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ON INTEGRABILITY IN DYNAMIC PROBLEMS FOR A RIGID BODY INTERACTING WITH A MEDIUM

M. V. Shamolin

A complete analysis of the phase trajectories representing the motion of a rigid body in a resisting medium in quasistationary conditions is carried out. More general systems characterized by hidden nontrivial symmetries are studied

Keywords: rigid body, equations of motion, integrability, transcendental first integral

Introduction. When solving the problem of the motion of a rigid body in a resisting medium under quasistationary conditions, we encountered first integrals with nonstandard properties.

Namely, they were neither analytic, nor smooth; they could even be discontinuous on some sets. They were expressed in terms of a finite combination of elementary functions, which is important in applied problems of rigid-body mechanics.

A complete analysis of the phase paths of interest was carried out, and their coarseness was pointed out. Later, more general systems having some hidden nontrivial symmetries were studied.

It is of interest to study wide classes of dynamic systems with similar properties, including those related to the dynamics of a rigid body interacting with a medium.

1. Problem Statement for a Body Moving in a Resisting Medium. Consider the dynamics of a homogeneous axisymmetric rigid body of mass m with disk-shaped frontal area in a separated flow [3–5].

The remaining surface of the body is in a region bounded by the stream surface stemming from the disk's edge and is not affected by the medium. Similar conditions may arise, for example, after a homogeneous circular cylinder enters water by one of the ends [9].

Assume that no tangential forces act on the disk. Then the force S exerted by the medium on the body at a point N does not change its orientation relative to the body (is normal to the disk) and depends on the squared velocity of its center D (Fig. 1). It is also assumed that the weight of the body is negligible compared with the drag.

If the above conditions are satisfied, the motions of the body include retarded translation (undisturbed motion) similar to planar motion: the body can translate along the axis of symmetry, i.e., at a right angle to the disk. This case of motion is of the most applied interest (Fig. 2).

We choose a right-handed coordinate system $Dxyz$ fixed to the body (Fig. 1) and having the Dx-axis aligned with its axis of symmetry. The Dy- and Dz-axes are fixed to the disk. The components of the angular velocity Ω in the frame $Dxyz$ are denoted by $\Omega_x, \Omega_y, \Omega_z$. The inertia tensor of the dynamically symmetric body in the frame $Dxyz$ is $\mathrm{diag}\{I_1, I_2, I_2\}$.

Let us use the quasistationarity hypothesis [3–5, 9] and assume, for simplicity, that $R = DN$ is at least determined by the angle of attack α between the velocity vector v of the disk center D and the straight line Dx (Fig. 1).

Moreover, let $S = s_1(\alpha)v^2$, $v = |v|$. For the sake of simplicity, we introduce the following auxiliary alternating function $s_1(\alpha)$ instead of the drag coefficient $s(\alpha)$: $s_1 = s_1(\alpha) = s(\alpha)\,\mathrm{sgn}\cos\alpha \geq 0$. Thus, the pair of functions R and $s(\alpha)$ describes the effect of the medium on the disk.

Institute of Mechanics of Lomonosov Moscow State University, 1 Michurinskii Av., Moscow, Russia 119192, e-mail: shamolin@imec.msu.ru, shamolin@rambler.ru. Translated from Prikladnaya Mekhanika, Vol. 49, No. 6, pp. 44–54, November–December 2013. Original article submitted April 8, 2011.

