Dynamical systems with various dissipation: background, methods, applications

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Abstract

This work is devoted to the development of qualitative methods in the theory of nonconservative systems that arise, e.g., in such fields of science as the dynamics a rigid body interacting with a resisting medium, oscillation theory, etc. This material can call the interest of specialists in the qualitative theory of ordinary differential equations, in rigid body dynamics, as well as in fluid and gas dynamics, since the work uses the properties of motion of a rigid body in a medium under the streamline flow conditions. The author obtains the complete integrability cases for nonconservative dynamical systems having nontrivial symmetries.

1 Introduction

In this work, we study the problem on the body motion under the condition that the line of the force applied to the body does not change its orientation with respect to the body and can only displace parallel to itself depending on the angle of attack and, possibly, on other phase variables. Such conditions arise under the plate motion with the so-called "large" angles of attack in a medium under a streamline flow (in this case, the fluid is assumed to be ideal in general, although all this are also true for fluids of a small viscosity, first of all, for the water) or under a separation flow (which is justified by an experiment completely satisfactory). Therefore, the main objects of studying is a family of bodies such that a part of the surface of each of which has a plane part that is flowed by a medium according to the streamline flow laws [1, 2].

From the N. E. Zhukovskii studies, it is known an attempt to model the motion basing on experiments in self-rotations of plates falling in the air [3, 4] (the so-called "Hamburg cardboard"). Here, one needs to take into account such properties of interaction of the medium to the body as the resistance force and the body force. Precisely, the aerodynamic characteristics of a plate are also used for modeling the bird flight [4]. Among the studies of S. A. Chaplygin, we also mention the statement of the problem of heavy body motion in an incompressible fluid [5].

2 Physical assumptions, quasi-stationarity hypothesis, phase variables and linearized equations of motion

Assume that a rigid body of mass m executes a plane-parallel motion in a medium with quadratic resistance law and that a certain part of the exterior body surface is a plane plate being under the medium streamline flow conditions. This means that the action of the medium on the plate reduces to the force $\mathbf{S}$ (applied at the point N) whose line of action is orthogonal to the plate. Let the remained par of the body surface be situated in a volume bounded by the flow surface that goes away from the plate boundary and is not subjected by the medium action.

Let us relate to the body the right coordinate system $Dxyz$ whose axis z moves parallel to itself, and for simplicity, assume that the plane Dzx is the geometric symmetry plane of the body. This ensures the fulfillment of property 2) under the motion satisfying condition 1). To construct the dynamical model, let us introduce the following phase coordinates: the value $v = |\mathbf{v}|$ of the velocity $\mathbf{v}$ of the point D, the angle α between the vector $\mathbf{v}$ and the axis x, and the algebraic value Ω of the projection of the body absolute angular velocity on the axis z.

Assume that the value of the force $\mathbf{S}$ quadratically depend on v with nonnegative coefficient s_1 ($S = s_1 v^2$). As usual, one represents s_1 in the form $s_1 = \rho P c_x/2$, where c_x is now the dimension-free coefficient of the frontal resistance (ρ is the medium density and P is the plate area). This coefficient depends on the angle of attack, the Struchal number, and other quantities which are usually considered as parameters. In what follows, we also introduce the following additional phase variable of the "Struchal type": $\omega = \Omega D/v$, where D is the characteristic plate transversal size. We restrict ourselves to the dependence of c_x on the pair (α, ω) of variables, i.e., we assume that s_1 (as well as y_N) is a function of the pair (α, ω) of dimension-free variables.

Let us define (purely formally for now) the dependence of s_1 and the ordinate y_N of the point N on the phase coordinates (α, ω). The system of dynamical equations must admit a particular solution of the form $\alpha(t) \equiv 0, \quad \omega(t) \equiv 0$. Therefore, we have the condition $y_N(0,0) = 0$ for the function $y_N(\alpha, \omega)$, and in the linear case, we need to assume that $y_N = D(k\alpha - h\omega)$, where k and h are certain constants. Because the approximation is linear, we can ignore the dependence of s_1 on α and ω.

