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CASES OF COMPLETE INTEGRABILITY IN TRANSCENDENTAL FUNCTIONS IN DYNAMICS AND CERTAIN INVARIANT INDICES

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Abstract
The results of this work appeared in the process of studying a certain problem on the rigid body motion in a medium with resistance [Chaplygin, 1933], [Chaplygin, 1976], [Shamolin-1, 2007], [Shamolin-2, 2007], where we needed to deal with first integrals having non-standard properties. Precisely, they are not analytic, not smooth, and on certain sets, they can be even discontinuous. Moreover, they are expressed through a finite combination of elementary functions. However, the latter circumstances allowed us to carry out a complete analysis of all phase trajectories and show those their properties which have a ”roughness” and are preserved for systems of a more general form having certain symmetries of latent type. Therefore, it is interesting to study sufficiently wide classes of dynamical systems that have analogous properties and moreover, coming from the dynamics of a rigid body interacting with a medium.

Key words
rigid body, resisting medium, integrability

1 Introduction

The results in”coding” countably many topologically non-equivalent phase portraits were already obtained previously for systems of various types (see, e.g., [Shamolin-1, 2007]). The material of this section in the presented form was not published previously, and the authors decided to present it. The invariant indices introduced in this case were again apperaed from concrete problems of dynamics (see also [Shamolin-1, 2007]).

As is known, the problem of the rigid body motion in an infinite medium volume, owing to its difficulty, requires a whole number of simplifying assumptions and, moreover, the main point is the introduction of those assumptions which allows us to study the rigid body motion independently of the motion of a medium in which the body is placed. Such actions were incorporated in the classical Kirchhoff problem, but it does not exhaust the possibilities of modeling of such a type.

This work is the study of the problem of the plane-parallel motion of a symmetric rigid body that interacts with the medium only through a plane part (cavitator) of its exterior surface. In constructing the force field, we use the information about the properties of streamline flow around under quasi-stationarity conditions, and the medium motion is not studied in this case. In contrast to the previous works (see [Shamolin-1, 2007], [Shamolin-2, 2007]) where the dependence of the force moment on the body angular velocity is neglected, in this work, in accordance with experiment, we take into account the effects of influence of the rotational derivatives of hydro-aero-dynamical forces in components of the body angular velocity [Shamolin-1, 2007].

From the practical viewpoint, it is important the problem od studying the stability of the rectilinear progressive motion under which the velocities of body points are perpendicular to the cavitator (the angle of attack is identically equal to zero in this case).

The necessity of a complete nonlinear study is justified by the importance of finding those conditions under which there exist oscillations of bounded amplitude near the rectilinear progressive motion, which is unstable with respect to the angle of attack and the angular velocity.

2 Dynamical part of the equations of motion

Let us consider the plane-parallel motion of a symmetric homogeneous rigid body of mass m with front plane end-wall (plate) in a resisting medium (see also [Shamolin-1, 2007]). In the case where there are no tangent forces, the medium action on the body reduces to a force $\mathbf{S}$ (applied to a certain point N) orthogonal to the plate.

In the dynamical model, we naturally introduce the following three phase coordinates: $v = |\mathbf{v}|$ is the value of the plate center D with respect to the medium, α is the angle of attack, Ω is the algebraic value of the projection of the angular velocity on the axis orthogonal to the motion. The value of the force $\mathbf{S}$ quadratically depends on v, $S = s_1 v^2$, with certain coefficient s_1 (Newton's resistance). The medium action on the body is determined by the following two sign-alternating functions of the phase variables: $y_N = DN$ and $s_1 = s \operatorname{sgn} \cos\alpha$.

We assume that s is a function of α, and y_N is a function of the pair of dimensionless variables (α, ω), $\omega = \Omega\Delta/v$, where Δ is the characteristic size.

The phase state of the system is determined by six functions, three of these functions, v, α, ω "— are considered as quasi-velocities of the system, and the other three functions (kinematic variable) are cyclic, which leads to reducing the order of the general system of equations of motion. Also, it includes functions $y_N(\alpha, \omega)$ and $s(\alpha)$ determining the medium action on the body. As was already mentioned, the function y_N depends on the angle of attack and on the reduced angular velocity ω. If the latter dependence can be neglected as was in a number of the previous works), then y_N is a function of only the angle of attack: $y_N = y(\alpha)$, and its dependence on a single argument is found by using the experimental information about the properties of streamline flow around. Then, in what follows, the method of "embedding" the problem in the general class of problems is applied.

