$11^{TH}$ CONFERENCE
on
**DYNAMICAL SYSTEMS THEORY AND APPLICATIONS**
December 5-8, 2011. Łódź, Poland

# Variety of the cases of integrability in dynamics of a 2D-, 3D-, and 4D-rigid body interacting with a medium

Maxim V. Shamolin

A vast number of papers are devoted to studying the complete integrability of equations of fourdimensional rigid-body motion. Although in studying low-dimensional equations of motion of quite concrete (two- and three-dimensional) rigid bodies in a nonconservative force field, the author arrived at the idea of generalizing the equations to the case of a four-dimensional rigid body in an analogous nonconservative force field. As a result of such a generalization, he obtained the variety of cases of integrability in the problem of body motion in a resisting medium that fills the four-dimensional space in the presence of a certain tracing force that allows one to reduce the order of the general system of dynamical equations of motion in a methodical way.

## 1. Introduction

A huge number of works is devoted to studying the complete integrability cases of the equations of motion of a four-dimensional rigid body. In studying the "low-dimensional" equations of motion of quite concrete (two- and three-dimensional) rigid bodies in a non-conservative force field, he arrived at the idea to generalize the equations to the case of motion of a four-dimensional rigid body in an analogously constructed field. As a result of such a generalization, he obtained several cases of integrability in the problem of body motion in a resisting medium that fills a four-dimensional space under the presence of a certain tracing force, which allows one to methodologically reduce the order of the general system of dynamical equations of motion.

Moreover, to the author opinion, the obtained results are original from the viewpoint that a pair of non-conservative force exists in the system.

Previously, in [1–3], the author showed the complete integrability of the equation of plane-parallel body motion in a resisting medium under the streamline flow around conditions, when the system of dynamical equations has a first integral that is a transcendental function (in the sense of theory of functions of one complex variable, having essentially singular points) of quasi-velocities.In this case, it was assumed that the whole interaction of the medium and the body is concentrated on a part of the body surface that has the form of a (one-dimensional) plate. Later the plane problem was generalized to the spatial (three-dimensional) case where the system of of dynamical equations has a complete tuple of transcendental first integrals. It was assumed here that the whole interaction of the medium and the body is concentrated on a part of the body surface that has the form of a plane (two-dimensional) disk.

## 2. Motion on Two-Dimensional Plane

### 2.1. A more general problem of motion with tracing force

Let us consider the plane-parallel motion of a body with forward plane endwall in the resistance force field under the quasi-stationary conditions [1–3]. If $(v, \alpha)$ are the polar coordinates of a certain characteristic point of the rigid body, $\Omega$ is its angular velocity"— and $I$ and $m$ are the inertia-mass characteristics, then the dynamical part of the equations of motion (including the case of Chaplygin analytical functions of medium action; see below) takes the form

$$\begin{aligned} \dot v \cos\alpha - \dot\alpha v \sin\alpha - \Omega v \sin\alpha + \sigma\Omega^2 &= F_x, \\ \dot v \sin\alpha + \dot\alpha v \cos\alpha + \Omega v \cos\alpha - \sigma\dot\Omega &= 0, \\ I\dot\Omega &= y_N(\alpha, \Omega, v) s(\alpha), \end{aligned} \tag{1}$$

where $F_x = -s(\alpha)v^2/m,\ \sigma > 0$. If we consider a more general problem on the body motion under the existence of a certain tracing force $\mathbf{T}$ passing through the center of masses and ensuring the fulfilment of the relation

$$\mathbf{V}_C \equiv \mathbf{const} \tag{2}$$

during all the time of motion($\mathbf{V}_C$ is the velocity of the center of masses; then in system (1), instead of $F_x$, we have a quantity identically equal to zero, since a non-conservative par of forces acts on the body.

In the case of Chaplygin analytical functions, we take the dynamical functions $s$ and $y_N$ in the form $s(\alpha) = B\cos\alpha,\ y_N(\alpha, \Omega, v) = A\sin\alpha - h_1\Omega/v,\ h_1 > 0,\ A, B > 0,\ v \neq 0$, which shows that in the system considered, there also exists an additional damping (and breaking in some domains of the phase space) non-conservative force moment.

Owing to constraint (2), under certain condition, system (1) reduces to the following system on the three-dimensional cylinder $W_1 = \mathbf{R}^1_+\{v\} \times \mathbf{S}^1\{\alpha \bmod 2\pi\} \times \mathbf{R}^1\{\omega\}$ :

$$v' = v\Psi(\alpha, \omega), \tag{3}$$

$$\dot{\alpha} = -\omega + \sigma n_0^2 \sin\alpha \cos^2\alpha + \sigma\omega^2 \sin\alpha - \frac{\sigma h_1 B}{I} \cos^2\alpha,$$
$$\dot{\omega} = n_0^2 \sin\alpha\cos\alpha - \sigma n_0^2 \omega \sin^2\alpha \cos\alpha + \sigma\omega^3 \cos\alpha + \frac{\sigma h_1 B}{I}\omega^2 \sin\alpha\cos\alpha - \frac{h_1 B}{I}\omega\cos\alpha, \tag{4}$$

$$\Psi(\alpha, \omega) = -\sigma\omega^2 \cos\alpha + \sigma n_0^2 \sin^2\alpha \cos\alpha - \frac{\sigma h_1 B}{I}\omega \sin\alpha\cos\alpha,$$

where $\Omega = \omega v$, $n_0^2 = AB/I$, $\langle\cdot\rangle = v\langle'\rangle$.

