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SPATIAL MOTION OF A RIGID BODY IN A RESISTING MEDIUM

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The paper studies the spatial motion of a rigid body in a resisting medium under the action of a follower force that causes the center of mass to move rectilinearly and uniformly. The body, which is axisymmetric and homogeneous, interacts with the medium by its frontal area that has the form of a flat circular disk. Since there is no exact analytic description of the forces and torques exerted by the medium on the disk, the problem is "immersed" in a wider class of problems. Partial solutions and phase portraits in the three-dimensional space of quasivelocities are obtained for the dynamic systems under consideration. The transcendental first integrals of the dynamic part of the equations of motion are listed

Keywords: rigid body, resisting medium, equations of motion, complete integrability

Introduction. Problems of motion of rigid bodies interacting with the medium only by a flat frontal area are of current importance. To describe the forces exerted by the medium on the body, use is made of the properties of a separated flow under quasistationary conditions [1–9]. The motion of the medium is not studied, but rather the typical time of motion of the rigid body about its center of mass is considered commensurable with the typical time of motion of the center itself. Since nonlinear analysis is complicated, its initial stage is to neglect the dependence of the drag torque on the angular velocity of the body, not neglecting its dependence on the angle of attack [12]. Of practical importance is to analyze the stability of so-called unperturbed (translational) motion during which the velocities of particles of the body are perpendicular to the plate (cavitator).

Dynamics of a rigid body in a medium is just the field that usually studies systems with variable dissipation with zero or nonzero mean [12–14]. The present study continues the research conducted earlier. We will use a technique that allows analytic solution to some model problem of spatial motion of a rigid body.

1. Problem Formulation and Equation of Motion.

1.1. The Forces and Torques Exerted by the Medium on a Dynamically Symmetric Body. Consider the dynamics of a homogeneous axisymmetric rigid body of mass m with disk-shaped frontal area in a separated flow [3–9]. The remaining surface of the body is in a region bounded by the stream surface stemming from the disk's edge and is not affected by the medium. Similar conditions may arise, for example, after a homogeneous circular cylinder enters water or during the flight of a parachute [6].

Assume that there are no tangential forces to the disk (cf. [1, 2]). Then the force $\boldsymbol{S}$ exerted by the medium on the body does not change its orientation relative to the body (is normal to the disk) and depends on the squared velocity of its center (Fig. 1). It is also assumed that the weight of the body is negligible compared with the drag.

If the above conditions are satisfied, the motions of the body include retarded translation: the body can translate along the axis of symmetry, i.e., at a right angle to the disk. The point N of application of the force $\boldsymbol{S}$ coincides with the geometrical center D of the disk (Fig. 1).

When the translation of the body is disturbed, the velocity vector $\boldsymbol{v}_D = \boldsymbol{v}$ of the point D is generally deflected from the axis of geometrical symmetry by some angle (of attack) α. Then the point N shifts from the center of the disk by R and lies in the plane formed by the vector $\boldsymbol{v}$ and the axis of symmetry of the body.

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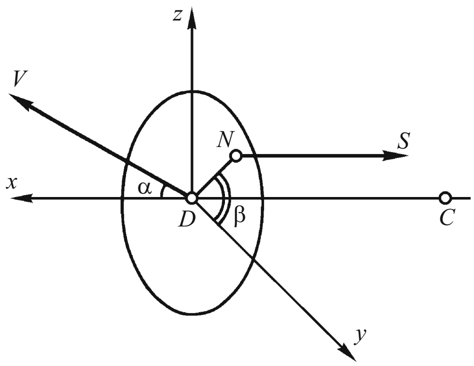

Fig. 1

We choose a right-handed frame of reference $Dxyz$ fixed to the body (Fig. 1) and having the Dx-axis aligned with the axis of geometrical symmetry of the body. The Dy- and Dz-axes are fixed to the disk so as to form a right-handed frame. The components of the angular velocity Ω are denoted by $\{\Omega_x, \Omega_y, \Omega_z\}$ in $Dxyz$. The inertia tensor of the dynamically symmetric body is diagonalized ($\mathrm{diag}\{I_1, I_2, I_2\}$) in the body-fixed frame $Dxyz$.

1.2. Quasistationarity Hypothesis. Let us use the quasistationarity hypothesis and assume, for simplicity, that R is only determined by one parameter, the angle of attack α between the velocity vector $\boldsymbol{v}$ of the disk center D and the straight line Dx. Thus, $DN = R(\alpha)$.

Moreover, let $S = s_1(\alpha)v^2$, $v = |\boldsymbol{v}|$. For the sake of simplicity, we introduce the following auxiliary alternating function $s(\alpha)$ instead of the drag coefficient $s_1(\alpha)$: $s_1 = s_1(\alpha) = s(\alpha)\,\mathrm{sign}\cos\alpha \ge 0$. Thus, the pair of functions $R(\alpha)$ and $s(\alpha)$ describes the effect of the medium on the disk.