In what follows, to take into account the action direction of the force $\mathbf{S}$, we introduce the following sigh-alternating auxiliary function $s(\alpha, \omega) = s_1(\alpha, \omega) sign \cos\alpha$.

Therefore, the linearized model of the force medium action contains three parameters s, k, and h, which are determined by the plate form in the plan. As was already mentioned, the first of these parameters, the coefficient s, is dimensional. The parameters k and h are dimension-free because of the method of their introduction [6].

The equations of motion of the center of masses in projections on the axes Dx and Dy of the related coordinate system and the equation of the kinetic moment variation with respect to the König axis have the following form with accuracy up to terms linear in α and Ω (here, σ is the distance DC and I is the central moment of inertia of the body):

$$\dot{v} = -sv^2/m \tag{1}$$

$$v\dot{\alpha} - sv^2\alpha/m + v\Omega - \sigma\dot{\Omega} = 0 \tag{2}$$

$$I\dot{\Omega} = sDv^2(k\alpha - hD\Omega/v) \tag{3}$$

Assuming that $v \neq 0$, introducing the natural parameter σ_1 ($v\,dt = D\,d\sigma_1$), which is usual for such systems, and using the change $\omega = D\Omega/v$ (see above) of the variable Ω and the obvious differentiation formula $D\dot{(\,)} = vd/d\sigma_1(\,) = v(\,)'$ we arrive at the system

$$v' = -\frac{sv}{m}D$$

$$I\omega' = \omega(I - mD^2h)\frac{Ds}{m} + sD^3k\alpha \tag{4}$$

$$\alpha' = -\omega\left(1 + \frac{s\sigma D^2}{I}h\right) + sD\left(\frac{1}{m} + \frac{k\sigma D}{I}\right)\alpha \tag{5}$$

in which two latter equations are separated from the first thus forming the independent second-order system 4, 5 and can be studied separately.

Let us transform these equations as follows: exclude α from them introducing the angle of turn ϕ by the formula $\phi' = \omega$; we obtain their linear integral in the form $\alpha - \omega(I/(kmD^2) + \sigma/D) + \phi(1 + \sigma s/m + Is/(km^2D) - hsD/km) = b = \text{const}$. With account for this, the equation for the angle of turn ϕ becomes $I\phi'' + \phi' sD(D^2h - D\sigma k - 2I/m) + \phi sD^3(k + k\sigma s/m + Is/m^2D - hsD/m) = ksD^3b$. It is easy to see that it has the form of the linear pendulum equation; under certain conditions, this pendulum executes oscillations near a certain position ϕ_* determined by the value b of the linear integral.

For $h = 0$, the coefficient of the so-called linear damping is negative, whereas the coefficients of the positional component positive; this allows us to speak about the oscillatory instability of the solution $\phi = \phi_*$. On the contrary, for a sufficiently large h, the solution $\phi = \phi_*$ can be made to be stable, and, moreover, it can loose its oscillatory character, since the coefficient of the positional component becomes negative.

But the main circumstance is that because this linear pendulum is multiparameter, the oscillatory stability of the solution $\phi = \phi_*$ is possible for certain finite values of h, since both mentioned coefficient can be positive in principle.

To describe the results and concrete properties of the body motion, in Institute of Mechanics of M. V. Lomonosov Moscow State University, V. A. Eroshin and V. M. Makarshin carried out experiments in registration of the motion of homogeneous circular cylinders in the water. Owing to the experiment, it becomes possible to find the dimension-free parameters k and h of the medium action on a rigid body.

The experiment allows one to make several important conclusions. The first of them is as follows: the rectilinear stationary free body drag (in the water) is unstable at least with respect to the angle of attack and the angular velocity. The second conclusion obtained from the carried out natural experiment is as follows: in modeling the medium action on the body, it is necessary to take into account the additional parameter characterizing the rotational derivative of the moment with respect to the body angular velocity. This parameter introduces the dissipation into the system. In our linear approximation, the accounting damping moment linearly depends on the body velocity.

