Cases of Integrability in Transcendental Functions in 3D Dynamics of a Rigid Body Interacting With a Medium

Maxim V. Shamolin

Lomonosov Moscow State University: shamolin@imec.msu.ru

Abstract

The results of this work appeared in the process of studying a certain problem on the rigid body motion in a medium with resistance, where we needed to deal with first integrals possessing nonstandard properties. Precisely, they are not analytic, not smooth, and can be even discontinuous on certain sets. Moreover, they are expressed through a finite combination of elementary functions. In this activity the obtained results on study of the equations of the motion of dynamically symmetrical three-dimensional rigid body which residing in a certain nonconservative field of the forces are systematized. Its type is unoriginal from dynamics of the real rigid bodies interacting with a resisting medium on the laws of a jet flow, under which the nonconservative tracing force acts onto the body and forces the value of the velocity of a certain typical point of the rigid body to remain as constant in all time, that means the presence in system of nonintegrable servo-constraint [2].

Keywords: *rigid body dynamics, integrability, transcendental first integral*

1 Preliminary information on integrability

As it is known, the concept of integrability is sufficiently broad and undeterminate in general. In its construction, it is necessary to take into account in what sense it is understood (it is meant that a certain criterion according to which one makes a conclusion that the structure of trajectories of the dynamical system considered is especially attractive), in which function classes the first integrals are sought for, etc.

In this activity, the author applies such an approach such that as first integrals, transcendental functions are elementary. Here, the transcendence is understood not in the sense of elementary functions but in the sense that they have essentially singular points (by the classification accepted in the theory of functions of one complex variable according to which a function has essentially singular points). In this case, it is necessary to continue them formally to the complex plane. As a rule, such systems are strongly nonconservative.

Previously, in [1], the author showed the complete integrability of the equations of body planeparallel motion in a resisting medium under the conditions of streamline flow around when the system of dynamical equations has a first integral that is a transcendental (having essentially singular points in the sense of the theory of functions of one complex variable) function of quasi-velocities. At that time, it was assumed that the interaction of the medium with the body is concentrated on the part of the body surface that has the form of a (one-dimensional) plate.

Later on, in [3], the plane problem was generalized to the spatial (three-dimensional) case where the system of dynamical equations has a complete tuple of transcendental first integrals. It was assumed here that the whole interaction of the medium and the body is concentrated on a part of the body surface that has the form of a plane (two-dimensional) disk.

2 More general problem of the motion with the tracing force

Let consider the spatial motion of a homogeneous axe-symmetric rigid body with the front flat end-wall (two-dimensional disk) in the field of resisting force under assumption of quasi-stationarity [4]. If (v, α, β_1) are the spherical coordinates of the vector velocity of a certain typical point D of a rigid body (D is the center of the disk and lies on the axe of symmetry of the body), $\Omega = \{\Omega_1, \Omega_2, \Omega_3\}$ are the projections of its angular velocity to the axes of the coordinate system $Dx_1x_2x_3$ related to the body, herewith, the axe of symmetry CD coincides with the axe Dx_1 (C is the center of mass), and the axes Dx_2, Dx_3 lie on the disk hyperplane, $I_1,\ I_2,\ I_3 = I_2,\ m$ are the inertia–mass characteristics then the dynamic part of the equations of the body motion (including and in the case of Chaplygin analytical functions [3], see below) under which the tangent forces of the interaction of a medium to the body are absent, has a type:

$$\dot{v}\cos\alpha - \dot{\alpha}v\sin\alpha + \Omega_2 v\sin\alpha\sin\beta_1 - \Omega_3 v\sin\alpha\cos\beta_1 + \sigma(\Omega_2^2 + \Omega_3^2) = \frac{F_x}{m},$$