1.3. Frames of Reference. To describe the position of the body in space, we choose the Cartesian coordinates (x_0, y_0, z_0) of the point D and three angles (θ, ψ, φ) similar to navigation angles. A rotation from $Dx_0y_0z_0$ to $Dxyz$ is represented by a composition of three rotations, with vectors with some coordinates in the basis (e_x, e_y, e_z) acquiring new coordinates in the basis $(e_{x_0}, e_{y_0}, e_{z_0})$. Such a transformation is defined by the following matrix in the basis (e_x, e_y, e_z):

$$\mathrm{T} = \begin{pmatrix} \cos\psi\cos\varphi & -\cos\psi\sin\varphi & \sin\psi \\ \cos\theta\sin\varphi + \sin\theta\sin\psi\cos\varphi & \cos\theta\cos\varphi - \sin\theta\sin\psi\sin\varphi & -\sin\theta\cos\psi \\ \sin\theta\sin\varphi - \cos\theta\sin\psi\cos\varphi & \sin\theta\cos\varphi + \cos\theta\sin\psi\sin\varphi & \cos\theta\cos\psi \end{pmatrix}$$

and the phase state of the system is characterized by twelve quantities: $(\dot x_0, \dot y_0, \dot z_0, \dot\theta, \dot\psi, \dot\varphi, x_0, y_0, z_0, \theta, \psi, \varphi)$.

The spherical coordinates (v, α, β) of the terminus of the velocity vector of the point D (the angle β is reckoned from the Dy-axis in the plane of the disk to the straight line DN) and the components of angular velocity $\{\Omega_x, \Omega_y, \Omega_z\}$ are related to the variables $(\dot x_0, \dot y_0, \dot z_0, \dot\theta, \dot\psi, \dot\varphi, \theta, \psi, \varphi)$ by nonintegrable kinematic equations. Thus, the phase state of the system can be defined in terms of the functions $(v, \alpha, \beta, \Omega_x, \Omega_y, \Omega_z, x_0, y_0, z_0, \theta, \psi, \varphi)$, and the first six quantities can be considered to be quasivelocities of the system.

The kinetic energy of the body and the drag of the medium do not depend on the position of the body in space. Therefore, the coordinates $(x_0, y_0, z_0, \theta, \psi, \varphi)$ are cyclic, which reduces the order of the dynamic system [10–15].

1.4. Closed System of Equations of Motion. Using the center-of-mass and angular-momentum theorems (in $Dxyz$), we obtain the dynamic part of the differential equations of motion in the six-dimensional quasivelocity space (σ is the distance DC):

$$\dot v\cos\alpha - \dot\alpha v\sin\alpha + \Omega_y v\sin\alpha\sin\beta - \Omega_z v\sin\alpha\cos\beta + \sigma(\Omega_y^2 + \Omega_z^2) = -s(\alpha)v^2/m,$$

$$\dot v\sin\alpha\cos\beta + \dot\alpha v\cos\alpha\cos\beta - \dot\beta v\sin\alpha\sin\beta + \Omega_z v\cos\alpha$$

$$-\Omega_x v\sin\alpha\sin\beta - \sigma\Omega_x\Omega_y - \sigma\dot\Omega_z = 0,$$

$$v^{\bullet}\sin\alpha\sin\beta+\alpha^{\bullet}v\cos\alpha\sin\beta+\beta^{\bullet}v\sin\alpha\cos\beta+\Omega_x v\sin\alpha\cos\beta$$

$$-\Omega_y v\cos\alpha-\sigma\Omega_x\Omega_z+\sigma\Omega_y^{\bullet}=0,\qquad I_1\Omega_x^{\bullet}=0,$$

$$I_2\Omega_y^{\bullet}+(I_1-I_2)\Omega_x\Omega_z=-F(\alpha)\sin\beta\cdot v^2,$$

$$I_2\Omega_z^{\bullet}+(I_2-I_1)\Omega_x\Omega_y=F(\alpha)\cos\beta\cdot v^2,\qquad F(\alpha)=R(\alpha)s(\alpha). \tag{1.1}$$

System (1.1) is supplemented with the following kinematic equations:

$$\theta^{\bullet}=\frac{1}{\cos\psi}[\Omega_x\cos\varphi-\Omega_y\sin\varphi],$$

$$\psi^{\bullet}=\Omega_x\sin\varphi+\Omega_y\cos\varphi,$$

$$\varphi^{\bullet}=\Omega_z+\frac{\sin\psi}{\cos\psi}[\Omega_y\sin\varphi-\Omega_x\cos\varphi],$$

$$\begin{pmatrix} x_0^{\bullet} \\ y_0^{\bullet} \\ z_0^{\bullet} \end{pmatrix}=T\begin{pmatrix} v\cos\alpha \\ v\sin\alpha\cos\beta \\ v\sin\alpha\sin\beta \end{pmatrix}. \tag{1.2}$$

Systems (1.1) and (1.2) constitute a closed system of equations for describing the spatial motion of a rigid body in a resisting medium under quasistationary conditions.

2. More General Class of Problems with Follower Force. The assumptions on the body–medium interaction remaining the same, we will consider the more general class of problems in which the body is subject to not only the drag, but also a follower force (thrusts) $\boldsymbol{T}$ along the straight line CD (Fig. 1). One of such problems was solved for a constant thrust in [6]. It was shown there that translation is unstable against changes in the angle of attack and angular velocity.