3 Beginning of nonlinear analysis

For different bodies, under motion in a medium and under certain conditions, the angles of attack can practically assume any value from the interval $(0, \pi/2)$, and only for the angles close to $\pi/2$, the so-called washing out of the lateral surface is inevitable. Therefore, there arises the necessity of extending the functions y_N and s to finite angles of attack, i.e., the expansion of the domain of the pair of dynamical functions up to the interval $(0, \pi/2)$. But, in fact, it is necessary to extend the dynamical functions to the whole numerical line; this is clear from the following arguments.

In order to pass to a more complete description of the free body motion, let us represent the dynamics equations $mw_c = F,\ I\dot{\Omega} = M$ obtained early in the linear form (see 1-3) as follows:

$$\dot{v}\cos\alpha - \dot{\alpha}v\sin\alpha - \Omega v\sin\alpha + \sigma\Omega^2 = F_x/m \tag{6}$$

$$\dot{v}\sin\alpha + \dot{\alpha}v\cos\alpha + \Omega v\cos\alpha - \sigma\dot{\Omega} = 0 \tag{7}$$

$$I\dot{\Omega} = y_N(\alpha,\omega)F_x, \quad \omega = D\Omega/v \tag{8}$$

As a rule, for various variants of the body motion considered below, the generalized force F_x is quadratic in velocities (v, Ω) and explicitly depends on the auxiliary sign-alternating function $s(\alpha, \omega)$ (for example, $F_x(\alpha, v, \Omega) = -s(\alpha, \omega)v^2$ in the case of the body free drag). Therefore, the class of conceptual bodies and their conceptual motions defines a certain pair of dynamical functions $\big(s(\alpha,\omega),\, y_N(\alpha,\omega)\big)$ belonging to definite function classes.

4 Classes of dynamical functions

The fist stage of the complete nonlinear study of the body motion in a medium under the quasi-stationarity conditions is the study of the corresponding dynamical systems in which the damping is not taken into account (in particular, $h = 0$ in the linear case). The account for the damping is the next labor-consuming stage of studying the problem, which is presented in this work in a sufficient detail. To begin with, we consider the case where the pair of dynamical functions (y_N, s) depends only on the angle of attack. In this case, to qualitatively describe this pair of functions, we use the experimental information about the streamline flow properties.

The classes of dynamical functions to be introduced are sufficiently wide. They consists of smooth, 2π-periodic ($y_N(\alpha)$ is odd and $s(\alpha)$ is even) functions satisfying the following conditions: $y_N(\alpha) > 0$ for $\alpha \in (0, \pi)$, and, moreover, $y'_N(0) > 0$ and $y'_N(\pi) < 0$ (the function class $\{y_N\} = Y$); $s(\alpha) > 0$ for $\alpha \in (0, \pi/2)$, $s(\alpha) < 0$ for $\alpha \in (\pi/2, \pi)$, and, moreover, $s(0) > 0$ and $s'(\pi/2) < 0$ (the function class $\{s\} = \Sigma$). Both y_N and s change the sign under the replacement of α on $\alpha + \pi$. Therefore, $y_N \in Y,\ s \in \Sigma$ In particular, the analytic functions

$$y_N(\alpha) = y_0(\alpha) = A\sin\alpha \in Y \tag{9}$$

$$s(\alpha) = s_0(\alpha) = B\cos\alpha \in \Sigma, \quad A, B > 0 \tag{10}$$

serve as typical representatives of the described classes and correspond to the medium interaction functions obtained by S. A. Chaplygin in studying the plane-parallel flow around of a plane infinite length by a homogeneous medium flow.

In what follows, there rises the product $F(\alpha) = y_N(\alpha)s(\alpha)$ in the dynamical systems considered. It follows from the conditions listed above that F is a sufficiently smooth odd π-periodic functions satisfying the following conditions: $F(\alpha) > 0$ for $\alpha \in (0, \pi/2)$, $F'(0) > 0$, and $F'(\pi/2) < 0$ (the function class $\{F\} = \Phi$). Therefore, $F \in \Phi$ In particular, the analytic function

$$F = F_0(\alpha) = AB \sin\alpha \cos\alpha \in \Phi \tag{11}$$

is also a typical representative of the a function class Φ arisen (and also corresponds to the S. A. Chaplygin case mentioned above).