Our primary goal is the account of the influence of rotational derivatives of medium force action moment in the body angular velocity, which requires the introduction of an additional argument in the medium action functions, which itself is a nontrivial problem of modelling. In this work, we restrict ourselves to only the introduction of the angular velocity as an argument to the function y_N, and neglect asimilar introduction to the reduced resistance coefficient s.

In what follows, we consider y_N in the form $y_N(\alpha, \omega) = y_N(\alpha, \Omega\Delta/v) = y(\alpha) - H\Omega/v$; in this case, $H > 0$ according to the experimental results. Then the equation for the kinetic moment variation is written as

$$I\dot{\Omega} = F(\alpha)v^2 - Hs(\alpha)\Omega v, \quad F(\alpha) = y(\alpha)s(\alpha);$$

moreover, changing the differentiation, we can reduce the dynamical part of equations to the form

$$v' = v\Psi(\alpha, \omega), \tag{1}$$

$$\alpha' = \omega + \sigma\omega^2 \sin\alpha + \frac{\sigma}{I}F(\alpha)\cos\alpha +$$

$$+ \frac{s(\alpha)}{m}\sin\alpha + \frac{\sigma}{I}h\omega s(\alpha)\cos\alpha, \tag{2}$$

$$\omega' = -\frac{1}{I}F(\alpha) + \sigma\omega^3\cos\alpha - \frac{\sigma}{I}\omega F(\alpha)\sin\alpha +$$

$$+\omega\frac{s(\alpha)}{m}\cos\alpha - \frac{B}{I}h\omega\cos\alpha - \frac{\sigma}{I}h\omega^2 s(\alpha)\sin\alpha, \tag{3}$$

where $\Psi(\alpha, \omega) = -\sigma\omega^2\cos\alpha + \sigma F(\alpha)\sin\alpha/I - s(\alpha)\cos\alpha/m + \sigma h\omega s(\alpha)\sin\alpha/I$.

The latter two equations (2), (3) of the system (1)–(3) compose an independent second-order system on the phase cylinder $\mathbf{S}^1\{\alpha \bmod 2\pi\} \times \mathbf{R}^1\{\omega\}$.

3 "Embedding" the problem in a wider class of problems

The system (1)–(3) contains the functions $F(\alpha)$ and $s(\alpha)$ whose explicit form is sufficiently difficult to analytically describe, even for plates of simple form. For this reason,we use the method for "embedding" this problem in a wider class of problems, which takes into account only the qualitative properties of the functions $F(\alpha)$ and $s(\alpha)$.

The support result for us is the result of S. A. Chaplygin who has obtained the functions $y(\alpha)$ and $s(\alpha)$ in the following analytical form for the parallel streamline flow around a plate of infinite length [Chaplygin, 1933]:

$$y(\alpha) = y_0(\alpha) = A\sin\alpha, \quad A > 0, \tag{4}$$

$$s(\alpha) = s_0(\alpha) = B\cos\alpha, \quad B > 0. \tag{5}$$

This result help us to construct functional classes $\{y\}$, $\{s\}$, and then $\{F\}$. Combining (4), (5) with the experimental information about the properties of the streamline flow around, we describe the necessary classes consisting of sufficiently smooth, 2π-periodic functions ($y(\alpha)$ is odd, and $s(\alpha)$ is even) that satisfy the following conditions:

$$y(\alpha) > 0,\ \alpha \in (0, \pi),\ y'(0) > 0,\ y'(\pi) < 0$$

(function class $\{y\}$),

$$s(\alpha) > 0,\ \alpha \in \left(0, \frac{\pi}{2}\right),$$

$$s(\alpha) < 0,\ \alpha \in \left(\frac{\pi}{2}, \pi\right),\ s(0) > 0,\ s'(0) < 0$$

(function class $\{s\}$). y, as well as s change the sign under the replacement of α with $\alpha + \pi$. Therefore, we have found how the embedding $y \in \{y\} = Y, s \in \{s\} = \Sigma$ is fulfilled. From the above properties, it follows that F is a sufficiently smooth odd π-periodic function that satisfies the conditions

$$F(\alpha) > 0,\ \alpha \in \left(0, \frac{\pi}{2}\right),\ F'(0) > 0,\ F'\left(\frac{\pi}{2}\right) < 0$$

(function class $\{F\}$). Therefore, we have found how the embedding $F \in \{F\} = \Phi$.

is fulfilled. In particular, the analytic function

$$F(\alpha) = F_0(\alpha) = AB \sin\alpha \cos\alpha \in \Phi \qquad (6)$$

is a typical representative of the function class Φ (see [Chaplygin, 1933]).