Noteworthy is the motion analyzed below in detail ($\boldsymbol{V_C}$ is the velocity of the center of mass):

$$\boldsymbol{V_C}=\text{const}\qquad (\text{if } \boldsymbol{T}=-\boldsymbol{S}). \tag{2.1}$$

It is easy to verify that if

$$T=\tau(\alpha)v^2 \tag{2.2}$$

(a more general condition for the function $T=|\boldsymbol{T}|$ may also be used [12–14]), then an independent subsystem may be separated from the closed system of dynamic equations, as in the case of free deceleration. As a first step, we rearrange system (1.1) into

$$v^{\bullet}+\sigma(z_1^2+z_2^2)\cos\alpha-\sigma\Omega_x z_1\sin\alpha-\sigma\left[\frac{I_1-I_2}{I_2}\Omega_x z_1+\frac{F(\alpha)v^2}{I_2}\right]\sin\alpha=\frac{T-s(\alpha)v^2}{m}\cos\alpha,$$

$$\alpha^{\bullet}v+vz_2-\sigma(z_1^2+z_2^2)\sin\alpha-\sigma\Omega_x z_1\cos\alpha-\sigma\left[\frac{I_1-I_2}{I_2}\Omega_x z_1+\frac{F(\alpha)v^2}{I_2}\right]\cos\alpha=-\frac{T-s(\alpha)v^2}{m}\sin\alpha,$$

$$z_1^{\bullet}=\frac{I_2-I_1}{I_2}\Omega_x z_2+\beta^{\bullet}z_2,\qquad z_2^{\bullet}=\frac{I_1-I_2}{I_2}\Omega_x z_1+\frac{F(\alpha)v^2}{I_2}-\beta^{\bullet}z_1,\qquad \Omega_x^{\bullet}=0,$$

$$\beta^{\bullet}v\sin\alpha+\Omega_x v\sin\alpha-z_1 v\cos\alpha-\sigma\Omega_x z_2+\sigma\left[\frac{I_2-I_1}{I_2}\Omega_x z_2\right]=0,$$

$$z_1 = \Omega_y \cos\beta + \Omega_z \sin\beta, \qquad z_2 = \Omega_z \cos\beta - \Omega_y \sin\beta. \tag{2.3}$$

Indeed, system (1.1), which has $(\tau(\alpha) - s(\alpha))v^2 / m$ on the right-hand side, is (by virtue of (2.1) and (2.2)) a Eulerian homogeneous system of the second degree of homogeneity in quasivelocities $(\Omega_x, \Omega_y, \Omega_z, v)$ because the change of the independent variable (time t) by the formula $dq = vdt$, $v \neq 0$, leads to a new system equivalent to (1.1) (see Sec. 3).

3. "Immersion" of the Problem in a More General Class of Problems. System (1.1) (as (2.3)) includes a function $F(\alpha)$ that is difficult to define analytically even for simple shapes such as a disk.

This is why we "immerse" (as in [12, 13]) the problem in a wider class of problems that takes into account only the qualitative properties of the functions $F(\alpha)$ (or $R(\alpha)$ and $s(\alpha)$).

A reference result is that obtained by Chaplygin who derived analytic expressions for the functions $R(\alpha)$, $s(\alpha)$ [8, 9] for plane-parallel flow past a plate of infinite length:

$$R(\alpha) = R_0(\alpha) = A \sin\alpha \in \{R\}, \quad A > 0, \tag{3.1}$$

$$s(\alpha) = s_0(\alpha) = B \cos\alpha \in \{s\}, \quad B > 0. \tag{3.2}$$

This result helps us to set up functional classes $\{R\}$, $\{s\}$, and $\{F\}$.

3.1. Formal Description of Classes of Influence Functions. Combining (3.1), (3.2) and experimental data on the properties of separated flows, we formally introduce classes of influence functions. They include functions that are sufficiently smooth, 2π-periodic ($R(\alpha)$ is odd, and $s(\alpha)$ is even), and satisfy the following conditions: $R(\alpha) > 0$ for $\alpha \in (0, \pi)$, and $R'(0) > 0, R'(\pi) < 0$ (class $\{R\}$); $s(\alpha) > 0$ for $\alpha \in (0, \pi/2)$, $s(\alpha) < 0$ for $\alpha \in (\pi/2, \pi)$, and $s(0) > 0$, $s'(\pi/2) < 0$ (class $\{s\}$). Both functions R and s reverse sign upon replacement of α by $\alpha + \pi$. Thus, we have

$$R \in \{R\}, \tag{3.3}$$

$$s \in \{s\}. \tag{3.4}$$

It follows from the conditions above that F is a sufficiently smooth, odd, π-periodic function such that $F(\alpha) > 0$ for $\alpha \in (0, \pi/2)$, $F'(0) > 0, F'(\pi/2) < 0$ (class $\{F\}$). Thus, we have

$$F \in \{F\}. \tag{3.5}$$

For example, the following analytic function is a typical representative of the class Φ [8, 9]:

$$F = F_0(\alpha) = AB \sin\alpha \cos\alpha \in \{F\} \tag{3.6}$$

3.2. Nonlinear Analysis (Finite Angles of Attack). The instability of retarded translation mentioned in [4, 7] brings up the question: Does the axis of symmetry of the body undergo angular oscillations of finite (limited) amplitude (for instance, self-oscillations) or, more generally, does there exist a pair of drag functions R and s such that the inequalities $0 < \alpha(t) < \alpha^* < \pi/2$ hold beginning from some time $t = t_1$ for some solution of the dynamic part of the equations of motion?

As already mentioned, one of the goals of the present study is to extend findings in the dynamics of plane-parallel motion of a body to spatial motion, i.e., to confirm the following negative answer to the question about limited-amplitude oscillations [10, 12]. The quasistationary description of the interaction of a medium with a axisymmetric body moving translationally (the functions R and s (or F) depend on the angle of attack alone) produces no oscillatory solutions of finite (limited) amplitude for any admissible pair of functions $R(\alpha)$ and $s(\alpha)$ (or $F(\alpha)$) over the entire range $(0 < \alpha < \pi/2)$ of finite angles of attack.