$$\begin{gathered}
\dot v\sin\alpha\cos\beta_1+\dot\alpha v\cos\alpha\cos\beta_1-\dot\beta_1 v\sin\alpha\sin\beta_1+\Omega_3 v\cos\alpha-\\
-\Omega_1 v\sin\alpha\sin\beta_1-\sigma\Omega_1\Omega_2-\sigma\dot\Omega_3=0,\\
\dot v\sin\alpha\sin\beta_1+\dot\alpha v\cos\alpha\sin\beta_1+\dot\beta_1 v\sin\alpha\cos\beta_1+\Omega_1 v\sin\alpha\cos\beta_1-\\
-\Omega_2 v\cos\alpha-\sigma\Omega_1\Omega_3+\sigma\dot\Omega_2=0,\\
I_1\dot\Omega_1=0,\\
I_2\dot\Omega_2+(I_1-I_2)\Omega_1\Omega_3=-z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha)v^2,\\
I_2\dot\Omega_3+(I_2-I_1)\Omega_1\Omega_2=y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha)v^2,
\end{gathered}\tag{1}$$

where

$$F_x=-S,\ S=s(\alpha)v^2,\ \sigma>0,\ v>0.\tag{2}$$

The first three equations of (1) describe the motion of a center of mass on three-dimensional Euclidean space $\mathbf{E}^3$ in the projections onto the system of coordinates $Dx_1x_2x_3$. And the second three equations of (1) are obtained from the theorem on the rigid body angular momentum on the Köenig axes.

Thus, the direct product of three-dimensional manifold on the Lie algebra so(3)

$$\mathbf{R}^1\times\mathbf{S}^2\times\mathrm{so}(3)\tag{3}$$

is the phase space of the system (1) of the sixth order.

We shall notice immediately that the system (1), by the virtue of available dynamical symmetry

$$I_2=I_3,\tag{4}$$

possesses the cyclic first integral

$$\Omega_1\equiv\Omega_1^0=\text{const}.\tag{5}$$

Herewith, hereinafter we shall consider the dynamics of the system on zero level:

$$\Omega_1^0=0.\tag{6}$$

And if there exists the more general problem of the body motion with the certain tracing force **T** which passing through the center of mass and providing the fulfillment of the following equality in all time of the motion (see also [5])

$$v\equiv\text{const},\tag{7}$$

then in the system (1) the value

$$T-s(\alpha)v^2,\ \sigma=DC,\tag{8}$$

will stand instead of F_x.

As a result of corresponding value choice T of the tracing force it is possible to obtain formally the fulfillment of the equality (7) in all time of the motion. Really, if we express formally the value T by virtue of the system (1) we shall obtain for $\cos\alpha\neq 0$:

$$\begin{gathered}
T=T_v(\alpha,\beta_1,\Omega)=m\sigma(\Omega_2^2+\Omega_3^2)+\\
+s(\alpha)v^2\left[1-\frac{m\sigma}{I_2}\frac{\sin\alpha}{\cos\alpha}\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1+y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1\right]\right].
\end{gathered}\tag{9}$$

The conditions (5)–(7) are used at the obtaining of the equality (9).

It makes possible to look at this procedure from two positions. In first, the transformation of the system has occurred at presence of the tracing (control) force in the system which provides the consideration of interesting class of the motion (7). In second, it makes possible to look at this like the procedure which allows to deflate the system. Really, the system (1) as a result of that action generates an independent system of the fourth order of the following type:

$$\begin{aligned}
&\dot\alpha v\cos\alpha\cos\beta_1-\dot\beta_1 v\sin\alpha\sin\beta_1+\Omega_3 v\cos\alpha-\sigma\dot\Omega_3=0,\\
&\dot\alpha v\cos\alpha\sin\beta_1+\dot\beta_1 v\sin\alpha\cos\beta_1-\Omega_2 v\cos\alpha+\sigma\dot\Omega_2=0,\\
&I_2\dot\Omega_2=-z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha)v^2,\\
&I_2\dot\Omega_3=y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha)v^2,
\end{aligned}\tag{10}$$

in which the parameter v is added to the constant parameters specified above.

The system (10) is equivalent to

$$\begin{gathered}
\dot{\alpha} v \cos\alpha + v\cos\alpha\left[\Omega_3\cos\beta_1 - \Omega_2\sin\beta_1\right] + \sigma\left[-\dot{\Omega}_3\cos\beta_1 + \dot{\Omega}_2\sin\beta_1\right] = 0,\\
\dot{\beta}_1 v\sin\alpha - v\cos\alpha\left[\Omega_2\cos\beta_1 + \Omega_3\sin\beta_1\right] + \sigma\left[\dot{\Omega}_2\cos\beta_1 + \dot{\Omega}_3\sin\beta_1\right] = 0,\\
\dot{\Omega}_2 = -\frac{v^2}{I_2} z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha),\\
\dot{\Omega}_3 = \frac{v^2}{I_2} y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)s(\alpha).
\end{gathered} \tag{11}$$