### 2.2. A complete list of first integrals

From the system (3), (4), the independent second-order system (4) is separated.

**Theorem 1.** *The system* (3), (4) *has a complete tuple of first integrals; one of them is an analytic function, and the other is a transcendental function of phase variables expressing through a finite combination of elementary functions.*

It is necessary to make an important remark here. The matter is that from the viewpoint of elementary function theory,the obtained first integral is transcendental (i.e., non-algebraic).In this case, the transcendence is understood in the sense of theory of functions of one complex variable, when after a formal continuation of a function to the complex domain, it has essentially singular points corresponding to attracting and repelling limit sets of the dynamical system considered.

Indeed, by (2), the value of the center-of-masses velocity if a first integral of system (1) under the condition $F_x \equiv 0$, precisely the function of phase variables

$$\Psi_0(v, \alpha, \Omega) = v^2 + \sigma^2\Omega^2 - 2\sigma\Omega v \sin\alpha = V_C^2 \tag{5}$$

is constant on phase trajectories.

By a nondegenerate change of the independent variable, the system (3), (4) also has an analytic integral, precisely, the function of phase variables

$$\Psi_1(v, \alpha, \omega) = v^2(1 + \sigma^2\omega^2 - 2\sigma\omega\sin\alpha) = V_C^2 \tag{6}$$

is constant on phase trajectories.

Relation (6) allows us to find the dependence of the velocity of a characteristic rigid body point on other phase variables not solving the system (3), (4); precisely, for $V_C \neq 0$, the following relation holds: $v^2 = (V_C^2)/(1 + \sigma^2\omega^2 - 2\sigma\omega\sin\alpha)$.

Since the phase space $W_1$ of the system (3), (4) is three-dimensional and there exist asymptotic limit sets in it,relation (6) defines a unique analytic (even continuous) first integral of the system (3), (4) on the whole phase space.

Let us examine the problem on the existence of the second (additional) first integral of the system (3), (4) in more detail. Its phase space is foliated into surfaces $\{(v, \alpha, \omega) \in W_1 : V_C = \text{const}\}$.

To justify the latter fact, let us introduce the dimensionless differentiation $\langle ' \rangle \mapsto n_0 \langle ' \rangle$ and the additional dimensionless parameter $H_1 = h_1 B / I n_0,\ n_0^2 = AB/I,\ \beta = \sigma n_0,\ \tau = \sin\alpha$, and to the separated second-order system (4), let us put in correspondence the differential equation

$$v' = v\Psi(\alpha, \omega), \tag{7}$$

$$\begin{cases} \dot{\alpha} = -\omega + \beta \sin\alpha \cos^2\alpha + \beta\omega^2 \sin\alpha - \beta H_1 \omega \cos^2\alpha, \\ \dot{\omega} = \sin\alpha\cos\alpha - \beta\omega\sin^2\alpha\cos\alpha + \beta\omega^3\cos\alpha + \beta H_1\omega^2 \sin\alpha\cos\alpha - H_1\omega\cos\alpha, \end{cases} \tag{8}$$

$$\Psi(\alpha, \omega) = -\beta\omega^2\cos\alpha + \beta\sin^2\alpha\cos\alpha - \beta H_1 \omega \sin\alpha\cos\alpha.$$

The analytic first integral (6) obtained above joins Eq. (3) (or (7)). To find the additional transcendental first integral, to the separated system (8), we put in correspondence the differential equation

$$\frac{d\omega}{d\tau} = \frac{\tau - \beta\omega[\omega^2 - \tau^2] + H_1\omega[\beta\omega\tau - 1]}{-\omega + \beta\tau + \beta\tau[\omega^2 - \tau^2] - \beta H_1 \omega[1 - \tau^2]}.$$

After introducing the homogeneous change of variables $\omega = t\tau, \quad d\omega = td\tau + \tau dt$, the integration of the latter equation reduces to the integration of the following Bernoulli equation: $a_1(t)d\tau/dt = a_2(t)\tau + a_3(t)\tau^3,\ a_1(t) = -(1+\beta H_1)t^2 + (\beta + H_1)t - 1,\ a_2(t) = (1+\beta H_1)t - \beta,\ a_3(t) = \beta - \beta H_1 t - \beta t^2$. Applying the classical change of variables $p = 1/\tau^2$, we reduce the equation studied to the linear homogeneous equation

$$\frac{dp}{dt} = c_1(t)p + c_2(t),$$

where

$$c_1(t) = \frac{2t(1+\beta H_1) - 2\beta}{(1+\beta H_1)t^2 - (\beta + H_1)t + 1}, \quad c_2(t) = \frac{2\beta - 2\beta H_1 t - 2\beta t^2}{(1+\beta H_1)t^2 - (\beta + H_1)t + 1}.$$

The solution $p_1$ of the homogeneous part of the equation studied is represented in the following form (three cases are possible):