5 On variable dissipation in system and mechanical and topological analogs

After certain simplifications, the general system 6–8 reduces to second-order pendulum systems in which there is a linear dissipative force with variable coefficient alternating the sign for different angles of attack.

In what follows, it is necessary to note at once an important mechanical analogy arising on the basis of qualitative properties of the free body stationary motion in the medium flow and the pendulum equilibrium. Such an analogy allows us to extend the properties of pendulum nonlinear dynamical systems to the dynamical systems of a free body and obtain some topological analogies. For example, under condition 11, the angle of turn of the pendulum is completely equivalent to the angle of attack in the free body motion. In condition 11 violates (the group of conditions 9 and 10), then the angle of attack of the free body and the angle of turn of the pendulum are trajectorially topologically equivalent.

And so, the results of the presented work are appeared owing to the study of the applied problem on the rigid body motion in a resisting medium [7, 8] where complete lists of transcendental first integrals expressed through a finite combination of elementary functions were obtained. This circumstance allows one to perform a complete analysis of phase trajectories and show those properties of them which exhibit the roughness and preserve for systems of a more general form. Note that similar problems were already appeared in applied aerodynamics in the studies of Central Aero- and Hydrodynamics Institute named after professor N. E. Zhukovskii.

Many results of this work were regularly reported at a number of workshops, including the workshop ‘Actual Problems of Geometry and Mechanics’ named after professor V. V. Trofimov, which is headed by D. V. Georgievskii and M. V. Shamolin.

Below, we highlight the classes of essentially nonlinear systems of the second and third orders integrable in transcendental (in the sense of theory of functions of one complex variable) elementary functions. For example such systems are five-parametric dynamical systems including the majority of systems that are studied early in the dynamics of a rigid body interacting with a medium:

$$\begin{aligned}
\dot\alpha &= a\sin\alpha + b\omega + \gamma_1 \sin^5\alpha + \gamma_2\omega\sin^4\alpha + \\
&\quad + \gamma_3\omega^2\sin^3\alpha + \gamma_4\omega^3\sin^2\alpha + \gamma_5\omega^4\sin\alpha \\
\dot\omega &= c\sin\alpha\cos\alpha + d\omega\cos\alpha + \gamma_1\omega\sin^4\alpha\cos\alpha + \gamma_2\omega^2\sin^3\alpha\cos\alpha + \\
&\quad + \gamma_3\omega^3\sin^2\alpha\cos\alpha + \gamma_4\omega^4\sin\alpha\cos\alpha + \gamma_5\omega^5\cos\alpha
\end{aligned}$$

So, in [7], the author studied and integrated two model variants of the body plane-parallel motion in a resisting medium, which are described by dynamical systems with zero mean variable dissipation.

For example a dynamical system of the form $\dot\alpha = \Omega + \beta\sin\alpha$, $\dot\Omega = -\beta\sin\alpha\cos\alpha$, is relatively structurally stable (relatively rough) and topologically equivalent to the system describing a clamped pendulum placed in the running-out medium flow.

We can find its first integral being a transcendental (in the sense of theory of functions of one complex variable such that it has essentially singular points after continuing it into the complex domain) functions of phase variables that is expressed through a finite combination of elementary functions.

As is seen, the phase cylinder $\mathbb{R}^2\{\alpha, \Omega\}$ of quasi-velocities of the system considered exhibits an interesting topological structure of partition into trajectories.

On the cylinder, there are two domains (whose closure is just the phase cylinder) filled in by trajectories of perfectly different character. The first domain called oscillatory or finitary (it is simply connected is entirely filled in by trajectories of the following type. Almost every such trajectory starts at the repelling point $(2\pi k, 0)$ and ended at the attracting point $\big((2k+1)\pi, 0\big)$, $k \in \mathbf{Z}$. An exception is the fixed points $(\pi k, 0)$ and separatrices that either emanate from the repelling point $(2\pi k, 0)$ and enter the saddles S_{2k} and S_{2k+1} or emanate from the saddles S_{2k+1} and S_{2k+2} and enter the attracting points $\big((2k+1)\pi, 0\big)$. Here, $S_k = (-\pi/2 + \pi k, (-1)^k\beta)$.

