}$$

$$\beta_1' = s_2\frac{\cos\alpha}{\sin\alpha}. \tag{192}$$

The given system can be considered on the tangent stratification $T\mathbf{S}^2$ to two-dimensional sphere $\mathbf{S}^2\{0 < \alpha < \pi,\ 0 \leq \beta_1 < 2\pi\}$, in which, in turn, it can be stand out the independent third

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order subsystem (189)–(191) on its own three-dimensional phase manifold.

And so, for the integration of the sixth order system at the beginning we used the system of discourses **I** (see above), when we did not yet take into account the existence of two independent analytical first integrals of the forms (31), (32). In consequence we have limited (reduced) the considered sixth order system on the levels (in consequence zero) of the assigned first integrals, i.e. the system of discourses **II** was used (see above).

C. Complete list of invariant relations

At the beginning we compare the third order system (189)–(191) to the nonautonomous second order system

$$\frac{ds_1}{d\alpha} = \frac{R_1\sin\alpha\cos\alpha - s_2^2\cos\alpha/\sin\alpha - R_2H_1s_1\cos\alpha}{-s_1+b\sin\alpha},$$
$$\frac{ds_2}{d\alpha} = \frac{s_1s_2\cos\alpha/\sin\alpha - R_2H_1s_2\cos\alpha}{-s_1+b\sin\alpha}. \tag{193}$$

Let rewrite the system (193) on algebraic form using the substitution $\tau = \sin\alpha$

$$\frac{ds_1}{d\tau} = \frac{R_1\tau - s_2^2/\tau - R_2H_1s_1}{-s_1+b\tau},\quad \frac{ds_2}{d\tau} = \frac{s_1s_2/\tau - R_2H_1s_2}{-s_1+b\tau}. \tag{194}$$

Later on, if we introduce the uniform variables by the formulas

$$s_1 = t_1\tau,\ \ s_2 = t_2\tau, \tag{195}$$

we shall reduce the system (194) to the following form:

$$\tau\frac{dt_1}{d\tau} + t_1 = \frac{R_1 - t_2^2 - R_2H_1t_1}{-t_1+b},\quad \tau\frac{dt_2}{d\tau} + t_2 = \frac{t_1t_2 - R_2H_1t_2}{-t_1+b}, \tag{196}$$

that is equivalent to

$$\tau\frac{dt_1}{d\tau} = \frac{t_1^2 - t_2^2 - (b+R_2H_1)t_1 + R_1}{-t_1+b},\quad \tau\frac{dt_2}{d\tau} = \frac{2t_1t_2 - (b+R_2H_1)t_2}{-t_1+b}. \tag{197}$$

Let compare the second order system (197) to the nonautonomous first order equation

$$\frac{dt_1}{dt_2} = \frac{t_1^2 - t_2^2 - (b+R_2H_1)t_1 + R_1}{2t_1t_2 - (b+R_2H_1)t_2}, \tag{198}$$

which is reduced uncomplicated to the complete differential:

$$d\left(\frac{t_1^2 + t_2^2 - (b+R_2H_1)t_1 + R_1}{t_2}\right) = 0. \tag{199}$$

And so, the equation (198) has the following first integral:

$$\frac{t_1^2 + t_2^2 - (b+R_2H_1)t_1 + R_1}{t_2} = C_1 = \text{const}, \tag{200}$$

which in former variables is looked like

$$\frac{s_1^2 + s_2^2 - (b+R_2H_1)s_1\sin\alpha + R_1\sin^2\alpha}{s_2\sin\alpha} = C_1 = \text{const}. \tag{201}$$

Remark 8.3. Let consider the system (189)–(191) with zero mean variable dissipation which becomes the conservative for $b = R_2H_1$:

$$\alpha' = -s_1 + b\sin\alpha,$$
$$s_1' = R_1\sin\alpha\cos\alpha - s_2^2\frac{\cos\alpha}{\sin\alpha} - bs_1\cos\alpha, \tag{202}$$
$$s_2' = s_1s_2\frac{\cos\alpha}{\sin\alpha} - bs_2\cos\alpha.$$