Of practical importance is to analyze the dynamic equations only in the neighborhood of retarded translation because the lateral surface is wetted at some angles of attack, which makes the medium–body interaction model under consideration invalid. However, first, the critical angles of attack for bodies with different shapes of lateral surface are generally speaking different and unknown. Therefore, the entire range of angles has to be examined. Second, the original system (1.1) is a mechanical pendulum-type system having interesting nonlinear properties. This compels us to carry out a comprehensive nonlinear analysis. Thus, the second part of the present study is of independent methodical interest.

4. Dynamic Part of the Equations of Motion. Analytic System.

4.1. Analytic First Integral. As already mentioned, the choice of phase variables allows us to consider the system of dynamic equations of the sixth order as independent. Moreover, it can be seen from the equations of motion that the longitudinal component of angular velocity remains:

$$\Omega_x = \Omega_{x0} = \text{const}. \tag{4.1}$$

For simplicity, we will restrict the consideration to the translational motion of the body, i.e., $\Omega_{x0} = 0$.

4.2. Equations of Translational Motion. We introduce the following notation: $z_i = Z_i v$, $i = 1, 2$, $\alpha^{\bullet} = \alpha' v$, $\beta^{\bullet} = \beta' v$, $v^{\bullet} = v' v$, $(\,')= d/dq$.

With (4.1) where $\Omega_{x0} = 0$, system (2.3) can be transform to the form

$$v' = v\Psi(\alpha, Z_1, Z_2), \tag{4.2}$$

$$\alpha' = -Z_2 + \sigma(Z_1^2 + Z_2^2)\sin\alpha + \frac{\sigma}{I_2}F(\alpha)\cos\alpha, \tag{4.3}$$

$$Z_2' = \frac{F(\alpha)}{I_2} - Z_2\Psi(\alpha, Z_1, Z_2) - Z_1^2\frac{\cos\alpha}{\sin\alpha}, \tag{4.4}$$

$$Z_1' = -Z_1\Psi(\alpha, Z_1, Z_2) + Z_1 Z_2\frac{\cos\alpha}{\sin\alpha}, \tag{4.5}$$

$$\beta' = Z_1\frac{\cos\alpha}{\sin\alpha}, \quad \Psi(\alpha, Z_1, Z_2) = -\sigma(Z_1^2 + Z_2^2)\cos\alpha + \frac{\sigma}{I_2}F(\alpha)\sin\alpha.$$

Equations (4.2)–(4.5) constitute a closed subsystem of the fourth order, while Eqs. (4.3)–(4.5) a closed subsystem of the third order.

For simplicity, we will consider system (4.3)–(4.5) with reference conditions (3.1), (3.2) (or (3.6)). Under some natural conditions, it describes all the topological properties of the phase space of the more general system (4.2)–(4.5) with (3.3), (3.4) (or (3.5)) partitioned into phase paths. The resulting reference system (to be analyzed qualitatively in the first place) has the following analytic form:

$$v' = v\Psi(\alpha, Z_1, Z_2), \tag{4.6}$$

$$\alpha' = -Z_2 + \sigma(Z_1^2 + Z_2^2)\sin\alpha + \sigma n_0^2\sin\alpha\cos^2\alpha, \tag{4.7}$$

$$Z_2' = n_0^2\sin\alpha\cos\alpha - Z_2\Psi(\alpha, Z_1, Z_2) - Z_1^2\frac{\cos\alpha}{\sin\alpha}, \tag{4.8}$$

$$Z_1' = -Z_1\Psi(\alpha, Z_1, Z_2) + Z_1 Z_2\frac{\cos\alpha}{\sin\alpha}, \tag{4.9}$$

$$\Psi(\alpha, Z_1, Z_2) = -\sigma(Z_1^2 + Z_2^2)\cos\alpha + \sigma n_0^2\sin^2\alpha\cos\alpha, \quad n_0^2 = \frac{AB}{I_2}, \quad \beta' = Z_1\frac{\cos\alpha}{\sin\alpha}. \tag{4.10}$$

5. Some Partial Solutions. Let us choose fixed points of the dynamic systems under consideration. The fixed points of the third-order subsystem (4.3)–(4.5) may be projections of nonsingular phase paths of the fourth-order system (4.2)–(4.5).

Therefore, the issue of fixed points is resolved for system (4.2)–(4.5) in its four-dimensional phase space and for the truncated system (4.3)–(4.5) in its three-dimensional phase space. As will be shown below, the latter has equilibrium positions that fill one-dimensional manifolds.

The original system (1.1) has the following partial solutions:

$$v(t) \equiv v^0 = v(0), \quad \alpha(t) \equiv \pi k, k \in \{0, 1\}, \quad \Omega_x \equiv \Omega_y(t) \equiv \Omega_z(t) \equiv 0, \quad \beta = \beta(t),$$

which formally vanish after the system is reduced to normal form (such as (4.2)–(4.5)). Such a transformation will lead to an equivalent system if we define the continuity of system (4.3)–(4.5) at the points (0; 0; 0) and (π; 0; 0), which become its (added) fixed points.

5.1. Isolated and Nonisolated Fixed Points of the Third-Order System. After the redefinition of system (4.3)–(4.5), it has the following isolated fixed points (see also [11]):

$$\alpha = \pi k, \quad k \in \{0, 1\}, \quad Z_1 = Z_2 = 0 \quad \text{(isolated)} \tag{5.1}$$

on the set $\{(\alpha, Z_1, Z_2) \in R^3 : 0 < \alpha < \pi, Z_1 \geq 0\}$.

When $k = 0$, the fixed point (5.1) coincides with the origin of coordinates and corresponds to translation with the center of mass in the rear. When $k = 1$, the fixed point corresponds to translation with the center of mass at the front (parachute).