1. for$D = (\beta - H_1)^2 - 4 > 0$,

$$p_1 = k[(1+\beta H_1)t^2 - (\beta + H_1)t + 1] \cdot \left| \frac{2(1+\beta H_1)t - (\beta - H_1) - \sqrt{D}}{2(1+\beta H_1)t - (\beta - H_1) + \sqrt{D}} \right|^{\frac{H_1 - \beta}{\sqrt{D}}};$$

2. for $D = (\beta - H_1)^2 - 4 < 0$,

$$p_1 = k[(1+\beta H_1)t^2 - (\beta + H_1)t + 1] \cdot \exp\left\{\arctan \frac{2(1+\beta H_1)t - (\beta + H_1)}{\sqrt{-D}}\right\};$$

3. for $D = (\beta - H_1)^2 - 4 = 0$,

$$p_1 = k\big[(1+\beta H_1)t^2 - (\beta + H_1)t + 1\big] \cdot \exp\left\{\frac{2L_1}{\sqrt{1+\beta H_1} \pm 1}\right\}, \quad L_1 = \frac{\beta}{\sqrt{1+\beta H_1}} \pm 1.$$

It is clear that to find a particular solution of the equation studied, applying the variation-of-constant method, we need to assume that $k$ is a function of $t$, which is certainly solvable in the class of elementary functions. In this work, we do not present the corresponding calculations.

## 3. Motion in Three-Dimensional Space

### 3.1. General problem of motion with tracing force

Let us consider the spatial motion of a homogeneous axially-symmetric rigid body of mass $m$ with forward round endwall in the resistance force field under the quasi-stationarity condition. If $(v, \alpha, \beta)$ are the spherical coordinates of a certain characteristic point of the rigid body, $\{\Omega_x, \Omega_y, \Omega_z\}$ are components of its angular velocity, and $I_1, I_2$, and $I_2$ are the principal moments of inertia in a certain coordinate system related to the body, then the dynamical part of the equations of motion in the case of Chaplygin functions [1] of medium action has the form

$$\begin{gathered}
\dot v \cos\alpha - \dot\alpha v \sin\alpha + \Omega_y v \sin\alpha \sin\beta - \Omega_z v \sin\alpha\cos\beta + \sigma(\Omega_y^2 + \Omega_z^2) = F_x,\\
\dot v \sin\alpha\cos\beta + \dot\alpha v\cos\alpha\cos\beta - \dot\beta v \sin\alpha\sin\beta + \Omega_z v\cos\alpha - \Omega_x v \sin\alpha\sin\beta -\\
-\sigma\Omega_x\Omega_y - \sigma\dot\Omega_z = 0,\\
\dot v \sin\alpha\sin\beta + \dot\alpha v\cos\alpha\sin\beta + \dot\beta v\sin\alpha\cos\beta + \Omega_x v\sin\alpha\cos\beta - \Omega_y v\cos\alpha -\\
-\sigma\Omega_x\Omega_z + \sigma\dot\Omega_y = 0,\\
\dot\Omega_x = 0, \quad I_2\dot\Omega_y + (I_1 - I_2)\Omega_x\Omega_z = -ABv^2\sin\alpha\cos\alpha\sin\beta - \frac{h\Omega_y}{v},\\
I_2\dot\Omega_z + (I_2 - I_1)\Omega_x\Omega_y = ABv^2\sin\alpha\cos\alpha\cos\beta - \frac{h\Omega_z}{v},
\end{gathered} \tag{9}$$

where $F_x = -Bv^2/m\cos\alpha$, $A, B, \sigma, h > 0$. If we consider a more general problem of body motion in a resisting medium under the existence of a certain tracing force $\mathbf{T}$ passing through the symmetry axis and ensuring the fulfilment of relation (2) during all the motion time, then in system (9), instead of $F_x$, we have the quantity $(T - B\cos\alpha)v^2/m$; moreover, owing to condition (2), under certain condition, system (9) reduces to a system of a lower order.

It is seen that the choice of phase variables allows us to consider the six-order system (9) of dynamical equations as an independent system. Moreover, as is seen from the equations of motion, the component of the longitudinal angular velocity component is conserved:

$$\Omega_x = \Omega_{x0} = \text{const.} \tag{10}$$

In what follows, we restrict ourselves to the body motion without proper rotation, ie., to the case where $\Omega_{x0} = 0$; moreover, for simplicity, let $h = 0$.

Introduce the following notation: $z_1 = \Omega_y \cos\beta + \Omega_z \sin\beta$, $z_2 = -\Omega_y \sin\beta + \Omega_z \cos\beta$, $z_i = Z_i v$, $i = 1, 2$, $\dot\alpha = v\alpha'$, $\dot\beta = v\beta'$, $\dot v = vv'$. Then system (9) in case (2) for $\Omega_{x0} = 0$ can be transformed into the following form:

$$v' = v\Psi(\alpha, Z_1, Z_2), \tag{11}$$

$$\begin{cases} \alpha' = -Z_2 + \sigma n_0^2 \sin\alpha\cos^2\alpha + \sigma(Z_1^2 + Z_2^2)\sin\alpha, \\ Z_2' = n_0^2 \sin\alpha\cos\alpha - Z_2\Psi(\alpha, Z_1, Z_2) - Z_1^2 \dfrac{\cos\alpha}{\sin\alpha}, \\ Z_1' = -Z_1\Psi(\alpha, Z_1, Z_2) + Z_1 Z_2 \dfrac{\cos\alpha}{\sin\alpha}, \end{cases} \tag{12}$$

$$\beta' = Z_1 \frac{\cos\alpha}{\sin\alpha}, \tag{13}$$

where

$$\Psi(\alpha, Z_1, Z_2) = -\sigma(Z_1^2 + Z_2^2)\cos\alpha + \sigma n_0^2 \sin^2\alpha\cos\alpha, \quad n_0^2 = \frac{AB}{I_2}.$$