The second domain called rotational (it is two-connected is entirely filled in by rotational motions similar to rotations on the mathematical pendulum plane. These phase trajectories envelope the phase cylinder and are periodic on it.

Although the dynamical system considered is not conservative, in the rotational domain of its phase plane $\mathbb{R}^2\{\alpha, \Omega\}$ it admits the preservation of an invariant measure with variable density. This property characterizes the system considered as a system with zero mean variable dissipation.

As a result of all this, we obtain a family of three-dimensional phase portraits analogous to the plane-parallel dynamics (see [7]).

6 One of the definitions of systems with zero mean variable dissipation

We study the system of ordinary differential equations having a periodic phase coordinate. The systems under study have those symmetries for which in the mean during the period of the periodic coordinate, their phase volume preserves. So, for example, the following pendulum system with smooth right-hand side $\mathbf{V}(\alpha, \omega)$ periodic in α of period T:

$$\dot\alpha = -\omega + f(\alpha), \dot\omega = g(\alpha)$$
$$f(\alpha + T) = f(\alpha), \quad g(\alpha + T) = g(\alpha)$$

preserves its phase area on the phase cylinder during the period T: $\int\limits_0^T \mathrm{div}\mathbf{V}(\alpha, \omega)\, d\alpha =$

$$\int\limits_0^T \left(\frac{\partial}{\partial\alpha}\big(-\omega + f(\alpha)\big) + \frac{\partial}{\partial\omega} g(\alpha)\right) d\alpha = \int\limits_0^T f'(\alpha)\, d\alpha = 0.$$

The system considered is equivalent to the pendulum equation $\ddot\alpha - f'(\alpha)\dot\alpha + g(\alpha) = 0$ in which the integral if the coefficient $f'(\alpha)$ standing by the dissipative term $\dot\alpha$ vanishes in the mean during the period.

It is seen that the system considered has those symmetries for which it becomes a system with zero mean variable dissipation in the sense of the following definition.

Definition. Let us consider a smooth autonomous system of the $(n+1)$th order and normal form defined on the cylinder $\mathbb{R}^n\{x\} \times \mathbf{S}^1\{\alpha \bmod 2\pi\}$, where α is a periodic coordinate of period $T > 0$. Denote by $\mathrm{div}(x, \alpha)$ the divergence of the right-hand side (which is a function of all phase variables in general and is not identically equal to zero) of this system. Such a system is called a system with zero (nonzero) mean variable dissipation if the function $\int\limits_0^T \mathrm{div}(x, \alpha)\, d\alpha$ is (is not) identically equal to zero. Moreover, in some cases (for example, when at certain points of the circle $\mathbf{S}^1\{\alpha \bmod 2\pi\}$, there arise singularities), this integral is understood in the principal value sense .

Let us consider systems of the form (the dot denotes the derivative in time)

$$\dot\alpha = f_\alpha(\omega, \sin\alpha, \cos\alpha) \tag{12}$$
$$\dot{\omega_k} = f_k(\omega, \sin\alpha, \cos\alpha),\ k = 1, \ldots, n, \tag{13}$$

defined on the set $\mathbf{S}^1\{\alpha \bmod 2\pi\}\backslash K\times\mathbb{R}^n\{\omega\}$, $\omega=(\omega_1,\ldots,\omega_n)$, where the functions $f_\lambda(u_1,u_2,u_3)$, $\lambda=\alpha,1,\ldots,n$, of three variables$u_1$, u_2, and u_3 are as follows:

$$\begin{aligned}&f_\lambda(-u_1,-u_2,u_3)=-f_\lambda(u_1,u_2,u_3)\\&f_\alpha(u_1,u_2,-u_3)=f_\alpha(u_1,u_2,u_3)\\&f_k(u_1,u_2,-u_3)=-f_k(u_1,u_2,u_3)\end{aligned}$$

The set K is either empty or consists of finitely many points of the circle $\mathbf{S}^1\{\alpha \bmod 2\pi\}$.