The fixed points of system (4.2)–(4.5) are similarly generated by the fixed points of system (1.1) that has not been reduced to normal form. System (5.1) defines points into which the partial solutions of system (4.2)–(4.5) are orthogonally projected (from the four-dimensional space to the three-dimensional one): $v(q) \equiv v^0 = v(0)$, $\alpha(q) \equiv \pi k, k \in \{0, 1\}$, $Z_1(q) \equiv Z_2(q) \equiv 0$.

Moreover, system (4.2)–(4.5) has nonisolated fixed points that fill the circle defined by the equations

$$\alpha = \frac{\pi}{2}, \quad \left(Z_2 - \frac{1}{2\sigma}\right)^2 + Z_1^2 = \frac{1}{4\sigma^2} \quad \text{(nonisolated).} \tag{5.2}$$

System (5.2) defines fixed points in the phase space of system (4.3)–(4.5) into which the manifolds of fixed points of system (4.2)–(4.5) are projected.

The fixed points defined by system (5.2) have no physical meaning because the model degenerates (the drag is formally zero at $\alpha = \pi/2$) at them.

Due to the easy definition of the fixed points (5.1), (5.2) and in the case (5.1) of explicit mechanical interpretation of the steady motions corresponding to these points, we will call them trivial fixed points.

The following Proposition is natural.

Proposition 1. System (4.3)–(4.5) has no equilibrium positions other than those mentioned above.

5.2. Nonisolated Fixed Points of the Fourth-Order System. Such points of system (4.2)–(4.5) are defined by the following relations with a positive parameter v_1:

$$\alpha = \frac{\pi}{2}, \quad \left(Z_2 - \frac{1}{2\sigma}\right)^2 + Z_1^2 = \frac{1}{4\sigma^2}, \quad v = v_1. \tag{5.3}$$

In the four-dimensional phase space of system (4.2)–(4.5), system (5.3) defines a one-dimensional manifold (circle) fully filled with fixed points.

5.3. Interpretation of the Fixed Points of the Four-Dimensional Phase Space. The fixed points of system (4.6)–(4.9) (as well as (4.2)–(4.5)) play an important role in the classification of global phase portraits of system (4.7)–(4.9) ((4.2)–(4.5)). Therefore, we will formally interpret them, because, as already mentioned, the mechanical interpretation at $\alpha = \pi/2$ is open to question. The formal interpretation gives a way to integrate system (1.2) (i.e., integrate the kinematic equations).

According to (5.3), we will consider a formal partial solution of system (4.2)–(4.5): $\alpha = \frac{\pi}{2}$, $v = v_0$, $\Omega_y = \Omega_{y0}$, $\Omega_z = \Omega_{z0}$.

An independent subsystem of the second order (last two equations) that has an analytic first integral can be separated from the kinematic equations

$$\theta^{\bullet} = -\frac{\Omega_{y0}}{\cos\psi} \sin\varphi,$$

$$\psi^{\bullet} = \Omega_{y0}\cos\varphi,$$

$$\varphi^{\bullet} = \Omega_{z0} + \frac{\Omega_{y0}}{\cos\psi}\sin\varphi\sin\psi \tag{5.4}$$

derived from (1.2).

Proposition 2. The second-order system (5.4) has the following first integral:

$$\Omega_{y0}\sin\varphi\cos\psi - \Omega_{z0}\sin\psi = C_1 = \text{const}, \tag{5.5}$$

which means that the projection of the absolute angular velocity of the rigid body onto the Ox_0-axis remains.

According to Proposition 2 and the last two equations in (5.4), which constitiute an independent second-order system, the angle ψ can be found from

$$\sin\psi = \pm\frac{\Omega_{y0}\sqrt{(\Omega_{y0}^2 + \Omega_{z0}^2) - C_1^2}}{\Omega_{y0}^2 + \Omega_{z0}^2}\sin\left\{\sqrt{\Omega_{y0}^2 + \Omega_{z0}^2}(t + C_2)\right\} - \frac{C_1\Omega_{z0}}{\Omega_{y0}^2 + \Omega_{z0}^2}, \tag{5.6}$$

where $C_2 = \text{const}$.

Identities (5.4), (5.5) indicate that the angle φ (and hence θ) depends on time through a finite combination of elementary functions.

On the phase paths involved, the following equality holds by virtue of (1.2):

$$x_0^{\bullet} = -v_0\cos\psi\sin\varphi = \frac{v_0}{\Omega_{y0}}[-C_1 - \Omega_{z0}\sin\psi],$$

and the following equality holds by virtue of (5.6):

$$x_0^{\bullet} = \frac{v_0}{\Omega_{y0}}[A_1 \mp A_2\sin(A_3 t + A_4)], \quad A_k = A_k(\Omega_{y0}, \Omega_{z0}, C_1, C_2), \quad k = 1, \ldots, 4.$$

Thus, when $C_1 = 0$ (plane-parallel motion, where $\Omega_{z0} = 1/\sigma$, $\alpha = \pi/2$), the coordinate x_0 varies periodically with time. The other coordinates (y_0 and z_0) of the configuration space can be found in a similar way.

6. Symmetries of the Phase Space of the Reference Dynamic System. Initial Topological Analysis.

6.1. Topological Classification of Fixed Points. Introduce two dimensionless parameters: $\mu_1 = 2B/(mn_0)$, $\mu_2 = \sigma n_0$. For simplicity, we will consider the analytic system (4.7)–(4.9).