Let introduce new quasivelocities in the system:

$$z_1 = \Omega_2\cos\beta_1 + \Omega_3\sin\beta_1,\ z_2 = -\Omega_2\sin\beta_1 + \Omega_3\cos\beta_1. \tag{12}$$

As is seen from (11), on the manifold

$$O = \left\{(\alpha,\beta_1,\Omega_2,\Omega_3)\in \mathbf{R}^4:\ \alpha = \frac{\pi}{2}k,\ k\in\mathbf{Z}\right\} \tag{13}$$

it is impossible to resolve the system uniquely relatively to $\dot{\alpha},\ \dot{\beta}_1$. Thus, the violation of the uniqueness theorem is happened on the manifold (13) formally. Moreover, the indefiniteness is happened for even k by the reason of degeneration of the spherical coordinates (v,α,β_1), and it is happened the evident violation of the uniquiness theorem for odd k because of the first equation of (11) degenerates for this case.

It follows that the system (11) outside of and only outside of the manifold (13) is equivalent to the system

$$\begin{gathered}
\dot{\alpha} = -z_2 + \frac{\sigma v}{I_2}\frac{s(\alpha)}{\cos\alpha}\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1 + y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1\right],\\
\dot{z}_2 = \frac{v^2}{I_2}s(\alpha)\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1 + y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1\right] -\\
-z_1^2\frac{\cos\alpha}{\sin\alpha} - \frac{\sigma v}{I_2}\frac{s(\alpha)}{\sin\alpha}z_1\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1 - y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right],\\
\dot{z}_1 = z_1 z_2\frac{\cos\alpha}{\sin\alpha} + \left[-\frac{v^2}{I_2}s(\alpha) + \frac{\sigma v}{I_2}\frac{s(\alpha)}{\sin\alpha}z_2\right]\times\\
\times\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1 - y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right],\\
\dot{\beta}_1 = z_1\frac{\cos\alpha}{\sin\alpha} + \frac{\sigma v}{I_2}\frac{s(\alpha)}{\sin\alpha}\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1 - y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1\right].
\end{gathered} \tag{14}$$

Hereafter, the dependence on the groups of the variables $(\alpha,\beta_1,\Omega/v)$ is understood like the complicated dependence on $(\alpha,\beta_1,z_1/v,z_2/v)$ by virtue of (12).

The violation of the uniqueness theorem is happened for the system (11) for odd k on the manifold (13) in following sense: the regular phase trajectory of the system (14) passes through nearly any point from the manifold (13) for odd k intersecting the manifold (13) under right angle, and also there exist the phase trajectory which coincides completely with the specified point in all moments of time. But those are the different trajectories physically since the different values of the tracing force correspond them. Let show this.

As it is shown above, it is necessary to choose the value T for $\cos\alpha\neq 0$ in the form of (9) to fulfill the constraint (7). Let

$$\lim_{\alpha\to\pi/2}\frac{\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1 + y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1\right]s(\alpha)}{\cos\alpha} = L\left(\beta_1,\frac{\Omega}{v}\right). \tag{15}$$

Let note that $|L| < +\infty$ iff, when

$$\lim_{\alpha\to\pi/2}\left|\frac{\partial}{\partial\alpha}\left(\left[z_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\sin\beta_1 + y_N\left(\alpha,\beta_1,\frac{\Omega}{v}\right)\cos\beta_1\right]s(\alpha)\right)\right| < +\infty. \tag{16}$$

The necessary value of the tracing force for $\alpha = \pi/2$ should be found from the equality

$$T = T_v\left(\frac{\pi}{2}, \beta_1, \Omega\right) = m\sigma(\Omega_2^2 + \Omega_3^2) - \frac{m\sigma L v^2}{I_2}. \tag{17}$$

where the values of Ω_2, Ω_3 are arbitrary.

On the other hand, if we make the rotation around a certain point W by means of the tracing force it will be necessary to choose the tracing force in the form of

$$T = T_v\left(\frac{\pi}{2}, \beta_1, \Omega\right) = \frac{mv^2}{R_0}, \tag{18}$$

where R_0 is the distance CW.