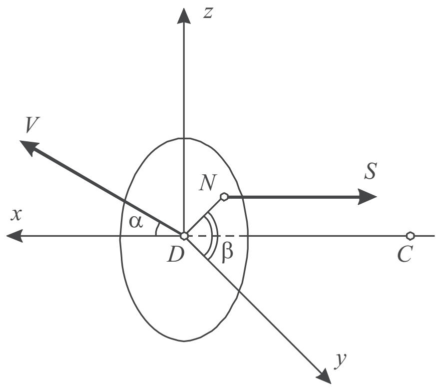

Fig. 1

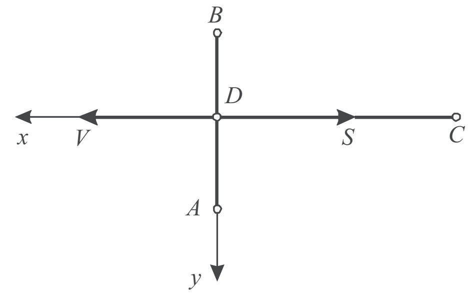

Fig. 2

We will use the spherical coordinates (v, α, β) of the tip of the velocity vector $v = v_D$ of the point D relative to the flow in which the angle β in the disk plane is measured (Fig. 1). Expressing the quantities (v, α, β) in terms of cyclic kinematic variables and velocities by nonintegrable formulas, we will use them as quasivelocities, together with the components $(\Omega_x, \Omega_y, \Omega_z)$ of the angular velocity in the frame $Dxyz$ in which

$$v_D = \{v\cos\alpha, v\sin\alpha\cos\beta_1, v\sin\alpha\sin\beta_1\}.$$

Using the center-of-mass and angular-momentum theorems (in $Dxyz$), we obtain the independent dynamic part of the equations of motion in the six-dimensional quasivelocity space ($\sigma = DC$):

$$v^{\bullet}\cos\alpha - \alpha^{\bullet} v\sin\alpha + \Omega_y v\sin\alpha\sin\beta_1 - \Omega_z v\sin\alpha\cos\beta_1 + \sigma(\Omega_y^2 + \Omega_z^2) = -s(\alpha)v^2 / m,$$

$$v^{\bullet}\sin\alpha\cos\beta_1 + \alpha^{\bullet} v\cos\alpha\cos\beta_1 - \beta_1^{\bullet} v\sin\alpha\sin\beta_1 + \Omega_z v\cos\alpha$$

$$-\Omega_x v\sin\alpha\sin\beta_1 - \sigma\Omega_x\Omega_y - \sigma\Omega_z^{\bullet} = 0,$$

$$v^{\bullet}\sin\alpha\sin\beta_1 + \alpha^{\bullet} v\cos\alpha\sin\beta_1 + \beta_1^{\bullet} v\sin\alpha\cos\beta_1$$

$$+\Omega_x v\sin\alpha\cos\beta_1 - \Omega_y v\cos\alpha - \sigma\Omega_x\Omega_z + \sigma\Omega_y^{\bullet} = 0,$$

$$I_1\Omega_x^{\bullet} = 0, \qquad I_2\Omega_y^{\bullet} + (I_1 - I_2)\Omega_x\Omega_z = -z_N s(\alpha)v^2,$$

$$I_2\Omega_z^{\bullet} + (I_2 - I_1)\Omega_x\Omega_y = y_N s(\alpha)v^2, \tag{1.1}$$

where y_N and z_N are the Cartesian coordinates of the point N in the disk plane at which the force S is applied.

2. Motion of a Body in a Resisting Medium under a Follower Force. Let us address the more general class of problems where a follower force T acts along the axis of geometrical symmetry (straight line CD, Fig. 1) of a body moving in a medium and, under certain conditions, generates motions (imposed constraints) that are of interest. The follower force itself is the reaction of these constraints. In the absence of the follower force, the body undergoes free deceleration in the resisting medium [11].

Let us consider two classes of motion of a body acted upon by a follower force ($v = |v_D|$, V_C is the velocity of the center of mass); namely, we will consider the following two cases:

I) $v \equiv \text{const}$;

II) $V_C \equiv \text{const}$.

2.1. Case I. In the presence of nonintegrable constraint I, the follower force can be well defined [4, 8, 10]. Moreover, the invariant relation $\Omega_x = \Omega_{x_0} =$const holds all the time due to Eqs. (1.1). In what follows, we will examine the case of zero rotation of the body about its axis of symmetry, i.e.,

$$\Omega_{x_0} = 0. \tag{2.1}$$

Then (in view of I, (2.1)) the order of the system decreases by two and the independent dynamic part of the equations of motion in the four-dimensional phase space takes the form

$$\alpha^{\bullet} v \cos\alpha \cos\beta_1 - \beta_1^{\bullet} v \sin\alpha \sin\beta_1 + \Omega_z v \cos\alpha - \sigma\Omega_z^{\bullet} = 0,$$

$$\alpha^{\bullet} v \cos\alpha \sin\beta_1 + \beta_1^{\bullet} v \sin\alpha \cos\beta_1 - \Omega_y v \cos\alpha + \sigma\Omega_y^{\bullet} = 0,$$

$$I_2 \Omega_y^{\bullet} = -z_N s(\alpha) v^2, \quad I_2 \Omega_z^{\bullet} = y_N s(\alpha) v^2, \tag{2.2}$$

where y_N, z_N, s are functions describing the effect of the medium; they can be described qualitatively based on experimental data on the properties of separated flows [3–6].