Proposition 3. (i) The nonisolated fixed points filling the circle defined by Eqs. (5.2) are saddles in each perpendicular elementary area for $Z_2 < 1/(2\sigma)$ (Fig. 2) and are centers (stable fixed points) for $Z_2 > 1/(2\sigma)$.

(ii) The isolated fixed point defined by Eqs. (5.1) is an attractor when $k = 0$ (Fig. 3).

(iii) The isolated fixed point defined by Eqs. (5.1) is a repeller when $k = 1$ (Fig. 3).

6.2. Symmetries of the Phase Space. Since an independent third-order subsystem is separated from a fourth-order system, the phase paths of system (4.2)–(4.5) lie on three-dimensional cylindrical surfaces in the four-dimensional phase space. In particular, if system (4.2)–(4.5) has the complete set of first integrals in the phase space, at least one of them is a function of the variables (α, Z_1, Z_2) and, thus, defines a family of cylinders in the space $R_+^1\{v\} \times R^3\{\alpha, Z_1, Z_2\}$.

The phase portrait of system (4.2)–(4.5) in the four-dimensional phase space can be drawn using the phase portrait of system (4.3)–(4.5) in the three-dimensional phase space. The vector field involved has such geometry that it becomes possible to transfer the phase paths from the three-dimensional space to the corresponding four-dimensional phase space.

Since $v > 0$, the motion is possible only in the following domain: $W = \{(\alpha, Z_1, Z_2, v) \in R^4 : v > 0\}$.

After the formal change of variable $p = \ln v$ in the domain W, the vector field in the four-dimensional space $R^4\{\alpha, Z_1, Z_2, p\}$ becomes independent of p and is uniquely orthogonally projected onto the family of planes

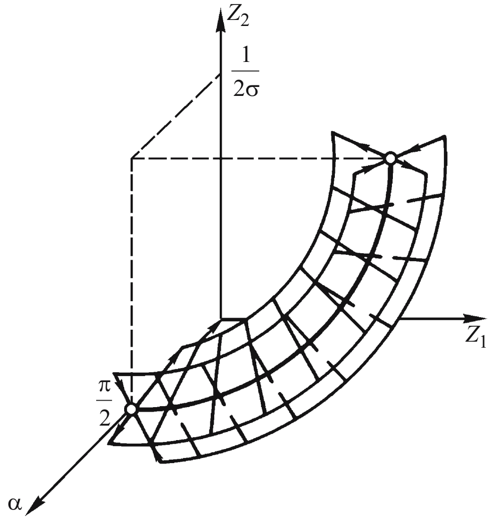

Fig. 2

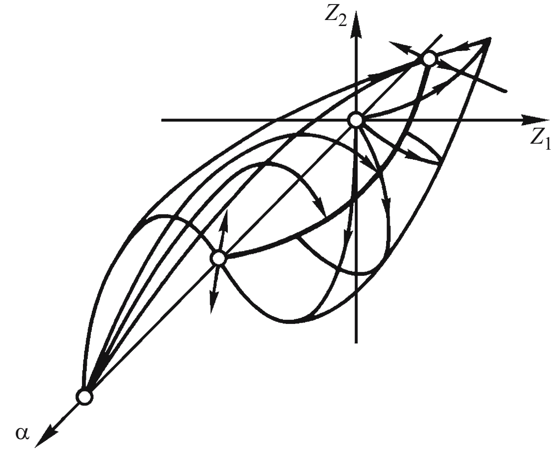

Fig. 3

$$\{(\alpha, Z_1, Z_2, p) \in R^4 : p = \text{const}\}.$$

The fixed points of system (4.3)–(4.5) in the space $R^3\ \{\alpha, Z_1, Z_2\}$ still coincide with the union of either projections of manifolds of singular points or projections of nonsingular phase paths of the domain W.

For any $F \in \{F\}$, the vector field of system (4.3)–(4.5) has:

(i) central symmetry about the points $(\pi k, 0, 0), k \in \{0, 1\}$, i.e., the vector field changes direction upon the following replacement in the coordinates (α, Z_1, Z_2):

$$\begin{pmatrix} \pi k + \alpha \\ Z_1, Z_2 \end{pmatrix} \to \begin{pmatrix} \pi k - \alpha \\ -Z_1, -Z_2 \end{pmatrix}, \quad k \in \{0, 1\};$$

(ii) the property that $\{(\alpha, Z_1, Z_2) \in R^3 : Z_1 = 0\}$ is an integral plane, and the vector field of the system has the following symmetry: its α- and Z_2-components remain, while the Z_1-component reverses sign upon the replacement

$$\begin{pmatrix} \alpha \\ Z_1, Z_2 \end{pmatrix} \to \begin{pmatrix} \alpha \\ -Z_1, Z_2 \end{pmatrix}, \quad k \in \{0, 1\};$$

(iii) some reflection symmetry. Such symmetry about the planes $\Lambda_i, i \in Z$, where $\Lambda_i = \left\{ (\alpha, Z_2, Z_1) \in R^3 : \alpha = \frac{\pi}{2} + \pi i \right\}$, is because the α-component of the vector field remains upon the following replacement in the coordinates (α, Z_2, Z_1):

$$\begin{pmatrix} \pi/2 + \pi i - \alpha \\ Z_2 \\ Z_1 \end{pmatrix} \to \begin{pmatrix} \pi k + \pi i + \alpha \\ Z_2 \\ Z_1 \end{pmatrix},$$

while the Z_2- and Z_1- components reverse sign.