### 3.2. A complete list of first integrals

As above, let us consider the problem of complete integrability (in elementary functions) for the dynamical system (11)–(13) with analytic right-hand sides.

Since we consider the class of body motions for which property (2) holds, the fifth-order system (11)–(13) has (along with (10)) an analytic first integral.

Indeed, in the coordinate system considered, we can represent the center-of-masses velocity in the form $\mathbf{V}_C = \left\{ v\cos\alpha,\ v\sin\alpha\cos\beta - \sigma\Omega_z,\ v\sin\alpha\sin\beta + \sigma\Omega_y \right\}$. Then the following relation is invariant for system (9) under conditions (10) ($\Omega_{x0} = 0$) and (2):

$$v^2 - 2\sigma v z_2 \sin\alpha + \sigma^2(z_1^2 + z_2^2) = V_{C0}^2. \tag{14}$$

Moreover, relation (14) in which the linear and angular velocities compose a homogeneous form of degree 2 allows us to write the polynomial integral in the above velocities for the system (11)–(13):

$$v^2(1 - 2\sigma Z_2 \sin\alpha + \sigma^2(Z_1^2 + Z_2^2)) = V_{C0}^2, \tag{15}$$

and relation (15) allows us to explicitly find the dependence of $v$ on the other quasi-velocities:

$$v^2 = \frac{V_{C0}^2}{1 - 2\sigma Z_2 \sin\alpha + \sigma^2(Z_1^2 + Z_2^2)}. \tag{16}$$

It is seen that relation (16) allows us to consider the problems of integrability in elementary functions of the system (11)–(13), which is just of lower order, the fourth order.

Let us rewrite the third-order system (12) in the form

$$\begin{aligned} \alpha' &= -Z_2 + b\sin\alpha\cos^2\alpha + b(Z_1^2 + Z_2^2)\sin\alpha, \\ Z_2' &= \sin\alpha\cos\alpha + bZ_2(Z_1^2 + Z_2^2)\cos\alpha - bZ_2\sin^2\alpha\cos\alpha - Z_1^2\frac{\cos\alpha}{\sin\alpha}, \\ Z_1' &= bZ_1(Z_1^2 + Z_2^2)\cos\alpha - bZ_1\sin^2\alpha\cos\alpha + Z_1Z_2\frac{\cos\alpha}{\sin\alpha}, \end{aligned} \tag{17}$$

where $b = \sigma n_0$ and the new dimensionless differentiation $\langle ' \rangle \mapsto n_0\langle ' \rangle$ is also introduced.

Furthermore, applying the substitution $\tau = \sin\alpha$, which is often used, er reduce system (17)) to the following form with algebraic right-hand sides:

$$\begin{aligned} \frac{dZ_2}{d\tau} &= \frac{\tau + bZ_2(Z_1^2 + Z_2^2) - bZ_2\tau^2 - Z_1^2/\tau}{-Z_2 + b\tau(1-\tau^2) + b\tau(Z_1^2 + Z_2^2)}, \\ \frac{dZ_1}{d\tau} &= \frac{bZ_1(Z_1^2 + Z_2^2) - bZ_1\tau^2 + Z_1Z_2/\tau}{-Z_2 + b\tau(1-\tau^2) + b\tau(Z_1^2 + Z_2^2)}. \end{aligned} \tag{18}$$

Let us pass to homogeneous coordinates $u_k$, $k = 1, 2$, by the formulas $u_k = Z_k\tau$. Then system (18) reduces to the form

$$\begin{aligned} \tau\frac{du_2}{d\tau} &= \frac{1 - bu_2 + u_2^2 - u_1^2}{-u_2 + b\tau^2(u_1^2 + u_2^2) + b(1-\tau^2)}, \\ \tau\frac{du_1}{d\tau} &= \frac{2u_1u_2 - bu_1}{-u_2 + b\tau^2(u_1^2 + u_2^2) + b(1-\tau^2)}. \end{aligned} \tag{19}$$

To system (19), we can put in correspondence the following first-order equation:

$$\frac{du_2}{du_1} = \frac{1 - bu_2 + u_2^2 - u_1^2}{2u_1u_2 - bu_1}. \tag{20}$$