In connection with the instability of the rectilinear progressive braking, it is natural to pose the following question: do there exist the angular oscillations of the body symmetry axis of a finite (bounded) amplitude? Let us formulate this question in a more general form: does there exist a pair of functions y and s of medium action such that for a certain solution of the dynamical part of the equations of motion, the constraint $0 < \alpha(t) < \alpha^* < \pi/2$ holds starting from a certain instant of time $t = t_1$?

Under the simplest assumption on the functions y_N and s of the medium action on the body, it was previously shown [Shamolin-1, 2007] that for the quasi-stationary description of the interaction of the medium with the symmetric body (when y_N and s depend only on the angle of attack ($H = 0$)), for any admissible pair of functions $y(\alpha)$ and $s(\alpha)$ of the medium action, in the whole range ($0 < \alpha < \pi/2$)of finite angles of attack, the system has no any oscillatory solutions of finite(bounded) amplitude.

Therefore, for a possible positive answer to the question posed above , we need to "use" the dependence of the medium action force moment on the reduced angular velocity. as will be shown below, under certain assumptions,in principle, we can expect a positive answer to this question.

Of course, from the practical viewpoint the analysis of dynamical equations is important only in a neighborhood of the rectilinear progressive motion, since for certain angles of attack, there occur a washing away of the lateral surface, and this model of medium action on the body is no longer true. But, first, for bodies with lateral surface of different form, the values of critical angles of attack are different and unknown in general. Therefore, we need to study the whole range of angles. Second, the initial system (1)–(3) is a mechanical pendulum-like system having interesting nonlinear properties, which forces us to perform a complete nonlinear analysis.

Therefore, to study the plane-parallel flow around a plate by a medium, we use the classes of dynamical systems defined by the pair of medium action functions, which considerably complicate the performance of the qualitative analysis.

4 Multiparameter family of system phase portraits on the two-dimensional cylinder

The dynamical system (2), (3) on the two-dimensional phase cylinder has the following equilibrium states:

$$(0,0), \quad (\pi, 0), \qquad (7)$$

$$\left(\frac{\pi}{2}, 0\right), \quad \left(\frac{3\pi}{2}, 0\right), \qquad (8)$$

$$\left(\frac{\pi}{2}, \frac{1}{\sigma}\right), \quad \left(\frac{-\pi}{2}, \frac{-1}{\sigma}\right). \qquad (9)$$

it is necessary to note that the equilibrium states (8) are saddles for any admissible parameters of the problem, and the equilibrium states (9) are attracting.

For a system of the form (1)–(3), it is convenient to introduce the following three-dimensionless parameters:

$$\mu_1 = 2\frac{B}{mn_0},\ \mu_2 = \sigma n_0,\ \mu_3 = \frac{Bh}{In_0}, \qquad (10)$$

$$n_0^2 = \frac{AB}{I},\ B = s(0),\ AB = F'(0).$$

Let us introduce the notation for bands on the phase cylinder

$$\Pi = \left\{(\alpha, \omega) \in \mathbf{R}^2\colon -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right\},$$

$$\Pi' = \left\{(\alpha, \omega) \in \mathbf{R}^2\colon \frac{\pi}{2} < \alpha < \frac{3\pi}{2}\right\}.$$

Theorem 1. In the infinite-dimensional parameter space of system (2), (3) there exists a domain $\mathbf{J}$ of positive measure which corresponds to the following behavior of trajectories of this system:

1) system (2), (3) has no other equilibrium states except for (7)–(9);

2) in the band Π', system (2), (3) has no closed phase characteristics;

3) in the band Π, near the equilibrium state $(0,0)$, under the variation of parameters (10), there can be a bifurcation of birth of a unique stable limit cycle from a weak focus

Let us consider the following subdomain of the domain $\mathbf{J}$:

$$J_1 = \{(\mu_1, \mu_2, \mu_3) \in \mathbf{R}^3\colon 0 < \mu_3 - \mu_2 < 2\}.$$

In this work, we consider only the following infinite parameter domain of system (2), (3):

$$\mathbf{J} \cap J_1. \qquad (11)$$

Remark 1.