Let us now define a family of three-dimensional layers:

$$\Pi_{(\alpha_1, \alpha_2)} = \{(\alpha, Z_1, Z_2) \in R^3 : \alpha_1 < \alpha < \alpha_2\},$$

where $\Pi_{(-\pi/2, \pi/2)} = \Pi$, $\Pi_{(\pi/2, 3\pi/2)} = \Pi'$. In fact, the phase space of system (4.3)–(4.5) is the set

$$\Pi_{(0,\pi)} \cap \{(\alpha, Z_1, Z_2) \in R^3 : Z_1 > 0\}. \qquad (6.1)$$

7. Complete Set of Transcendental First Integrals. Let us now look into the complete integrability (in terms of elementary functions) of the dynamic system of equations (4.6)–(4.10) with analytic right-hand sides. In the general case, system (4.2)–(4.5), (4.10) is not integrable in terms of elementary functions because it is reduced to the Riccati equations [10, 12].

7.1. First Integral Polynomial in Linear and Angular Velocities. Since the class of motions under consideration has property (2.1), the fifth-order system (4.2)–(4.5), (4.10) has an analytic first integral.

Indeed, the velocity of the center of mass C in the frame $Dxyz$ can be represented as

$$V_C = \{v\cos\alpha, v\sin\alpha\cos\beta - \sigma\Omega_z, v\sin\alpha\sin\beta + \sigma\Omega_y\}.$$

Then the following relation is invariant to system (2.3) if conditions (4.1) ($\Omega_{x0} = 0$) and (2.2) are satisfied:

$$v^2 - 2\sigma v z_2 \sin\alpha + \sigma^2 (z_1^2 + z_2^2) = V_{C0}^2 = \text{const}. \qquad (7.1)$$

Moreover, formula (7.1), in which the linear and angular velocities form a homogeneous quadratic form, allows us to write an integral polynomial in these velocities for system (4.2)–(4.5), (4.10)

$$v^2 (1 - 2\sigma Z_2 \sin\alpha + \sigma^2 (Z_1^2 + Z_2^2)) = V_{C0}^2. \qquad (7.2)$$

Formula (7.2) can be used to derive an explicit relation between v and the other quasivelocities:

$$v^2 = \frac{V_{C0}^2}{1 - 2\sigma Z_2 \sin\alpha + \sigma^2 (Z_1^2 + Z_2^2)}. \qquad (7.3)$$

Formula (7.3) permits an analysis of the integrability (in terms of elementary functions) of system (4.3)–(4.5), (4.10), which is of lower (fourth) order.

7.2. Three Additional Transcendental First Integrals. Let us rearrange the third-order system (4.7)–(4.9) as follows:

$$\alpha' = -Z_2 + b(Z_1^2 + Z_2^2)\sin\alpha + b\sin\alpha\cos^2\alpha,$$

$$Z_2' = \sin\alpha\cos\alpha + bZ_2(Z_1^2 + Z_2^2)\cos\alpha - bZ_2\sin^2\alpha\cos\alpha - Z_1^2 \frac{\cos\alpha}{\sin\alpha},$$

$$Z_1' = bZ_1(Z_1^2 + Z_2^2)\cos\alpha - bZ_1\sin^2\alpha\cos\alpha + Z_1 Z_2 \frac{\cos\alpha}{\sin\alpha}, \qquad (7.4)$$

where $b = \sigma n_0$, and "$'$" $\to n_0$ "$'$".

Next, changing the variable by $\tau = \sin\alpha$, we reduce system (7.4) to the following system with algebraic right-hand sides:

$$\frac{dZ_2}{d\tau} = \frac{\tau + bZ_2(Z_1^2 + Z_2^2) - bZ_2\tau^2 - Z_1^2/\tau}{-Z_2 + b\tau(Z_1^2 + Z_2^2) + b\tau(1 - \tau^2)},$$

$$\frac{dZ_2}{d\tau} = \frac{bZ_1(Z_1^2 + Z_2^2) - bZ_1\tau^2 + Z_1 Z_2/\tau}{-Z_2 + b\tau(Z_1^2 + Z_2^2) + b\tau(1 - \tau^2)}. \qquad (7.5)$$

Changing over to the homogeneous coordinates u_k, $k = 1, 2$, by the formulas $Z_k = u_k \tau$, we reduce system (7.5) to

$$\tau\frac{du_2}{d\tau} + u_2 = \frac{\tau + bu_2\tau^3(u_1^2 + u_2^2) - bu_2\tau^3 - u_1^2\tau}{-u_2\tau + b\tau^3(u_1^2 + u_2^2) + b\tau(1 - \tau^2)},$$

$$\tau \frac{du_1}{d\tau} + u_1 = \frac{bu_1 \tau^3 (u_1^2 + u_2^2) - bu_1 \tau^3 + u_1 u_2 \tau}{-u_2 \tau + b\tau^3 (u_1^2 + u_2^2) + b\tau(1-\tau^2)},$$

or

$$\tau \frac{du_2}{d\tau} = \frac{1 - bu_2 + u_2^2 - u_1^2}{-u_2 + b\tau^2 (u_1^2 + u_2^2) + b(1-\tau^2)},$$

$$\tau \frac{du_1}{d\tau} = \frac{2u_1 u_2 - bu_1}{-u_2 + b\tau^2 (u_1^2 + u_2^2) + b(1-\tau^2)}. \tag{7.6}$$

System (7.6) can be associated with the following first-order equation:

$$\frac{du_2}{du_1} = \frac{1 - bu_2 + u_2^2 - u_1^2}{2u_1 u_2 - bu_1}. \tag{7.7}$$