We will restrict ourselves to the analysis of system (2.2) for the following functions (obtained by Chaplygin [6]):

$$y_N = A \sin\alpha \cos\beta_1 - h\Omega_z / v,$$

$$z_N = A \sin\alpha \sin\beta_1 + h\Omega_y / v,$$

$$s(\alpha) = B \cos\alpha \quad (A, B, h > 0), \tag{2.3}$$

where the coefficient h appears in terms proportional to rotary derivatives of the moment of the force exerted by the medium with respect to the angular velocity of the body [1, 2, 7].

System (2.2) is a dynamic system with variable dissipation with zero mean (over the angle of attack). This means that the integral (responsible for the variation of the phase volume) of the divergence of its right-hand side over the period of the angle of attack is equal to zero (after some reduction of the system). The system is “semiconservative” in a sense [9].

Projecting the angular velocities onto the moving axes not fixed to the body so that $z_1 = \Omega_y \cos\beta_1 + \Omega_z \sin\beta_1$ and $z_2 = -\Omega_y \sin\beta_1 + \Omega_z \cos\beta_1$ and introducing dimensionless variables w_k $(k = 1, 2)$, $z_k = n_0 v w_k$, parameters $n_0^2 = AB / I_2$, $H_1 = Bh / I_2 n_0$, $b = \sigma n_0$, and derivative $\langle\bullet\rangle = n_0 v \langle'\rangle$, we obtain the following analytic system of the fourth order:

$$\alpha' = -(1 + bH_1) w_2 + b \sin\alpha,$$

$$w_2' = \sin\alpha \cos\alpha - (1 + bH_1) w_1^2 \frac{\cos\alpha}{\sin\alpha} - H_1 w_2 \cos\alpha, \tag{2.4}$$

$$w_1' = (1 + bH_1) w_1 w_2 \frac{\cos\alpha}{\sin\alpha} - H_1 w_1 \cos\alpha,$$

$$\beta' = (1 + bH_1) w_1 \frac{\cos\alpha}{\sin\alpha}, \tag{2.5}$$

which includes the independent subsystem of the third order (2.4).

If $b = H_1$ and $w_1^* = \ln|w_1|$, the divergence of the right-hand side of system (2.4) ((2.4), (2.5)) is identically equal to zero, which means that the system(s) is (are) conservative.

Theorem 1. System (2.4), (2.5) has a complete set of invariant relations that are elementary transcendental (in the sense of complex analysis) functions of phase variables. Two of them form a complete set of first integrals of system (2.4).

Indeed, system (2.4), (2.5) is associated with the following nonautonomous system of the second order:

$$\frac{dw_2}{d\alpha}=\frac{\sin\alpha\cos\alpha-(1+bH_1)w_1^2\cot\alpha-H_1w_2\cos\alpha}{-(1+bH_1)w_2+b\sin\alpha},$$

$$\frac{dw_1}{d\alpha}=\frac{(1+bH_1)w_1w_2\cot\alpha-H_1w_1\cos\alpha}{-(1+bH_1)w_2+b\sin\alpha}. \tag{2.6}$$

With $\tau=\sin\alpha$, system (2.6) becomes:

$$\frac{dw_2}{d\tau}=\frac{\tau-(1+bH_1)w_1^2/\tau-H_1w_2}{-(1+bH_1)w_2+b\tau},$$

$$\frac{dw_1}{d\tau}=\frac{(1+bH_1)w_1w_2/\tau-H_1w_1}{-(1+bH_1)w_2+b\tau}. \tag{2.7}$$

With $w_k=u_k\tau\ (k=1,2)$, which is a typical substitution for homogeneous systems, system (2.7) is associated with the following nonautonomous differential equation:

$$\frac{du_2}{du_1}=\frac{1+(1+bH_1)(u_2^2-u_1^2)-(H_1+b)u_2}{2(1+bH_1)u_1u_2-(H_1+b)u_1}. \tag{2.8}$$