The equations (9) and (18) define, generally speaking, the different values of the tracing force T for almost all the points of the manifold (13), and that is proved the suitable remark.

3 Case of the absence of the dependence of the moment of the nonconservative forces on the angular velocity

3.1 Reduced system

Similarly to the choice of the Chaplygin analytical functions [1], we shall accept the dynamic functions s, y_N and z_N as the following form:

$$s(\alpha) = B\cos\alpha,\ y_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right) = y_0(\alpha, \beta_1) = A\sin\alpha\cos\beta_1,$$
$$z_N\left(\alpha, \beta_1, \frac{\Omega}{v}\right) = z_0(\alpha, \beta_1) = A\sin\alpha\sin\beta_1,\ A, B > 0,\ v \neq 0, \tag{19}$$

which convinces us that the dependence of the moment of the nonconservative forces on the angular velocity is absent in considered system (and there exist the dependences on the angles α, β_1 only).

Then the dynamic part of the motion equations (the system (14)) will have the form as the following analytical system by means of the nonintegrable constraint (7) outside of and only outside of the manifold (13)

$$\begin{aligned}
\dot{\alpha} &= -z_2 + \frac{\sigma ABv}{I_2}\sin\alpha,\\
\dot{z}_2 &= \frac{ABv^2}{I_2}\sin\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha},\\
\dot{z}_1 &= z_1 z_2\frac{\cos\alpha}{\sin\alpha},\\
\dot{\beta}_1 &= z_1\frac{\cos\alpha}{\sin\alpha}.
\end{aligned} \tag{20}$$

If we introduce the dimensionless variables, parameters and differentiability as follows:

$$z_k \mapsto n_0 v z_k,\ k = 1, 2,\ n_0^2 = \frac{AB}{I_2},\ b = \sigma n_0,\ \ <\cdot> = n_0 v <'>, \tag{21}$$

we shall reduce the system (20) to the form

$$\begin{aligned}
\alpha' &= -z_2 + b\sin\alpha,\\
z_2' &= \sin\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha},\\
z_1' &= z_1 z_2\frac{\cos\alpha}{\sin\alpha},
\end{aligned} \tag{22}$$

$$\beta_1' = z_1\frac{\cos\alpha}{\sin\alpha}. \tag{23}$$

As is seen, the independent third order system (22) was formed on its own three-dimensional manifold in the fourth order system (22), (23) which, as it will be shown later, can be considered on the tangent stratification $T\mathbf{S}^2$ of two-dimensional sphere $\mathbf{S}^2$.

3.2 Complete list of invariant relations

At the beginning we compare the third order system (22) to the nonautonomous second order system

$$\begin{aligned} \frac{dz_2}{d\alpha} &= \frac{\sin\alpha\cos\alpha - z_1^2\cos\alpha/\sin\alpha}{-z_2 + b\sin\alpha}, \\ \frac{dz_1}{d\alpha} &= \frac{z_1 z_2\cos\alpha/\sin\alpha}{-z_2 + b\sin\alpha}. \end{aligned} \tag{24}$$

Let rewrite the system (24) on algebraic form using the substitution $\tau = \sin\alpha$

$$\begin{aligned} \frac{dz_2}{d\tau} &= \frac{\tau - z_1^2/\tau}{-z_2 + b\tau}, \\ \frac{dz_1}{d\tau} &= \frac{z_1 z_2/\tau}{-z_2 + b\tau}. \end{aligned} \tag{25}$$

Later on, if we introduce the uniform variables by the formulas

$$z_k = u_k\tau, \ k = 1, 2, \tag{26}$$

we shall reduce the system (25) to the following form:

$$\begin{aligned} \tau\frac{du_2}{d\tau} + u_2 &= \frac{1 - u_1^2}{-u_2 + b}, \\ \tau\frac{du_1}{d\tau} + u_1 &= \frac{u_1 u_2}{-u_2 + b}, \end{aligned} \tag{27}$$

that is equivalent to

$$\begin{aligned} \tau\frac{du_2}{d\tau} &= \frac{1 - u_1^2 + u_2^2 - bu_2}{-u_2 + b}, \\ \tau\frac{du_1}{d\tau} &= \frac{2u_1u_2 - bu_1}{-u_2 + b}. \end{aligned} \tag{28}$$