It has two the analytical first integrals of the forms

$$s_1^2 + s_2^2 - 2bs_1\sin\alpha + R_1\sin^2\alpha = C_1^* = \text{const}, \tag{203}$$

$$s_2\sin\alpha = C_2^* = \text{const}. \tag{204}$$

It is obviously that the ratio of two the first integrals (203), (204) is also the first integral of the system (202). But for $b \neq R_2H_1$ each of functions

$$s_1^2 + s_2^2 - (b+R_2H_1)s_1\sin\alpha + R_1\sin^2\alpha \tag{205}$$

and (204) are not the first integrals of the system (189)–(191) separately. However, the ratio of the functions (205), (204) is the first integral of the system (189)–(191) for any b, R_2H_1.

Later on, let find the evident form of the additional first integral of the third order system (189)–(191). At the beginning for this we shall transform the invariant relation (200) for $u_1 \neq 0$ as follows:

$$\left(t_1 - \frac{b+R_2H_1}{2}\right)^2 + \left(t_2 - \frac{C_1}{2}\right)^2 = \frac{(b+R_2H_1)^2 + C_1^2 - 4R_1}{4}. \tag{206}$$

As is seen, the parameters of given invariant relation should satisfy the condition

$$(b+R_2H_1)^2 + C_1^2 - 4R_1 \geq 0, \tag{207}$$

and the phase space of the system (189)–(191) is stratified on the family of the surfaces which is assigned by the equality (206).

Thus, by virtue of the relation (200) the first equation of the system (197) has the form

$$\tau\frac{dt_1}{d\tau} = \frac{2t_1^2 - 2(b+R_2H_1)t_1 + 2R_1 - C_1U_1(C_1,t_1)}{b-t_1}, \tag{208}$$

where

$$U_1(C_1,t_1) = \frac{1}{2}\{C_1 \pm U_2(C_1,t_1)\}, \tag{209}$$

$$U_2(C_1,t_1) = \sqrt{C_1^2 - 4(R_1 - (b+R_2H_1)t_1 + t_1^2)},$$

herewith, the constant of the integration C_1 is chosen from the condition (207).

Therefore, the quadrature for the search of the additional first integral of the system (189)–(191) has the form

$$\int\frac{d\tau}{\tau} = \int\frac{(b-t_1)dt_1}{2(R_1 - (b+R_2H_1)t_1 + t_1^2) - C_1\{C_1 \pm U_2(C_1,t_1)\}/2}. \tag{210}$$

The left-hand side (accurate to the additive constant), obviously, is equal to

$$\ln|\sin\alpha|. \tag{211}$$

If

$$t_1 - \frac{b+R_2H_1}{2} = w_1,\ b_1^2 = (b+R_2H_1)^2 + C_1^2 - 4R_1, \tag{212}$$

then the right-hand side of the equality (210) has the form

$$-\frac{1}{4}\int \frac{d(b_1^2-4w_1^2)}{(b_1^2-4w_1^2) \pm C_1\sqrt{b_1^2-4w_1^2}} -$$

$$-(b+R_2H_1)\int \frac{dw_1}{(b_1^2-4w_1^2) \pm C_1\sqrt{b_1^2-4w_1^2}} =$$

$$= -\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2-4w_1^2}}{C_1} \pm 1\right| \pm \frac{b+R_2H_1}{2} I_1, \tag{213}$$

where

$$I_1 = \int \frac{dw_3}{\sqrt{b_1^2-w_3^2}(w_3 \pm C_1)},\ w_3 = \sqrt{b_1^2-4w_1^2}. \tag{214}$$

Three cases are possible for the calculation of the integral (214).