Its first integral is

$$\frac{(1+bH_1)u_2^2-(H_1+b)u_2+(1+bH_1)u_1^2+1}{u_1}=C_1. \tag{2.9}$$

It follows from (2.8) that the first integral of system (2.4), (2.5) is

$$\frac{(1+bH_1)w_2^2-(H_1+b)w_2\sin\alpha+(1+bH_1)w_1^2+\sin^2\alpha}{w_1\sin\alpha}=C_1. \tag{2.10}$$

As indicated above, when $b=H_1$, the dynamic system (2.4) (as well as (2.4), (2.5)) is conservative. Indeed, relation (2.10) transforms into the invariant relation

$$\frac{w_2^2+(1+b^2)w_1^2+[bw_2-\sin\alpha]^2}{w_1\sin\alpha}=C_1. \tag{2.11}$$

Moreover, it is easy to verify that both the numerator and the denominator in (2.11) are the first integrals of system (2.4) when $b=H_1$:

$$w_2^2+(1+b)w_1^2+b[w_2-\sin\alpha]^2=C_1^*=\text{const},$$

$$w_1\sin\alpha=C_2^*=\text{const}.$$

If, however, $b\neq H_1$, system (2.4) is no longer conservative, and the numerator and the denominator in (2.10) are not first integrals. The latter fact may not be checked because system (2.4) has attracting and repelling limit sets [7, 11, 12] that forbid the system to have a complete set of first integrals, even if continuous.

An additional first integral for system (2.4) can be found from the quadrature

$$\int\frac{d\tau}{\tau}=\int\frac{[1-(1+bH_1)u_2]du_2}{1-(H_1+b)u_2+(1+bH_1)[u_2^2-U(u_1,C_1)]}, \tag{2.12}$$

where $U(u_1, C_1) = \frac{1}{2(1+bH_1)}\{C_1 \pm \sqrt{C_1^2 - 4D_1}\}$ for $C_1 > 4(1+bH_1)D_1$ in (2.10) ($D_1 = (1+bH_1)u_2^2 - (H_1+b)u_2 + 1$).

The general form of the additional first integral of system (2.4), (2.5) can be found from (2.12):

$$\Phi_1\left(\frac{w_1}{\sin\alpha}, \frac{w_2}{\sin\alpha}, \sin\alpha\right) = C_2 = \text{const}.$$

In view of (2.5) and (2.9), the additional first integral for system (2.4), (2.5) of the fourth order, which attaches Eq. (2.5), can be found from the solution of the equation

$$\frac{du_1}{d\beta} + \left[\frac{1-(1+bH_1)u_2}{1+bH_1}\right] = u_2 - \frac{H_1}{1+bH_1},$$

resulting in

$$\sin^2\{2(1+bH_1)^2(\beta + C_3)\} = \frac{(2(1+bH_1)w_1 - 2C_1\sin\alpha)^2}{[(H_1+b)^2 - 4b(1+bH_1) + C_1^2]\sin^2\alpha}.$$

Remark. The integrated system (2.4) is considered in the three-dimensional domain $S^1\{\alpha \operatorname{bmod} 2\pi\} \setminus \{\alpha = 0, \alpha = \pi\} \times R^2\{w_1, w_2\}$ (such a system is reduced to an equivalent system on a tangent bundle of a two-dimensional sphere S^2 [9]).

Thus, system (2.4) is a system with variable dissipation with zero mean that has two first integrals (i.e., a complete set) being transcendental functions and expressed in terms of a finite combination of elementary functions. The latter has become possible, as mentioned above, due to the association with the (generally nonautonomous) system of equations (2.7) with an algebraic (polynomial) right-hand side.

2.2. Case II. If we deal with the more general problem of the motion of a body in a resisting medium under a follower force T acting along the axis of symmetry and satisfying condition II throughout the entire period of motion, then the first equation of system (1.1) includes a quantity that identically equals zero and causing the body to move in such a way that a couple of forces acts on it. Under certain conditions, system (1.1) is reduced to a system from which a system of lower order can be separated.