Let compare the second order system (28) to the nonautonomous first order equation

$$\frac{du_2}{du_1} = \frac{1 - u_1^2 + u_2^2 - bu_2}{2u_1u_2 - bu_1}, \tag{29}$$

which is reduced uncomplicated to the complete differential:

$$d\left(\frac{u_2^2 + u_1^2 - bu_2 + 1}{u_1}\right) = 0. \tag{30}$$

And so, the equation (29) has the following first integral:

$$\frac{u_2^2 + u_1^2 - bu_2 + 1}{u_1} = C_1 = \text{const}, \tag{31}$$

which in former variables is looked like

$$\frac{z_2^2 + z_1^2 - bz_2\sin\alpha + \sin^2\alpha}{z_1\sin\alpha} = C_1 = \text{const}. \tag{32}$$

Remark 1. Let consider the system (22) with zero mean variable dissipation [3] which becomes the conservative for $b = 0$:

$$\begin{aligned} \alpha' &= -z_2, \\ z_2' &= \sin\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha}, \\ z_1' &= z_1 z_2\frac{\cos\alpha}{\sin\alpha}. \end{aligned} \tag{33}$$

It has two the analytical first integrals of the forms

$$z_2^2 + z_1^2 + \sin^2\alpha = C_1^* = \text{const}, \tag{34}$$

$$z_1 \sin\alpha = C_2^* = \text{const}. \tag{35}$$

It is obviously that the ratio of two the first integrals (34), (35) is also the first integral of the system (33). But for $b \neq 0$ each of functions

$$z_2^2 + z_1^2 - bz_2\sin\alpha + \sin^2\alpha \tag{36}$$

and (35) are not the first integrals of the system (22) separately. However, the ratio of the functions (36), (35) is the first integral of the system (22) for any b.

Later on, let find the evident form of the additional first integral of the third order system (22). At the beginning for this we shall transform the invariant relation (31) for $u_1 \neq 0$ as follows:

$$\left(u_2 - \frac{b}{2}\right)^2 + \left(u_1 - \frac{C_1}{2}\right)^2 = \frac{b^2 + C_1^2}{4} - 1. \tag{37}$$

As is seen, the parameters of given invariant relation should satisfy the condition

$$b^2 + C_1^2 - 4 \geq 0, \tag{38}$$

and the phase space of the system (22) is stratified on the family of the surfaces which is assigned by the equality (37).

Thus, by virtue of the relation (31), the first equation of the system (28) has the form

$$\tau\frac{du_2}{d\tau} = \frac{2(1 - bu_2 + u_2^2) - C_1U_1(C_1, u_2)}{-u_2 + b}, \tag{39}$$

where

$$U_1(C_1, u_2) = \frac{1}{2}\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + 1)}\}, \tag{40}$$

herewith, the constant of the integration C_1 is chosen from the condition (38).

Therefore, the quadrature for the search of the additional first integral of the system (22) has the form

$$\int \frac{d\tau}{\tau} = \int \frac{(b - u_2)du_2}{2(1 - bu_2 + u_2^2) - C_1\{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + 1)}\}/2}. \tag{41}$$

The left-hand side (accurate to the additive constant), obviously, is equal to

$$\ln|\sin\alpha|. \tag{42}$$

If

$$u_2 - \frac{b}{2} = w_1,\ b_1^2 = b^2 + C_1^2 - 4, \tag{43}$$

then the right-hand side of the equality (41) has the form

$$-\frac{1}{4}\int \frac{d(b_1^2 - 4w_1^2)}{(b_1^2 - 4w_1^2) \pm C_1\sqrt{b_1^2 - 4w_1^2}} - b\int \frac{dw_1}{(b_1^2 - 4w_1^2) \pm C_1\sqrt{b_1^2 - 4w_1^2}} =$$

$$= -\frac{1}{2}\ln\left|\frac{\sqrt{b_1^2 - 4w_1^2}}{C_1} \pm 1\right| \pm \frac{b}{2}I_1, \tag{44}$$

where

$$I_1 = \int \frac{dw_3}{\sqrt{b_1^2 - w_3^2}(w_3 \pm C_1)},\ w_3 = \sqrt{b_1^2 - 4w_1^2}. \tag{45}$$

Three cases are possible for the calculation of the integral (45).