This equation is integrable in terms of elementary functions because so is the identity following from Eq. (7.7):

$$d\left(\frac{1 - bu_2 + u_2^2}{u_1}\right) + du_1 = 0.$$

In the coordinates (τ, Z_1, Z_2), it corresponds to the following transcendental first integral [10, 12]:

$$\frac{Z_1^2 + Z_2^2 - bZ_2 \tau + \tau^2}{Z_1 \tau} = \text{const.} \tag{7.8}$$

Otherwise, system (4.7)–(4.9) has the following transcendental first integral represented by a finite combination of elementary functions:

$$\frac{Z_1^2 + Z_2^2 - bZ_2 \sin\alpha + \sin^2\alpha}{Z_1 \sin\alpha} = C_1 = \text{const.} \tag{7.9}$$

The first integral found above can be used to rearrange the first equation in (7.6) in the form

$$\tau \frac{du_2}{d\tau} = \frac{2 - 2bu_2 + 2u_2^2 - C_1 U_1(C_1, u_2)}{-u_2 + b - 2b\tau^2 + b\tau^2 (C_1 U_1(C_1, u_2) + bu_2)},$$

$$U_1(C_1, u_2) = \{C_1 \pm \sqrt{C_1^2 - 4(u_2^2 - bu_2 + 1)}\}/2, \tag{7.10}$$

or into Bernoulli's equation

$$\frac{d\tau}{du_2} = \frac{(b - u_2)\tau + b\tau^3 (C_1 U_1(C_1, u_2) + bu_2 - 2)}{2 - 2bu_2 + 2u_2^2 - C_1 U_1(C_1, u_2)}. \tag{7.11}$$

Equation (7.11) can easily be made linear and inhomogeneous:

$$\frac{dp}{du_2} = \frac{2(u_2 - b)p - 2b(C_1 U_1(C_1, u_2) + bu_2 - 2)}{2 - 2bu_2 + 2u_2^2 - C_1 U_1(C_1, u_2)}, \quad p = \frac{1}{\tau^2}. \tag{7.12}$$

The latter fact means that we can find one more transcendental first integral explicitly (i.e., in terms of a finite combination of quadratures). The general solution of Eq. (7.12) depends on an arbitrary constant C_2.

Let us now find the last additional first integral of system (4.5)–(4.10) (i.e., an integral relating the equation to the angle β). Since

$$\frac{d\beta}{d\tau} = \frac{Z_1/\tau}{-Z_2 + b\tau(Z_1^2 + Z_2^2) + b\tau(1-\tau^2)},$$

we supplement the equality

$$\frac{d\beta}{d\tau} = \frac{u_1}{-u_2\tau + b\tau^3(u_1^2 + u_2^2) + b\tau(1-\tau^2)} \tag{7.13}$$

with

$$\tau\frac{du_1}{d\tau} = \frac{2u_1u_2 - bu_1}{-u_2 + b\tau^2(u_1^2 + u_2^2) + b(1-\tau^2)}, \tag{7.14}$$

following from (7.6).

The resulting system (7.13), (7.14) leads to an equation from which the sought integral follows:

$$\frac{du_1}{d\beta} = 2u_2 - \beta. \tag{7.15}$$

Using the first integral of Eq. (7.7) (C_1 is its constant of integration), we obtain

$$\frac{du_1}{d\beta} = \pm\sqrt{b^2 - 4[u_1^2 - C_1u_1 + 1]}. \tag{7.16}$$

Hence, the sought quadrature is

$$\pm\int\frac{du_1}{\sqrt{b^2 - 4[u_1^2 - C_1u_1 + 1]}} = \beta + C_3, \quad C_3 = \text{const}. \tag{7.17}$$

The left-hand side of (7.17) (without sign) is

$$\frac{1}{2}\arcsin\frac{\left(u_1 - \frac{b}{2}\right)^2}{\sqrt{C_1^2 + (b^2 - 4)}}. \tag{7.18}$$

After substitutions, we have the required invariant relation

$$\cos^2[2(\beta + C_3)] = \frac{\left(u_2 - \frac{b}{2}\right)^2 u_1^2}{G'}, \tag{7.19}$$

where $G' = [u_2^2 - bu_2]^2 + 2[u_2^2 - bu_2][u_1^2 + 1] + [u_1^2 + 1]^2 + b^2u_1^2$.

If $b = 2$, Eq. (7.19) becomes

$$\cos^2[2(\beta + C_3)] = \frac{(Z_2 - \sin\alpha)Z_1}{(Z_2 - \sin\alpha)^2 + Z_1^2}. \tag{7.20}$$

The right-hand side (as an odd function of $\zeta = (Z_2 - \sin\alpha)/Z_1$) has a global maximum of 1/2 at $\zeta = 1$.

Conclusions. We have constructed a nonlinear model describing the effect of a medium on a rigid body. The model takes into account the dependence of the arm of the force on the reduced angular velocity of the body, the torque of this force being a function of the angle of attack. Experimental data on the motion of homogeneous circular cylinders in water suggest that these facts should be accounted for in modeling. The initial stage of the study was to neglect the dependence of the drag torque on the angular velocity, not neglecting its dependence on the angle of attack. All the results found under this elementary assumption lead us to the conclusion that it is impossible to establish conditions in which the systems would have solutions describing angular oscillations of the body with a limited amplitude (see also [16, 17]).

We have listed all the first integrals of the dynamic part of the equations of motion of a rigid body in a resisting medium under the action of a follower force causing its center of mass to move rectilinearly and uniformly. It was assumed that the body is homogeneous and axisymmetric and interacts with the medium only by the disk-shaped frontal area.

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