Indeed, the choice of phase variables allows us to consider the system of dynamic equations of the sixth order as independent. Moreover, it can be seen from the equations of motion that the longitudinal component of angular velocity remains: $\Omega_x = \Omega_{x_0} = \text{const}$.

Let also the moving body do not rotate, i.e., condition (2.1) is satisfied.

Then, introducing quasivelocities z_1, z_2, dimensionless variables Z_k ($k = 1, 2$), $z_k = n_0 v Z_k$, parameters $n_0^2 = AB/I_2$, $H_1 = Bh/I_2 n_0$, $b = \sigma n_0$, and derivative $\langle\bullet\rangle = n_0 v\langle'\rangle$ and using conditions (2.3), we obtain the following analytic system of the fourth order:

$$v' = v\Psi(\alpha, Z_1, Z_2), \tag{2.13}$$

$$\alpha' = -Z_2 + b(Z_1^2 + Z_2^2)\sin\alpha + b\sin\alpha\cos^2\alpha - bH_1 Z_2\cos^2\alpha,$$

$$Z_2' = \sin\alpha\cos\alpha - Z_2\Psi(\alpha, Z_1, Z_2) - (1+bH_1)Z_1^2\frac{\cos\alpha}{\sin\alpha} - H_1 Z_2\cos\alpha,$$

$$Z_1' = -Z_1\Psi(\alpha, Z_1, Z_2) + (1+bH_1)Z_1 Z_2\frac{\cos\alpha}{\sin\alpha} - H_1 Z_1\cos\alpha, \tag{2.14}$$

$$\beta' = (1+bH_1)Z_1\frac{\cos\alpha}{\sin\alpha}, \tag{2.15}$$

$$\Psi(\alpha, Z_1, Z_2) = -b(Z_1^2 + Z_2^2)\cos\alpha + b\sin^2\alpha\cos\alpha - bH_1 Z_2 \sin\alpha\cos\alpha.$$

Let us now look (as above) into the complete integrability (in terms of elementary functions) of the dynamic system of equations (2.13)–(2.15) with analytic right-hand sides.

Since the class of motions under consideration has property II, system (2.13)–(2.15) of the fifth order has an analytic first integral.

Indeed, the velocity of the center of mass C in the frame of reference under consideration can be represented as $V_C = \{v\cos\alpha, v\sin\alpha\cos\beta - \sigma\Omega_z, v\sin\alpha\sin\beta + \sigma\Omega_y\}$. Then the following relation is invariant to system (1.1) if conditions (2.1) and II are satisfied:

$$v^2 - 2\sigma v z_2 \sin\alpha + \sigma^2(z_1^2 + z_2^2) = V_{C0}^2 = \text{const}. \tag{2.16}$$

Formula (2.16), in which the linear and angular velocities form a homogeneous quadratic form, allows us to write an integral polynomial in these velocities for system (2.13)–(2.15):

$$v^2(1 - 2bZ_2\sin\alpha + b^2(Z_1^2 + Z_2^2)) = V_{C0}^2. \tag{2.17}$$

Formula (2.17) can be used to derive an explicit relation between v and the other quasivelocities:

$$v^2 = \frac{V_{C0}^2}{1 - 2bZ_2\sin\alpha + b^2(Z_1^2 + Z_2^2)}. \tag{2.18}$$

Formula (2.18) permits an analysis of the integrability (in terms of elementary functions) of system (2.13)–(2.15), which is of lower (fourth) order.

With $\tau = \sin\alpha$, system (2.14) can be reduced to the following system with algebraic right-hand sides:

$$\frac{dZ_2}{d\tau} = \frac{\tau + bZ_2(Z_1^2 + Z_2^2) - bZ_2\tau^2 - (1 + bH_1)Z_1^2/\tau + bH_1 Z_2^2\tau - H_2 Z_2}{-Z_2 + b\tau(Z_1^2 + Z_2^2) + b\tau(1 - \tau^2) - bH_1 Z_2(1 - \tau^2)},$$

$$\frac{dZ_2}{d\tau} = \frac{bZ_1(Z_1^2 + Z_2^2) - bZ_1\tau^2 + (1 + bH_1)Z_1 Z_2/\tau + bH_1 Z_1 Z_2\tau - H_2 Z_1}{-Z_2 + b\tau(Z_1^2 + Z_2^2) + b\tau(1 - \tau^2) - bH_1 Z_2(1 - \tau^2)}. \tag{2.19}$$