I. $(b+R_2H_1)^2 - 4R_1 > 0$.

$$I_1 = -\frac{1}{2W_1}\times$$

$$\times \ln\left|\frac{W_1+\sqrt{b_1^2-w_3^2}}{w_3 \pm C_1} \pm \frac{C_1}{W_1}\right| +$$

$$+\frac{1}{2W_1}\ln\left|\frac{W_1-\sqrt{b_1^2-w_3^2}}{w_3 \pm C_1} \mp \frac{C_1}{W_1}\right| + \tag{215}$$

$$+\text{const},$$

$$W_1 = \sqrt{(b+R_2H_1)^2-4R_1}.$$

II. $(b+R_2H_1)^2 - 4R_1 < 0$.

$$I_1 = \frac{1}{\sqrt{4R_1-(b+R_2H_1)^2}}\arcsin\frac{\pm C_1w_3+b_1^2}{b_1(w_3 \pm C_1)} + \text{const}. \tag{216}$$

III. $(b+R_2H_1)^2 - 4R_1 = 0$.

$$I_1 = \mp\frac{\sqrt{b_1^2-w_3^2}}{C_1(w_3 \pm C_1)} + \text{const}. \tag{217}$$

When we return to the variable

$$w_1 = \frac{s_1}{\sin\alpha} - \frac{b+R_2H_1}{2}, \tag{218}$$

we shall have the final form for the value I_1:

I. $(b+R_2H_1)^2 - 4R_1 > 0$.

$$I_1 =$$

$$= -\frac{1}{2W_1}\ln\left|\frac{W_1 \pm 2w_1}{\sqrt{b_1^2-4w_1^2} \pm C_1} \pm \frac{C_1}{W_1}\right| +$$

$$+\frac{1}{2W_1}\ln\left|\frac{W_1 \mp 2w_1}{\sqrt{b_1^2-4w_1^2} \pm C_1} \mp \frac{C_1}{W_1}\right| + \tag{219}$$

$$+\text{const}.$$

II. $(b+R_2H_1)^2 - 4R_1 < 0$.

$$I_1 = \frac{1}{\sqrt{4R_1-(b+R_2H_1)^2}}\times$$

$$\times \arcsin\frac{\pm C_1\sqrt{b_1^2-4w_1^2}+b_1^2}{b_1(\sqrt{b_1^2-4w_1^2} \pm C_1)} + \text{const}. \tag{220}$$

III. $(b+R_2H_1)^2 - 4R_1 = 0$.

$$I_1 = \mp\frac{2w_1}{C_1(\sqrt{b_1^2-4w_1^2} \pm C_1)} + \text{const}. \tag{221}$$

So, the additional first integral was found right before for the third order system (189)–(191), i.e. it was presented the complete tuple of the first integrals which are the transcendental functions of its own phase variables.

Remark 8.4. It is necessary to substitute formally the left-hand side of the first integral (200) instead of C_1 in the expression of the found first integral.

Then the obtained additional first integral has the following structural form (which is similar to the transcendental first integral from the planeparallel dynamics):

$$\ln|\sin\alpha| + G_2\left(\sin\alpha, \frac{s_1}{\sin\alpha}, \frac{s_2}{\sin\alpha}\right) = C_2 = \text{const}. \tag{222}$$

Thus, there are already found two the independent first integrals for the integration of the fourth order system (189)–(192). And for the complete its integrability, as specified above, it is sufficient to find the additional first integral which ”connects” the equation (192).