I. $b > 2$.

$$I_1 = -\frac{1}{2\sqrt{b^2 - 4}}\ln\left|\frac{\sqrt{b^2 - 4} + \sqrt{b_1^2 - w_3^2}}{w_3 \pm C_1} \pm \frac{C_1}{\sqrt{b^2 - 4}}\right| +$$

$$+\frac{1}{2\sqrt{b^2-4}}\ln\left|\frac{\sqrt{b^2-4}-\sqrt{b_1^2-w_3^2}}{w_3\pm C_1}\mp\frac{C_1}{\sqrt{b^2-4}}\right|+\text{const.} \tag{46}$$

II. $b<2$.

$$I_1=\frac{1}{\sqrt{4-b^2}}\arcsin\frac{\pm C_1w_3+b_1^2}{b_1(w_3\pm C_1)}+\text{const.} \tag{47}$$

III. $b=2$.

$$I_1=\mp\frac{\sqrt{b_1^2-w_3^2}}{C_1(w_3\pm C_1)}+\text{const.} \tag{48}$$

When we return to the variable

$$w_1=\frac{z_2}{\sin\alpha}-\frac{b}{2}, \tag{49}$$

we shall have the final form for the value I_1:

I. $b>2$.

$$I_1=-\frac{1}{2\sqrt{b^2-4}}\ln\left|\frac{\sqrt{b^2-4}\pm 2w_1}{\sqrt{b_1^2-4w_1^2}\pm C_1}\pm\frac{C_1}{\sqrt{b^2-4}}\right|+$$

$$+\frac{1}{2\sqrt{b^2-4}}\ln\left|\frac{\sqrt{b^2-4}\mp 2w_1}{\sqrt{b_1^2-4w_1^2}\pm C_1}\mp\frac{C_1}{\sqrt{b^2-4}}\right|+\text{const.} \tag{50}$$

II. $b<2$.

$$I_1=\frac{1}{\sqrt{4-b^2}}\arcsin\frac{\pm C_1\sqrt{b_1^2-4w_1^2}+b_1^2}{b_1(\sqrt{b_1^2-4w_1^2}\pm C_1)}+\text{const.} \tag{51}$$

III. $b=2$.

$$I_1=\mp\frac{2w_1}{C_1(\sqrt{b_1^2-4w_1^2}\pm C_1)}+\text{const.} \tag{52}$$

So, the additional first integral was found right before for the third order system (22), i.e. it was presented the complete tuple of the first integrals which are the transcendental functions of its own phase variables.

Remark 2. It is necessary to substitute formally the left-hand side of the first integral (31) instead of C_1 in the expression of the found first integral.

Then the obtained additional first integral has the following structural form (which is similar to the transcendental first integral from the planeparallel dynamics):

$$\ln|\sin\alpha|+G_2\left(\sin\alpha,\frac{z_2}{\sin\alpha},\frac{z_1}{\sin\alpha}\right)=C_2=\text{const.} \tag{53}$$

Thus, there are already found two the independent first integrals for the integration of the forth order system (22), (23). And for the complete its integrability it is sufficient to find one more (additional) first integral which "connects" the equation (23).

Since

$$\frac{du_1}{d\tau}=\frac{u_1(2u_2-b)}{(b-u_2)\tau},\ \frac{d\beta_1}{d\tau}=\frac{u_1}{(b-u_2)\tau}, \tag{54}$$

then

$$\frac{du_1}{d\beta_1}=2u_2-b. \tag{55}$$

It is obvious that for $u_1\neq 0$ the following equality is fulfilled

$$u_2=\frac{1}{2}\left(b\pm\sqrt{b_1^2-4\left(u_1-\frac{C_1}{2}\right)^2}\right),\ b_1^2=b^2+C_1^2-4, \tag{56}$$

then the integration of the following quadrature:

$$\beta_1+\text{const}=\pm\int\frac{du_1}{\sqrt{b_1^2-4\left(u_1-\frac{C_1}{2}\right)^2}} \tag{57}$$

will bring to the invariant relation

$$2(\beta_1 + C_3) = \pm \arcsin \frac{2u_1 - C_1}{\sqrt{b^2 + C_1^2 - 4}},\ C_3 = \text{const}. \tag{58}$$