This equation is integrated in elementary functions, since we integrate the following identity obtained from Eq. (20):

$$d\left(\frac{1 - bu_2 + u_2^2}{u_1}\right) + du_1 = 0,$$

and in the coordinates $(\tau, Z_1, Z_2)$, it corresponds to the transcendental first integral of the following form

$$\frac{Z_1^2 + Z_2^2 - bZ_2\tau + \tau^2}{Z_1\tau} = \text{const}. \tag{21}$$

Using relation (21), we conclude that system (12) has the following transcendental first integral, which is expressed through a finite combination of elementary functions:

$$\frac{Z_1^2 + Z_2^2 - bZ_2 \sin\alpha + \sin^2\alpha}{Z_1 \sin\alpha} = \text{const.} \tag{22}$$

Now, using the just found first integral (22), we write the first equation of system (19) in the form

$$\begin{gathered}
\tau\frac{du_2}{d\tau} = \frac{2 - 2bu_2 + 2u_2^2 - C_1U_1(C_1, u_2)}{-u_2 + b - 2b\tau^2 + b\tau^2(C_1U_1(C_1, u_2) + bu_2)}, \\
U_1(C_1, u_2) = \frac{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + 1)}}{2},
\end{gathered} \tag{23}$$

or in the form of the Bernoulli equation

$$\frac{d\tau}{du_2} = \frac{(b - u_2)\tau + b\tau^3(C_1U_1(C_1, u_2) + bu_2 - 2)}{2 - 2bu_2 + 2u_2^2 - C_1U_1(C_1, u_2)}. \tag{24}$$

Equation (24) (by using (23)) easily reduces to the linear inhomogeneous equation

$$\frac{dp}{du_2} = \frac{2(u_2 - b)p - 2b(C_1U_1(C_1, u_2) + bu_2 - 2)}{2 - 2bu_2 + 2u_2^2 - C_1U_1(C_1, u_2)}, \quad p = \frac{1}{\tau^2}. \tag{25}$$

The latter fact means that we can find one more transcendental first integral in explicit form (i.e., through a combination of qyadratures). Moreover, the general solution of Eq. (25) depends on an arbitrary constant $C_2$; we do not present complete calculations.

To find the last additional integral of the system (11)–(13) (i.e., the integral, which connects the equation for the angle $\beta$) we note that since $d\beta/d\tau = (Z_1/\tau)/(-Z_2 + b\tau(Z_1^2 + Z_2^2) + b\tau(1 - \tau^2))$, it follows that to the relation

$$\frac{d\beta}{d\tau} = \frac{u_1}{-u_2\tau + b\tau^3(u_1^2 + u_2^2) + b\tau(1 - \tau^2)} \tag{26}$$

the relation

$$\tau\frac{du_1}{d\tau} = \frac{2u_1u_2 - bu_1}{-u_2 + b\tau^2(u_1^2 + u_2^2) + b(1 - \tau^2)} \tag{27}$$

taken from system (19) is added.

The obtained system (26), (27) allows us to write the following equation for obtaining the desired integral:

$$\frac{du_1}{d\beta} = 2u_1 - \beta. \tag{28}$$

Now, using the first integral of Eq. (20) ($C_1$ is its constant of integration) and Eq. (28), we can obtain that

$$\frac{du_1}{d\beta} = \pm\sqrt{b^2 - 4(u_1^2 - C_1u_1 + 1)}; \tag{29}$$

hence, by (29), the desired quadrature takes the form

$$\pm \int \frac{du_1}{\sqrt{b^2 - 4(u_1^2 - C_1 u_1 + 1)}} = \beta + C_3, \quad C_3 = \text{const.} \tag{30}$$

The left-hand side of (30) (without sign) has the form

$$\frac{1}{2} \arcsin \frac{\left(u_1 - \frac{b}{2}\right)^2}{\sqrt{C_1^2 + (b^2 - 4)}}. \tag{31}$$

After substitutions, from (31), we obtain the desired invariant relation

$$\cos^2[2(\beta + C_3)] = \frac{\left(u_2 - \frac{b}{2}\right)^2 u_1^2}{G_1}, \tag{32}$$

where $G_1 = [u_2^2 - bu_2]^2 + 2[u_2^2 - bu_2][u_1^2 + 1] + [u_1^2 + 1]^2 + b^2 u_1^2$.

In particular, if $b = 2$, then relation (32) takes the form

$$\cos^2[2(\beta + C_3)] = \frac{(Z_2 - \sin\alpha)Z_1}{(Z_2 - \sin\alpha)^2 + Z_1^2}.$$

The right hand side, as an odd function of $\zeta = (Z_2 - \sin\alpha)/(Z_1)$ has a global maximum for $\zeta = 1$, which is equal to $1/2$.

Therefore, we have proved the following assertion.

**Theorem 2.** *The system* (11)–(13) *has a complete tuple of first integrals; one of them is an analytic function, and two other are elementary transcendental functions of their phase own variables.*

In conclusion, we note that for searching for first integrals of the systems considered, we need to reduce them to the corresponding systems with polynomial right-hand sided; the form of the latter ones determines the possibility of integrating the initial system in elementary functions.