Changing over to the homogeneous coordinates u_k $(k = 1, 2)$ by the formulas $Z_k = u_k\tau$, we reduce system (2.19) to

$$\tau\frac{du_2}{d\tau} = \frac{1 - (b + H_1)u_2 + (1 + bH_1)u_2^2 - (1 + bH_1)u_1^2}{-u_2 + b\tau^2(u_1^2 + u_2^2) + b(1 - \tau^2) - bH_1 u_2(1 - \tau^2)},$$

$$\tau\frac{du_1}{d\tau} = \frac{2(1 + bH_1)u_1 u_2 - (b + H_1)u_1}{-u_2 + b\tau^2(u_1^2 + u_2^2) + b(1 - \tau^2) - bH_1 u_2(1 - \tau^2)}. \tag{2.20}$$

System (2.20) can be associated with the following first-order equation:

$$\frac{du_2}{du_1} = \frac{1 - (b + H_1)u_2 + (1 + bH_1)u_2^2 - (1 + bH_1)u_1^2}{2(1 + bH_1)u_1 u_2 - (b + H_1)u_1}. \tag{2.21}$$

This equation is integrable in terms of elementary functions. After simple transformations, we arrive at an invariant relation corresponding in coordinates (τ, Z_1, Z_2) to the transcendental first integral

$$\frac{(1 + bH_1)Z_1^2 + (1 + bH_1)Z_2^2 - (b + H_1)Z_2\tau + \tau^2}{Z_1\tau} = \text{const}. \tag{2.22}$$

It follows from (2.22) that system (2.14) has a transcendental first integral expressed in terms of a finite combination of elementary functions:

$$\frac{(1+bH_1)Z_1^2+(1+bH_1)Z_2^2-(b+H_1)Z_2\sin\alpha+\sin^2\alpha}{Z_1\sin\alpha}=C_1=\text{const.} \tag{2.23}$$

The first integral (2.23) can be used to rearrange the first equation in (2.20) to the form

$$\tau\frac{du_2}{d\tau}=\frac{1-(b+H_1)u_2+(1+bH_1)u_2^2-(1+bH_1)U_1^2(C_1,u_2)}{-u_2+b(1-\tau^2)+b\tau^2(U_1^2(C_1,u_2)+u_2^2)-bH_1u_2(1-\tau^2)},$$

$$U_1(C_1,u_2)=\{C_1\pm\sqrt{C_1^2-4(1+bH_1)((1+bH_1)u_2^2-(b+H_1)u_2+1)}\}/2 \tag{2.24}$$

or into Bernoulli's equation

$$\frac{d\tau}{du_2}=\frac{(b-(1+bH_1)u_2)\tau-b\tau^3(1-U_1^2(C_1,u_2)-u_2^2-H_1u_2)}{1-(b+H_1)u_2+(1+bH_1)u_2^2-(1+bH_1)U_1^2(C_1,u_2)}. \tag{2.25}$$

With (2.24), Eq. (2.25) can easily be made linear and inhomogeneous:

$$\frac{dp}{du_2}=\frac{2(-b+(1+bH_1)u_2)\tau+2b(1-U_1^2(C_1,u_2)-u_2^2-H_1u_2)}{1-(b+H_1)u_2+(1+bH_1)u_2^2-(1+bH_1)U_1^2(C_1,u_2)},$$

$$p=\frac{1}{\tau^2}. \tag{2.26}$$

This means that we can derive one more transcendental first integral in explicit form (i.e., in terms of a finite combination of quadratures). The general solution of Eq. (2.26) depends on an arbitrary constant C_2.