In other words, the equality

$$\sin[2(\beta_1 + C_3)] = \pm \frac{2u_1 - C_1}{\sqrt{b^2 + C_1^2 - 4}} \tag{59}$$

is fulfilled and under the transition to the old variables

$$\sin[2(\beta_1 + C_3)] = \pm \frac{2z_1 - C_1 \sin\alpha}{\sqrt{b^2 + C_1^2 - 4}\sin\alpha}. \tag{60}$$

In principle, it makes possible to stop on the latter equality to achieve the additional invariant relation "connecting" the equation (23), if we add to this equality that it is necessary to substitute formally the left-hand side of the first integral (31) instead of C_1 in the latter expression.

But we shall make the certain transformations which reduce to the obtaining of the following evident form of the additional first integral (herewith, the equality (31) is used):

$$\text{tg}^2[2(\beta_1 + C_3)] = \frac{(u_1^2 - u_2^2 + bu_2 - 1)^2}{u_1^2(4u_2^2 - 4bu_2 + b^2)}. \tag{61}$$

Returning to the old coordinates, we shall obtain the additional invariant relation as the form

$$\text{tg}^2[2(\beta_1 + C_3)] = \frac{(z_1^2 - z_2^2 + bz_2\sin\alpha - \sin^2\alpha)^2}{z_1^2(4z_2^2 - 4bz_2\sin\alpha + b^2\sin^2\alpha)}, \tag{62}$$

or finally

$$-\beta_1 \pm \frac{1}{2}\text{arctg}\frac{z_1^2 - z_2^2 + bz_2\sin\alpha - \sin^2\alpha}{z_1(2z_2 - b\sin\alpha)} = C_3 = \text{const}. \tag{63}$$

And so, the system of dynamic equations (1) under the condition (19) has five invariant relations in considered case: there exist the analytical nonintegrable constraint (7), the cyclic first integral (5), (6), the first integral (32) and also there exists the first integral expressed by the relations (46)–(53) which is the transcendental function of its phase variables (in sense of complex analysis also) and expresses in terms of finite combination of the elementary functions, and finally the transcendent first integral (63).

Theorem 1. *The system (1) under the conditions (7), (5), (6), (19) possesses five invariant relations (the complete tuple), three of which are the transcendental functions from the complex analysis view of point. Herewith, all the relations express in terms of the finite combination of the elementary functions.*

3.3 Topological analogies

Let consider the following third order system of the equations:

$$\begin{aligned} &\ddot\theta + b_*\dot\theta\cos\theta + \sin\theta\cos\theta - \dot\psi^2\frac{\sin\theta}{\cos\theta} = 0,\\ &\ddot\psi + b_*\dot\psi\cos\theta + \dot\theta\dot\psi\frac{1+\cos^2\theta}{\cos\theta\sin\theta} = 0,\ b_* > 0, \end{aligned} \tag{64}$$

describing the fixed spherical pendulum which is placed in a flow of the filling medium under the absence of the dependence of the moment of the forces on the angular velocity, i.e. the mechanical system in the nonconservative field of the forces [3], [5], [6]. In general, the order of such system should be equal to 4, but the phase variable ψ is the cyclic, that reduces to the stratification of the phase space and the deflation.

Its phase space is the tangent stratification

$$T\mathbf{S}^2\{\dot\theta, \dot\psi, \theta, \psi\} \tag{65}$$

to two-dimensional sphere $\mathbf{S}^2\{\theta, \psi\}$, herewith, the equation of the big circles

$$\dot\psi \equiv 0 \tag{66}$$

assigns the family of the integral manifolds.

It is not difficult to make sure that the system (64) is equivalent to the dynamic system with the zero mean variable dissipation on the tangent stratification (65) to two-dimensional sphere. Moreover, the following theorem is equitable.

Theorem 2. *The system (1) under the conditions (7), (5), (6), (19) is equivalent to the dynamic system (64).*

Really, it is sufficient to accept $\alpha = \theta,\ \beta_1 = \psi,\ b = -b_*$.

On more general topological analogies see also [3].