## 4. Motion in Four-Dimensional Space

### 4.1. Two case of dynamical symmetry of a four-dimensional body

Let a four-dimensional rigid body $\Theta$ of mass $m$ with smooth three-dimensional boundary $\partial\Theta$ move in a resisting medium that fills a four-dimensional domain of the Euclidean space. Assume that it is dynamically symmetric; in this case, there exist two logical possibilities of representation of its tensor of inertia: either in a certain coordinate system $Dx_1x_2x_3x_4$ related to the body, the tensor of inertia has the form

$$\text{diag}\{I_1, I_2, I_2, I_2\}, \tag{33}$$

or the form

$$\operatorname{diag}\{I_1, I_1, I_3, I_3\}. \tag{34}$$

In the second case, the two-dimensional planes $Dx_1x_2$ and $Dx_3x_4$ are planes of body dynamical symmetry.

### 4.2. Physical assumptions and equations on so(4)

Assume that the distance from the point $N$ of application of a non-conservative force $\mathbf{S}$ to a point $D$ is a function of only one parameter, the angle $\alpha$: $DN = R(\alpha)$ (in the case of motion in the three-dimensional space, this is the angle of attack. In case (33), this angle is measured between the velocity $\mathbf{v}_D$ of the point $D$ and the axis $Dx_1$. In case (34), the meaning of the angle will be clear from the equations.

The value of the non-conservative (resistance) force $\mathbf{S}$ is $S = s(\alpha)\operatorname{sgn}\cos\alpha \cdot v^2$, $|\mathbf{v}_D| = v$, where $s$ is a certain function , which is characterized as as scattering or pumping of energy in the system.

To obtain the explicit form of the dynamical part of the equations of motion,let us define two functions $R$ and $S$using the information about the motion of three-dimensional bodies as follows (in this case, we also use the known analytical result of S. A. Chaplygin): $R = R(\alpha) = A\sin\alpha$, $S = S_v(\alpha) = Bv^2\cos\alpha$; $A, B > 0$.

If $\Omega$ is the angular velocity tensor of the four-dimensional rigid body, $\Omega \in \mathrm{so}(4)$, then the part of the equations of motions,which corresponds to the algebra so(4), has the following form:

$$\dot{\Omega}\Lambda + \Lambda\dot{\Omega} + [\Omega,\ \Omega\Lambda + \Lambda\Omega] = M, \tag{35}$$

where $\Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, $\lambda_1 = (-I_1 + I_2 + I_3 + I_4)/2,\ \ldots,\ \lambda_4 = (I_1 + I_2 + I_3 - I_4)/2$, $M$ is the exterior force moment acting on the body in $\mathbf{R}^4$ and projected on the natural coordinates in the algebra so(4), and $[\cdot,\cdot]$ is the commutator in so(4). A skew-symmetric matrix $\Omega \in \mathrm{so}(4)$ is represented in the form

$$\begin{pmatrix} 0 & -\omega_6 & \omega_5 & -\omega_3 \\ \omega_6 & 0 & -\omega_4 & \omega_2 \\ -\omega_5 & \omega_4 & 0 & -\omega_1 \\ \omega_3 & -\omega_2 & \omega_1 & 0 \end{pmatrix},$$

where $\omega_i$, $i = 1, \ldots, 6$, are components of the angular velocity tensor in projections on the coordinates in the algebra so(4). In this case, it is obvious that for any $i, j = 1, \ldots, 4$, the following relations hold: $\lambda_i - \lambda_j = I_j - I_i$.

In calculating the exterior force moment, it is necessary to construct the mapping

$$\mathbf{R}^4 \times \mathbf{R}^4 \longrightarrow \mathrm{so}(4),$$

which transforms a pair of vectors from $\mathbf{R}^4$ into a certain element of the algebra so(4).

### 4.3. Dynamics in $\mathbf{R}^4$

As for the equation of motion of the center of masses $C$ of the four-dimensional rigid body, then it ie represented in the form

$$m\mathbf{w}_C = \mathbf{F}, \tag{36}$$

where, by the many-dimensional Rivals formula,

$$\mathbf{w}_C = \mathbf{w}_D + \Omega^2\mathbf{DC} + E\mathbf{DC}, \quad \mathbf{w}_D = \mathbf{v}_D + \Omega\mathbf{v}_D, \quad E = \dot{\Omega},$$

$\mathbf{F}$ is the exterior force acting on the body (in our case, $\mathbf{F} = \mathbf{S}$), $E$ is the angular acceleration tensor.

### 4.4. Generalized problem of body motion under tracing force action

In this work, we consider only the case (33) of distribution of principal moments of inertia.

Let us slightly extend the problem. Assume that along the line $Dx_1$ (in case (33)), a certain tracing force acts whose line of action passes through the center of masses $C$. The introduction of such a force is used for consideration of classes of motions interesting for us; as a result of which the order of the dynamical system can be reduced.

as in the previous sections, let us consider the class of motion of the four-dimensional rigid body in the case (2)), i. e., its center of masses moves rectilinear and uniformly.

### 4.5. Case (33)

By a completely definite choice of the tracing force,the fulfilment of condition (2) can be achieved.