The following behavior of phase trajectories near the equilibrium states (7) corresponds to the parameter domain (11):

1) the equilibrium state $(\pi, 0)$ is repelling;

2) the equilibrium state $(0, 0)$ is repelling if $\mu_1 > \mu_3 - \mu_2$ and attracting if $\mu_1 \le \mu_3 - \mu_2$; moreover, if $\mu_1 = \mu_3 - \mu_2$, then the equilibrium states are a weak attracting focus.

Closed curves consisting of phase trajectories of system (2), (3) for the parameter domain (11) can exist only in the band Π [Shamolin-1, 2007].

The main problem of the portrait classification is the problem on the behavior of stable and unstable separatrices of the existing hyperbolic saddles.

let us consider the key problems, the problems on the global behavior of the following separatrices:

a) the separatrix emanating from the point $(\pi/2, 0)$ to the band Π';

b) the separatrix entering the point $(-\pi/2, 0)$ from the band Π;

c) the separatrix emanating from the point $(\pi/2, 0)$ to the band Π.

By the independence of behavior of these separatrices we mean the situation where they have limit sets independently chosen from the domain of definition of all their logically possible limit sets with account of the character of location of all isoclines of the system and existing equilibrium states.

Theorem 2. The global behavior of any two separatrices a)–c) is independent, i.e., the behavior of the third separatrix is defined through the behavior of two other separatrices.

As the pair of key separatrices whose behavior is independent, let us choose the sepsrstrices a) and b).

Definition 1. The index k_1 of the separatrix) is the rational number from the set

$$\left\{ r \in \mathbf{Q} \colon r = \frac{1}{4} + m,\ r = \frac{1}{2} + m,\ m \in \mathbf{N}_0 \right\}.$$

We say that $k_1 = r$, if the separatrix a) has the point $(2\pi r, 1/\sigma)$ if $r = 1/4 + m$ and the point $(2\pi r - 0, +\infty)$ if $r = 1/2 + m$ as its ω-limit set.

Definition 2. The index k_2 of the separatrix b) is the natural number j from the set

$$\{ j \in \mathbf{N} \colon j = 1, 2, 3, 4, 5 \}.$$

We say that $k_2 = j$ if as the α-limit set, the separatrix b) has the point $(0, 0)$ or the stable limit cycle ($j = 1$); the point $(\pi/2, 0)$ ($j = 2$); the point $(\pi, 0)$ ($j = 3$); the point $(-0, -\infty)$ ($j = 4$); the point $(-\pi, 0)$ ($j = 5$).

Therefore, it is seen that the global behavior of the separatrix c) indeed depends on the indices k_1 and k_2, i.e., on the behavior of the separatrices a) and b) in each concrete case.

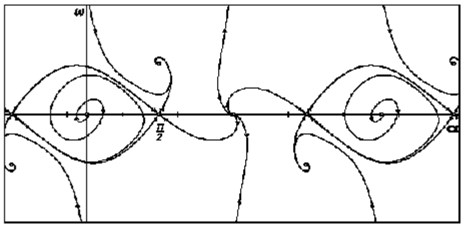

Figure 1. $(k_1, k_2) = (1/4, 2)^*$

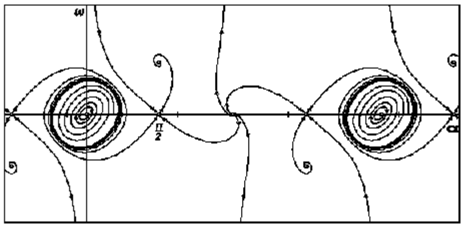

Figure 2. $(k_1, k_2) = (1/4, 4)^*$

Remark 2. If we fix the index k_1, then in some cases, the index k_2 can be chosen from a narrower set described in Definition 2.