Since

$$\frac{dt_2}{d\tau} = \frac{2t_1t_2-(b+R_2H_1)t_2}{(b-t_1)\tau},\ \frac{d\beta_1}{d\tau} = \frac{t_2}{(b-t_1)\tau}, \tag{223}$$

then

$$\frac{dt_2}{d\beta_1} = 2t_1 - (b+R_2H_1). \tag{224}$$

It is obvious that for $t_2 \neq 0$ the following equality is fulfilled

$$t_1 = \frac{1}{2}\left((b+R_2H_1) \pm \sqrt{b_1^2-(2t_2-C_1)^2}\right), \tag{225}$$

$$b_1^2 = (b+R_2H_1)^2 + C_1^2 - 4R_1,$$

then the integration of the following quadrature:

$$\beta_1 + \text{const} = \pm\int \frac{dt_2}{\sqrt{b_1^2-(2t_2-C_1)^2}} \tag{226}$$

will bring to the invariant relation

$$2(\beta_1+C_3) =$$

$$= \pm\arcsin\frac{2t_1-C_1}{\sqrt{(b+R_2H_1)^2+C_1^2-4R_1}},\ C_3 = \text{const}. \tag{227}$$

In other words, the equality

$$\sin[2(\beta_1+C_3)] = \pm\frac{2t_2-C_1}{\sqrt{(b+R_2H_1)^2+C_1^2-4R_1}} \tag{228}$$

is fulfilled and under the transition to the old variables

$$\sin[2(\beta_1+C_3)] = \pm\frac{2s_2-C_1\sin\alpha}{\sqrt{(b+R_2H_1)^2+C_1^2-4R_1}\sin\alpha}. \tag{229}$$

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In principle, it makes possible to stop on the latter equality to achieve the additional invariant relation "connecting" the equation (192), if we add to this equality that it is necessary to substitute formally the left-hand side of the first integral (200) instead of C_1 in the latter expression.

But we shall make the certain transformations which reduce to the obtaining of the following evident form of the additional first integral (herewith, the equality (200) is used):

$$\mathrm{tg}^2[2(\beta_1 + C_3)] = \\ = \frac{(t_2^2 - t_1^2 + (b + R_2H_1)t_1 - R_1)^2}{t_2^2(2t_1 - (b + R_2H_1))^2}. \qquad (230)$$

Returning to the old coordinates, we shall obtain the additional invariant relation as the form

$$\mathrm{tg}^2[2(\beta_1 + C_3)] = \\ = \frac{(s_2^2 - s_1^2 + (b + R_2H_1)s_1 \sin\alpha - R_1 \sin^2\alpha)^2}{s_2^2(2s_1 - (b + R_2H_1)\sin\alpha)^2}, \qquad (231)$$

or finally

$$-\beta_1 \pm \frac{1}{2}\times \\ \times \mathrm{arctg} \frac{s_2^2 - s_1^2 + (b + R_2H_1)s_1 \sin\alpha - R_1 \sin^2\alpha}{s_2(2s_1 - (b + R_2H_1)\sin\alpha)} = \qquad (232) \\ = C_3 = \mathrm{const}.$$

And so, the system of dynamic equations (4)–(7), (10)–(15) under the condition (138) has nine invariant relations in considered case: there exist the analytical nonintegrable constraints (20), the cyclic first integrals (18), (19), the analytical first integrals (31), (32), the first integral (201) and also there exists the first integral expressed by the relations (215)–(222) which is the transcendental function of its phase variables (in sense of complex analysis also) and expresses in terms of finite combination of the elementary functions, and finally the transcendent first integral (232).

Theorem 8.3. *The system (4)–(7), (10)–(15) under the conditions (20), (138), (19), (188) possesses nine invariant relations (the complete tuple), three of which are the transcendental functions from the complex analysis view of point. Herewith, all the relations express in terms of the finite combination of the elementary functions.*

We also note that in the similar theorem 8.1 of this section the question is on the complete tuple of the first integrals which consisting on eight the first integrals, although there are exist all nine the first integrals. But at proof of theorem 8.1 the system of discourses **I** is used (se above) which implies the introduction of such phase coordinates (in particular, $w_k,\ k = 1, \ldots, 4$), in which the system vector field allows the additional stratifications. Herewith, the analytical first integrals (31), (32) do not use directly, that is admit to dispense by the less quantity of the first integrals.