Let us now find the last additional first integral of system (2.13)–(2.15) (i.e., an integral relating the equation to the angle β). Since

$$\frac{d\beta}{d\tau}=\frac{(1+bH_1)Z_1/\tau}{-Z_2+b\tau(Z_1^2+Z_2^2)+b\tau(1-\tau^2)-bH_1Z_2(1-\tau^2)},$$

the equality

$$\frac{d\beta}{d\tau}=\frac{(1+bH_1)u_1}{-u_2\tau+b\tau^3(u_1^2+u_2^2)+b\tau(1-\tau^2)-bH_1\tau u_2(1-\tau^2)} \tag{2.27}$$

can be supplemented with the equality

$$\tau\frac{du_1}{d\tau}=\frac{2(1+bH_1)u_1u_2-(b+H_1)u_1}{-u_2\tau+b\tau^3(u_1^2+u_2^2)+b\tau(1-\tau^2)-bH_1\tau u_2(1-\tau^2)}, \tag{2.28}$$

following from (2.20).

The resulting system (2.27), (2.28) leads to an equation from which the sought integral follows:

$$\frac{du_1}{d\beta}=2u_2-\frac{b+H_1}{1+bH_1}. \tag{2.29}$$

Using the first integral of Eq. (2.21) (C_1 is its constant of integration) and Eq. (2.29), we can find the required first integral of the original system.

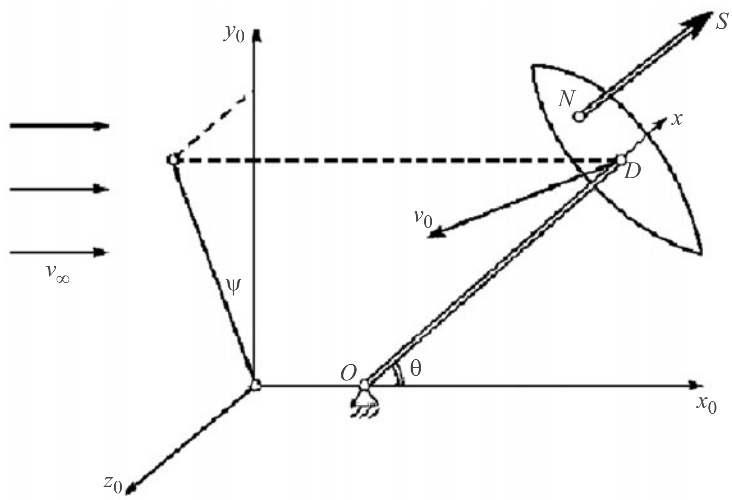

Fig. 3

Thus, the following statement has been proved.

Theorem 2. System (2.13)–(2.15) has a complete set of first integrals one of which is an analytic function, while the other two are elementary transcendental functions of the phase variables.

Note that the first integrals of the systems under consideration can be found by reducing them to systems with polynomial right-hand sides the form of which determines whether the original system can be integrated in terms of elementary functions.

Thus, we have demonstrated the relationship of the following three seemingly independent properties, which blend in systems encountered in rigid-body dynamics:

(i) classes of systems are separated;

(ii) these classes are characterized by variable dissipation with zero mean (with respect to the variable α), which makes it possible to consider them as "almost" conservative systems [9];

(iii) in some (even if low-dimensional) cases, they have a complete set of first integrals which are generally transcendental.

3. Spatial Pendulum in a Flow. By analogy with the planar motion of a free body, we will address the problem of the motion of a spatial pendulum in a homogeneous flow. The flow is incident on a disk fixed at its center perpendicularly to a holder that, in turn, is fixed by the other end to a ball joint. The effect of the medium on the disk is described by the same model.

Let the pendulum move in the flow without rotation (i.e., $\Omega_{x0} = 0$). The effects of the rotary derivatives of the moment of hydroaerodynamic forces with respect to the angular velocity of the body in the Chaplygin functions (2.3) (Fig. 3) are still taken into account.

The original equations of motion are

$$\dot{\Omega}_y = -\frac{1}{I_2} v_D^2 z_N s(\alpha), \quad \dot{\Omega}_z = \frac{1}{I_2} v_D^2 y_N s(\alpha),$$

where the functions y_N and z_N satisfy conditions (2.3).