Theorem 3. For any $k = (k_1, k_2)$ from the (possibly truncated) domain of definition, the corresponding global behavior of the separatrices a) and b) is admissible.

Therefore, Definitions 1 and 2 are correct, and we construct an infinite family of phase portraits containing portraits with limit cycles; moreover all these portraits have different qualitative properties.

Theorem 3 allows us to make the following conclusion: any sufficiently small perturbation yielding the desired system in the parameter domain considered described the physical pendulum of the plane infinitely many times reconstructs the global type of the Hamiltonian phase portrait of the physical pendulum.

Some of the portraits (the index k assumes the values $(1/4, 2)^*$, $(1/4, 4)^*$, $(1/4, 5)$, $(1/2, 3)^*$, $(1/4, 5)^*$, $(1/4, 3)^*$, $(1/4, 3)$, $(1/4, 4)$) are shown in Figs 1–8. Here, the star labels the phase portraits having simple or complicated limit cycles in the band Π.

For a system of the particular form (2), (3), under conditions (4), (5) (or (6)), we therefore have a certain three-parametric family of phase portraits.

Note that many assertions of this section hold in wider parameter domains.

5 Conclusion

The two-parameter family constructed in [Shamolin-1, 2007] doe not contain limit cycles, in contrast to the just constructed family. But these two families are

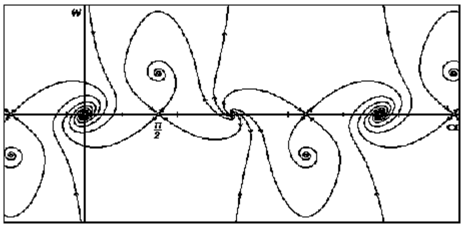

Figure 3. $(k_1, k_2) = (1/4, 5)$

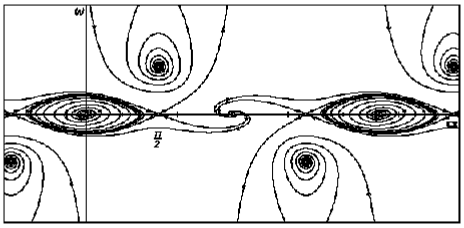

Figure 4. $(k_1, k_2) = (1/2, 3)^*$

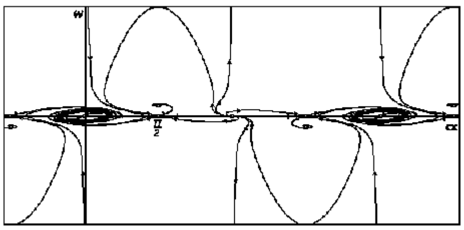

Figure 5. $(k_1, k_2) = (1/4, 5)^*$

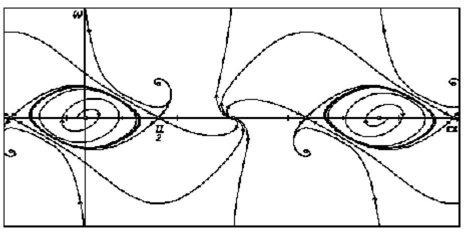

Figure 6. $(k_1, k_2) = (1/4, 3)^*$

united by the fact that to every values of dimensionless parameters of the problem, we put in correspondence a pair of independent indices (k_1 and k_2 in this case) ”coding” the topological type of the phase portrait.

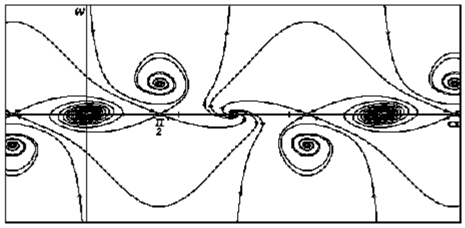

Figure 7. $(k_1, k_2) = (1/4, 3)$

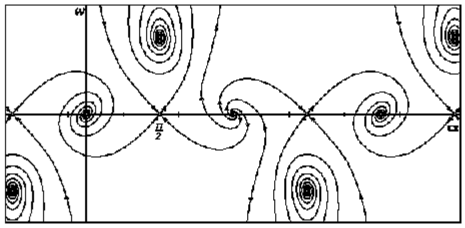

Figure 8. $(k_1, k_2) = (1/4, 4)$

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