If $(0, x_{2N}, x_{3N}, x_{4N})$ are coordinates — of the point $N$ in the system $Dx_1x_2x_3x_4$ and $\{-S, 0, 0, 0\}$ are coordinates of the resistance force vector in the same system, then to find the force moment, we construct the auxiliary matrix

$$\begin{pmatrix} 0 & x_{2N} & x_{3N} & x_{4N} \\ -S & 0 & 0 & 0 \end{pmatrix},$$

which allows us to obtain the resistance force moment in the projections on the coordinates in the algebra so(4): $\{0, 0, x_{4N}S, 0, -x_{3N}S, x_{2N}S\} \in \mathbf{R}^6 \cong M \in \mathrm{so}(4)$. Here,it is necessary to take into account that if $(v,\alpha,\beta_1,\beta_2)$ are the spherical coordinates in $\mathbf{R}^4$, then $x_{2N} = R(\alpha)\cos\beta_1, \quad x_{3N} = R(\alpha)\sin\beta_1\cos\beta_2, \quad x_{4N} = R(\alpha)\sin\beta_1\sin\beta_2$.

Taking into account all what was said, we can write Eq. (35) in the form

$$
\begin{aligned}
(\lambda_4+\lambda_3)\dot\omega_1 + (\lambda_3-\lambda_4)(\omega_3\omega_5+\omega_2\omega_4) &= 0,\\
(\lambda_2+\lambda_4)\dot\omega_2 + (\lambda_2-\lambda_4)(\omega_3\omega_6-\omega_1\omega_4) &= 0,\\
(\lambda_4+\lambda_1)\dot\omega_3 + (\lambda_4-\lambda_1)(\omega_2\omega_6+\omega_1\omega_5) &= x_{4N}S,\\
(\lambda_3+\lambda_2)\dot\omega_4 + (\lambda_2-\lambda_3)(\omega_5\omega_6+\omega_1\omega_2) &= 0,\\
(\lambda_1+\lambda_3)\dot\omega_5 + (\lambda_3-\lambda_1)(\omega_4\omega_6-\omega_1\omega_3) &= -x_{3N}S,\\
(\lambda_1+\lambda_2)\dot\omega_6 + (\lambda_1-\lambda_2)(\omega_4\omega_5+\omega_2\omega_3) &= x_{2N}S.
\end{aligned}
\tag{37}
$$

Obviously, in the case (33), equations (37) have three cyclic first integrals

$$
\omega_1 = \omega_1^0, \quad \omega_2 = \omega_2^0, \quad \omega_4 = \omega_4^0. \tag{38}
$$

For simplicity, let us consider the motions on zero levels $\omega_1^0 = \omega_2^0 = \omega_4^0 = 0$. The remained equations on the algebra so(4) take the following form (here, $n_0^2 = AB/2I_2$): $\dot\omega_3 = n_0^2v^2\sin\alpha\cos\alpha\sin\beta_1\sin\beta_2$, $\dot\omega_5 = -n_0^2v^2\sin\alpha\cos\alpha\sin\beta_1\cos\beta_2$, $\dot\omega_6 = n_0^2v^2\sin\alpha\cos\alpha\cos\beta_1$.

If we introduce the change of angular velocities by the formulas $z_1 = \omega_3\cos\beta_2 + \omega_5\sin\beta_2$, $z_2 = -\omega_3\sin\beta_2\cos\beta_1 + \omega_5\cos\beta_2\cos\beta_1 + \omega_6\sin\beta_1$, $z_3 = \omega_3\sin\beta_2\sin\beta_1 - \omega_5\cos\beta_2 \sin\beta_1 + \omega_6\cos\beta_1$, then the "compatible" equations of motion on the tangent bundle $T\mathbb{S}^3$ of the three-dimensional sphere (after taking into account four conditions (2) and (38),which help us to reduce the order of the general system of dynamical equations of the tenth order to the sixth order) take the following symmetric form ($\sigma = DC$):

$$
\dot v = \sigma\cos\alpha\Big[n_0^2v^2\sin^2\alpha - (z_1^2+z_2^2+z_3^3)\Big], \tag{39}
$$

$$
\begin{aligned}
&\dot\alpha = -z_3 + \sigma n_0^2 v\sin\alpha\cos^2\alpha + \sigma\sin\alpha(z_1^2+z_2^2+z_3^3)/v,\\
&\dot z_3 = n_0^2v^2\sin\alpha\cos\alpha - (z_1^2+z_2^2)\mathrm{ctg}\alpha,\\
&\dot z_2 = z_2z_3\mathrm{ctg}\alpha + z_1^2\mathrm{ctg}\alpha\mathrm{ctg}\beta_1,\\
&\dot z_1 = z_1z_3\mathrm{ctg}\alpha - z_1z_2\mathrm{ctg}\alpha\mathrm{ctg}\beta_1,\\
&\dot\beta_1 = z_2\mathrm{ctg}\alpha,
\end{aligned}
\tag{40}
$$

$$
\dot\beta_2 = -z_1\mathrm{ctg}\alpha\csc\beta_1. \tag{41}
$$

From the complete system of the seventh order (39)–(41), the independent system (40), (41), of the sixth order is separated,and, in turn, it has an independent subsystem (40) of the fifth order. To completely integrate this system, we need, in general, six independent first integrals. However, after changes of variables and introducing a new differentiation $z = \sqrt{z_1^2 + z_2^2}$, $z_* = \frac{z_2}{z_1}$, $z = n_0 v Z$, $z_k = n_0 v Z_k$, $k = 1, 2, 3$, $z_* = Z_*$, $n_0 v\ ' \mapsto\ '$, the system (39)–(41) reduces to the following form ($b = \sigma n_0$, $[b] = 1$):