Let the angles (θ, ψ) define the position of the spatial pendulum on the sphere S^2. The angle θ is measured from the x_0-axis to the holder, while the angle ψ from the projection of the holder onto the plane Oy_0z_0 to the y_0-axis (let $\psi = 0$ at the initial time). Then (v_D, α, β) and $(\theta, \psi, \Omega_y, \Omega_z)$ are related by the following formulas (in which l is the length of the holder):

$$v_D \cos\alpha = -v_\infty \cos\theta,$$

$$v_D \sin\alpha \cos\beta = l\Omega_z + v_\infty \sin\theta \cos\psi,$$

$$v_D \sin\alpha \sin\beta = -l\Omega_y - v_\infty \sin\theta \sin\psi.$$

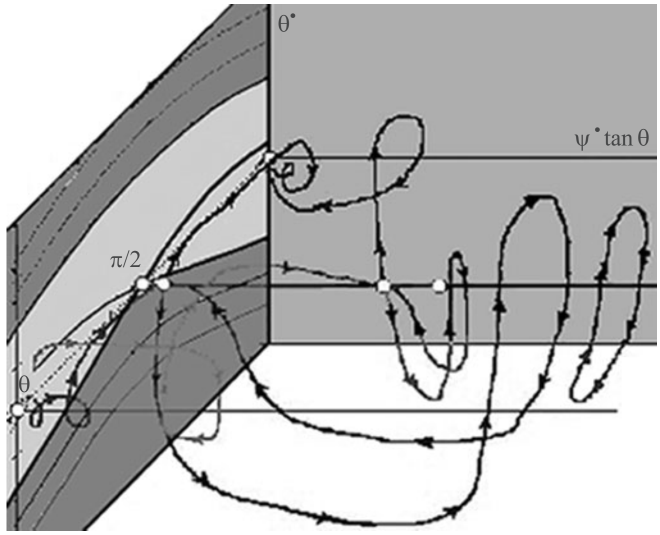

Fig. 4

The kinematic formulas similar to the Euler formulas yield

$$\Omega_y = \theta^\bullet \sin\psi + \psi^\bullet \frac{\sin\theta}{\cos\theta}\cos\psi, \Omega_z = \theta^\bullet \cos\psi - \psi^\bullet \frac{\sin\theta}{\cos\theta}\sin\psi.$$

Then the equations of motion of such a system on the tangent bundle of a two-dimensional sphere can be represented as

$$\theta^{\bullet\bullet} + (b - H_1)\theta^\bullet \cos\theta + \sin\theta\cos\theta - \psi^{\bullet 2}\frac{\sin\theta}{\cos\theta} = 0, \tag{3.1}$$

$$\psi^{\bullet\bullet} + (b - H_1)\psi^\bullet \cos\theta + \theta^\bullet\psi^\bullet\left[\frac{1+\cos^2\theta}{\cos\theta\sin\theta}\right] = 0, \tag{3.2}$$

where b and H_1 are dimensionless physical constants (see above), the coefficient H_1 being still proportional to the rotary derivatives of the moment of the hydroaerodynamic forces with respect to the components of the angular velocity of the spatial pendulum. The length of the holder is equivalent to the distance $\sigma = CD$ for the free body, and the constant velocity of the incident flow v_∞ is equivalent to the constant parameter v for the free body. The angle of attack α for the free body is equivalent to the angle θ between the pendulum and the velocity vector of the flow, and the angle β is equivalent to the cyclic variable, i.e., the angle ψ.

Eliminating w_1 and w_2 in system (2.4)–(2.5), we arrive at system (3.1), (3.2) where α and β replace θ and ψ, respectively. Therefore, the following theorem is true.

Theorem 3. System (2.4)–(2.5) is equivalent to system (3.1), (3.2).

Note that when $\cos\theta = 0$, the continuity of system (3.1), (3.2) can be additionally defined, and the singularity $\sin\theta = 0$ is purely kinematic because the spherical coordinates (v, α, β) degenerate at it.

Figure 4 shows the phase portrait of system (3.1), (3.2) (see also [9]).

Conclusions. Finding cases of complete integrability, especially in terms of elementary functions is always a challenge. It has been demonstrated that there is a close relationship among the following three seemingly independent properties, which blend in systems encountered in rigid-body dynamics: (i) a class of systems with certain symmetries is considered; (ii) this class of systems is characterized by variable dissipation with zero mean (with respect to the periodic phase variable), which makes it possible to consider them as “almost” conservative systems; (iii) in some cases, they have a complete set of first integrals which are generally transcendental.

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