And at proof of the theorem 8.3 the system of discourses **II** is used (see above) which implies the reduction of investigated system on (zero) levels of the analytical first integrals (31), (32). The latter fact takes into account in principal the complete tuple of the having first integrals.

D. Topological analogies

Let consider the following third order system of the equations:

$$\ddot{\xi} + (b_* - H_1^*)\dot{\xi}\cos\xi + R_3 \sin\xi\cos\xi - \dot{\eta_1}^2 \frac{\sin\xi}{\cos\xi} + \\ + H_1^{**}[W_1^0 \sin\eta_1 - W_2^0 \cos\eta_1] = 0, \\ \ddot{\eta_1} + (b_* - H_1^*)\dot{\eta_1}\cos\xi + \dot{\xi}\dot{\eta_1}\frac{1 + \cos^2\xi}{\cos\xi\sin\xi} + \\ + H_1^{**}[W_1^0 \cos\eta_1 + W_2^0 \sin\eta_1] = 0,\ b_* > 0,\ H_1^{**} > 0, \qquad (233)$$

describing the fixed spherical pendulum which is placed in a flow of the filling medium under the presence of the dependence of the moment of the forces on the angular velocity, i.e. the mechanical system in the nonconservative field of the forces. Unlike previous activities [1], [5], [6], the order of such system is equal to 4 (but not 3) since the phase variable η_1 is not the cyclic, that does not reduce to the stratification of the phase space and the deflation.

Its phase space is the tangent stratification

$$T\mathbf{S}^2\{\dot{\xi}, \dot{\eta_1}, \xi, \eta_1\} \qquad (234)$$

to two-dimensional sphere $\mathbf{S}^2\{\xi, \eta_1\}$, herewith, the equation of the big circles

$$\dot{\eta_1} \equiv 0 \qquad (235)$$

assigns the family of the integral manifolds for $W_1^0 = W_2^0 = 0$ only.

It is not difficult to make sure that the system (233) is equivalent to the dynamic system with the (zero mean) variable dissipation on the tangent stratification (234) to two-dimensional sphere. Moreover, the following theorem is equitable.

Theorem 8.4. *The system (4)–(7), (10)–(15) under the conditions (20), (138), (19) is equivalent to the dynamic system (233).*

Really, it is sufficient to accept $\alpha = \xi,\ \beta_1 = \eta_1,\ b = -b_*,\ H_1 = H_1^{**},\ R_2H_1 = -H_1^*,\ R_1 - bR_2H_1 = R_3$.

VII. Conclusion

In the previous studies of the author, the problems on the motion of the four-dimensional solid were already considered in a nonconservative force field in the presence of the following force. This study opens a new cycle of works on integration of a multidimensional solid in the nonconservative field because previously, as was already specified, we considered only such motions of a solid when the field of external forces was the potential.

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REFERENCES

[1] M. V. Shamolin, *Methods of analysis of dynamical systems with various disssipation in rigid body dynamics*, Moscow, Russian Federation: Ekzamen, 2007.

[2] M. V. Shamolin, *Some questions of the qualitative theory of ordinary differential equations and dynamics of a rigid body interacting with a medium*, Journal of Mathematical Sciences, Vol. 110, No. 2, 2002, p. 2526–2555.

[3] M. V. Shamolin, *Foundations of differential and topological diagnostics*, Journal of Mathematical Sciences, Vol. 114, No. 1, 2003, p. 976–1024.

[4] M. V. Shamolin, *New integrable cases and families of portraits in the plane and spatial dynamics of a rigid body interacting with a medium*, Journal of Mathematical Sciences, Vol. 114, No. 1, 2003, p. 919–975.

[5] M. V. Shamolin, *Classes of variable dissipation systems with nonzero mean in the dynamics of a rigid body*, Journal of Mathematical Sciences, Vol. 122, No. 1, 2004, p. 2841–2915.