$$v' = v\Psi(\alpha, Z, Z_3), \quad \Psi(\alpha, Z, Z_3) = b\cos\alpha[\sin^2\alpha - (Z^2 + Z_3^2)], \tag{42}$$

$$\begin{cases} \alpha' = -Z_3 + b\sin\alpha\cos^2\alpha + b\sin\alpha(Z^2 + Z_3^2), \\ Z_3' = \sin\alpha\cos\alpha - Z^2\mathrm{ctg}\alpha - Z_3\Psi(\alpha, Z, Z_3), \\ Z' = ZZ_3\mathrm{ctg}\alpha - Z\Psi(\alpha, Z, Z_3), \end{cases} \tag{43}$$

$$\begin{cases} Z_*' = Z\sqrt{1 + Z_*^2}\mathrm{ctg}\alpha\mathrm{ctg}\beta_1, \\ \beta_1' = \dfrac{ZZ_*}{\sqrt{1 + Z_*^2}}\mathrm{ctg}\alpha, \end{cases} \tag{44}$$

$$\beta_2' = -\frac{Z_1}{\sqrt{1 + Z_*^2}}\mathrm{ctg}\alpha\csc\beta_1. \tag{45}$$

It is seen that the fifth-order system (40) splits into independent subsystems of lower order: system (43) is of the third order and system (44) (of course, after the change of the independent variable ) is of the second order. Therefore, for the complete integrability of the system studied, it suffices to find two independent integrals of the system (43), one for system (44) and additional integrals "connecting" Eqs. (42) and (45).

Moreover, we note that system (43) can be considered on the tangent bundle $T\mathbb{S}^2$ of the two-dimensional sphere.

### 4.6. Complete list of first integrals

The complete system (42)–(45) has an analytic first integral of the form

$$v^2(1 - 2bZ_3\sin\alpha + (Z^2 + Z_3^2)) = V_C^2, \tag{46}$$

since property (2) holds. The latter invariant relation allows us to find $v$.

System (43) belongs to the class of systems arising in the three-dimensional rigid body dynamics and has two independent integrals, which are transcendental functions of their phase variables (in the sense of definitions of complex analysis) and are expressed through a finite combination of elementary functions:

$$\frac{Z^2 + Z_3^2 - bZ_3\sin\alpha + \sin^2\alpha}{Z\sin\alpha} = C_1 = \mathrm{const}, \tag{47}$$

$$G(Z, Z_3, \sin\alpha) = C_2 = \mathrm{const}. \tag{48}$$

System (44) has a first integral of the form

$$\frac{\sqrt{1 + Z_*^2}}{\sin\beta_1} = C_3 = \mathrm{const} \tag{49}$$

and, in turn, it has an additional first integral, which allows us to find $\beta_2$; it has the form

$$\pm\frac{\cos\beta_1}{\sqrt{C_3^2-1}} = \sin\{C_3(\beta_2+C_4)\}, \quad C_4 = \text{const}. \tag{50}$$

Also, it is necessary to note the fact that the denominators of the presented systems contain the functions $\sin\alpha$ and $\sin\beta_1$, which reflect only the information about the fact that the coordinates $(v,\alpha,\beta_1,\beta_2)$ are spherical, and for $\sin\alpha = 0$ and $\sin\beta_1 = 0$ they (kinematically) degenerate.

**Theorem 3.** *The dynamical system* (42)–(45) *has a complete list of first integrals* (46)–(50)*; one of them is an analytic function, and the other are transcendental functions of their variables (after their formal continuation to the complex domain).*

### 4.7. Conclusion

This work complements the previous studies and also opens a new series of works, since previously, only those motions of a four-dimensional body were considered in which the exterior force moment is identically equal to zero ($M \equiv 0$) or the exterior force field is potential; unfortunately, we cannot mention all the authors). In the present work, we continue the direction developed by the author in studying the equations of motion of rigid body on $\mathrm{so}(4)\times\mathbf{R}^4$ under the presence of a non-conservative exterior force moment.

The results listed above and also studies of related fields were already reported at the workshop "Actual Problems of Geometry and Mechanics" named after professor V. V. Trofimov led by D. V. Georgievskii and M. V. Shamolin at Department of Mechanics and Mathematics of M. V. Lomonosov Moscow State University.

### References


1. Shamolin M.V.: *Methods for Analysis of variable Dissipation Dynamical Systems in Rigid Body Dynamics[in Russian]*, Moscow, Ekzamen, 2007.
2. Shamolin M.V.: New Jacobi integrable cases in dynamics of a rigid body interacting with a medium, *Dokl. Ross. Akad. Nauk*, 364(5), 1999, 627–629.
3. Shamolin M.V.: On an integrable case in spatial dynamics of a rigid body interacting with a medium, *Izv. Ross. Akad. Nauk. Mekhanika Tverdogo Tela*, 2, 1997, 65–68.



Maxim V. Shamolin, Professor: Lomonosov Moscow State University, Institute of Mechanics, Michurinskii Ave., 1, 119899 Moscow, Russian Federation (*shamolin@rambler.ru*).