4 Conclusions

We develop the qualitative methods in the theory of nonconservative systems that arise, e.g., in such fields of science as the dynamics of a rigid body interacting with a resisting medium, oscillation theory, etc. This material can call the interest of specialists in the qualitative theory of ordinary differential equations, in rigid body dynamics, as well as in fluid and gas dynamics since the work uses the properties of motion of a rigid body in a medium under the streamline flow around conditions [2].

The author obtains new families of phase portraits of systems with variable dissipation on lower- and higher-dimensional manifolds. He discusses the problems of their absolute or relative roughness, He discovers new integrable cases of the rigid body motion, including those in the classical problem of motion of a spherical pendulum placed in the over-running medium flow [4].

The phase pattern of the Eqs. (64) is on the Fig. 1.

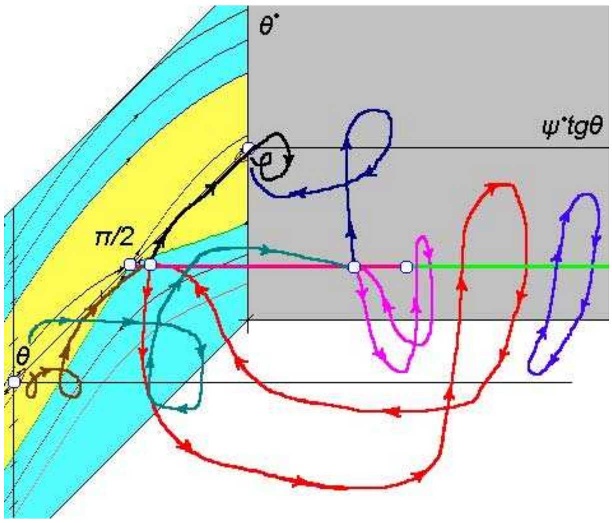

Figure 1. Phase pattern of spherical pendulum in a jet flow.

The assertions obtained in the work for variable dissipation system are a continuation of the Poincare–Bendixon theory for systems on closed two-dimensional manifolds and the topological classification of such systems.

The problems considered in the work stimulate the development of qualitative tools of studying, and, therefore, in a natural way, there arises a qualitative variable dissipation system theory.

Following Poincare, we improve some qualitative methods for finding key trajectories, i.e., the trajectories such that the global qualitative location of all other trajectories depends on the location and the topological type of these trajectories. Therefore, we can naturally pass to a complete qualitative study of the dynamical system considered in the whole phase space. We also obtain condition for existence of the bifurcation birth stable and unstable limit cycles for the systems describing the body motion in a resisting medium under the streamline flow around. We find methods for finding any closed trajectories in the phase spaces of such systems and also present criteria for the absence of any such trajectories. We extend the Poincare topographical plane system theory and the comparison system theory to the spatial case. We study some elements of the theory of monotone vector fields on orientable surfaces which form the so called invariant indices of relatively structural stable vector fields from dynamics of a rigid body interacting with a medium.

5 Acknowledgements

This work was supported by the Russian Foundation for basic Research, Grant No. 12-01-00020-a.

References

[1] V. A. Samsonov and M. V. Shamolin, To a problem on body motion in a resisting medium, *Vestn. MGU, Ser. 1, Mat., Mekh.*, **3**, 51–54, 1989.

[2] M. V. Shamolin, New integrable cases and families of portraits in the plane and spatial dynamics of a rigid body interacting with a medium. *Journal of Mathematical Sciences*, 114:1, 919–975, 2003.

[3] M. V. Shamolin, Dynamical systems with variable dissipation: approaches, methods, and applications. *Journal of Mathematical Sciences*, 162:6, 741–908, 2009.

[4] M. V. Shamolin, Integrability and nonintegrability in terms of transcendental functions in dynamics of a rigid body, *PAMM (Proc. Appl. Math. Mech.)*, 10:63-64, 2010.

[5] M. V. Shamolin, Variety of the cases of integrability in dynamics of a 2D-, 3D-, and 4D-rigid body interacting with a medium, *Proceedings of 11th Conf. on DYNAMICAL SYSTEMS (Theory and Applications) (DSTA 2011)*, pages 11–24.

[6] V. V. Trofimov and M. V. Shamolin, Geometric and dynamical invariants of integrable Hamiltonian and dissipative systems. *Journal of Mathematical Sciences*, 180:4, 365–530, 2012.