[6] M. V. Shamolin, *Structural stable vector fields in rigid body dynamics*, Proc. of 8th Conf. on Dynamical Systems (Theory and Applications) (DSTA 2005), Lodz, Poland, Dec. 12–15, 2005; Tech. Univ. Lodz, 2005, Vol. 1, p. 429–436.

[7] M. V. Shamolin, *The cases of integrability in terms of transcendental functions in dynamics of a rigid body interacting with a medium*, Proc. of 9th Conf. on Dynamical Systems (Theory and Applications) (DSTA 2007), Lodz, Poland, Dec. 17–20, 2007; Tech. Univ. Lodz, 2007, Vol. 1, p. 415–422.

[8] M. V. Shamolin, *Methods of analysis of dynamic systems with various dissipation in dynamics of a rigid body*, ENOC-2008, CD-Proc., June 30–July 4, 2008, Saint Petersburg, Russia, 6 p.

[9] M. V. Shamolin, *Some methods of analysis of the dynami systems with various dissipation in dynamics of a rigid body*, PAMM (Proc. Appl. Math. Mech.), **8**, 10137–10138 (2008) / DOI 10.1002/pamm.200810137.

[10] M. V. Shamolin, *Dynamical systems with variable dissipation: methods and applications*, Proc. of 10th Conf. on Dynamical Systems (Theory and Applications) (DSTA 2009), Lodz, Poland, Dec. 7–10, 2009; Tech. Univ. Lodz, 2009, p. 91–104.

[11] M. V. Shamolin, *New cases of integrability in dynamics of a rigid body with the cone form of its shape interacting with a medium*, PAMM (Proc. Appl. Math. Mech.), **9**, 139–140 (2009) / DOI 10.1002/pamm.200910044.

[12] M. V. Shamolin, *The various cases of complete integrability in dynamics of a rigid body interacting with a medium*, Multibody Dynamics, ECCOMAS Thematic Conf. Warsaw, Poland, 29 June–2 July 2009, CD-Proc.; Polish Acad. Sci., Warsaw, 2009, 20 p.

[13] M. V. Shamolin, *Dynamical systems with various dissipation: background, methods, applications* // CD-Proc. of XXXVIII Summer School-Conf. "Advances Problems in Mechanics" (APM 2010), July 1–5, 2010, St. Petersburg (Repino), Russia; St. Petersburg, IPME, 2010, p. 612–621.

[14] M. V. Shamolin, *Integrability and nonintegrability in terms of transcendental functions in dynamics of a rigid body*, PAMM (Proc. Appl. Math. Mech.), **10**, 63–64 (2010) / DOI 10.1002/pamm.201010024.

[15] M. V. Shamolin, *Cases of complete integrability in transcendental functions in dynamics and certain invariant indices*, CD-Proc. 5th Int. Sci. Conf. on Physics and Control PHYSCON 2011, Leon, Spain, September 5–8, 2011. Leon, Spain, 5 p.

[16] M. V. Shamolin, *Variety of the cases of integrability in dynamics of a 2D-, 3D-, and 4D-rigid body interacting with a medium*, Proc. of 11th Conf. on Dynamical Systems (Theory and Applications) (DSTA 2011), Lodz, Poland, Dec. 5–8, 2011; Tech. Univ. Lodz, 2011, p. 11–24.

[17] M. V. Shamolin, *Cases of integrability in dynamics of a rigid body interacting with a resistant medium*, CD-proc., 23th International Congress of Theoretitical and Applied Mechanics, August 19–24, 2012, Beijing, China; Beijing, China Science Literature Publishing House, 2012, 2 p.

[18] M. V. Shamolin, *Variety of the cases of integrability in dynamics of a 2D-, and 3D-rigid body interacting with a medium*, 8th ESMC 2012, CD-Materials (Graz, Austria, July 9–13, 2012), Graz, Graz, Austria, 2012, 2 p.