The latter two variables u_2 and u_3 in the functions $f_\lambda(u_1,u_2,u_3)$ depend on one parameter α, but they are distinguished in separate groups for the following reasons. First, not in the whole their domain, they are uniquely expressed from one another, and, second, the first of them is odd, whereas the second is an even function of α, which influences on the symmetry of system 12, 13 differently.

To this system, we put in correspondence the non-autonomous system

$$\frac{d\omega_k}{d\alpha}=\frac{f_k(\omega,\sin\alpha,\cos\alpha)}{f_\alpha(\omega,\sin\alpha,\cos\alpha)}$$

by the substitution $\tau=\sin\alpha$, it reduces to the form ($k=1,\ldots,n$)

$$\begin{aligned}&\frac{d\omega_k}{d\tau}=\frac{f_k(\omega,\tau,\varphi_k(\tau))}{f_\alpha(\omega,\tau,\varphi_\alpha(\tau))}\\&\varphi_\lambda(-\tau)=\varphi_\lambda(\tau),\lambda=\alpha,1,\ldots,n\end{aligned}$$

The following assertion immerses the class of systems 12, 13 in the class of dynamical systems with zero mean variable dissipation. The inverse embedding does not hold in general.

Lemma. Systems of the form 12, 13 are dynamical system with zero mean variable dissipation.

This proposition is proved by using the symmetries of system 12, 13 listed above.

In this work, we mainly consider the case where the functions $f_\lambda\big(\omega,\tau,\varphi_k(\tau)\big)$ $(\lambda=\alpha,1,\ldots,n)$ are polynomials in ω and τ.

To begin with, let us consider a certain class of autonomous systems on the two-dimensional circular cylinder $\mathbf{S}^1\{\alpha \bmod 2\pi\}\times\mathbb{R}^1\{\omega\}$. For example, to the following pendulum systems (arising in the dynamics of a rigid body interacting with a medium) with parameter $\beta>0$:

$$\dot\alpha=-\omega+\beta\sin\alpha,\quad \dot\omega=\sin\alpha\cos\alpha \tag{14}$$

$$\dot\alpha=-\omega+\beta\sin\alpha\cos^2\alpha+\beta\omega^2\sin\alpha \tag{15}$$

$$\dot\omega=\sin\alpha\cos\alpha-\beta\omega\sin^2\alpha\cos\alpha+\beta\omega^3\cos\alpha \tag{16}$$

in the variables (ω,τ), we put in correspondence the following equations with algebraic right-hand side, respectively:

$$\frac{d\omega}{d\tau}=\frac{\tau}{-\omega+\beta\tau},\quad \frac{d\omega}{d\tau}=\frac{\tau+\beta\omega[\omega^2-\tau^2]}{-\omega+\beta\tau+\beta\tau[\omega^2-\tau^2]}.$$

In this case, systems 14 and 15, 16 are dynamical systems with zero mean variable dissipation, which is easy to verify directly.

For example, system 14 has a first integral of the following form (depending on the value of β, three cases are possible that corresponds to the existence of foci, nodes, or degenerate nodes in the phase portrait of the system):

$$\begin{aligned}&\beta^2-4<0:\\&\{\Omega^2+\beta\Omega\sin\alpha+\sin^2\alpha\}\times\exp\left\{\frac{2\beta}{\sqrt{-\beta^2+4}}\arctan\frac{2\Omega+\beta\sin\alpha}{\sqrt{-\beta^2+4}\sin\alpha}\right\}=\mathrm{const}\end{aligned}$$

$$\beta^2 - 4 > 0:$$
$$\left|2\Omega + \left(\beta + \sqrt{\beta^2 - 4}\sin\alpha\right)\right|^{\sqrt{\beta^2-4}-\beta} \times$$
$$\times \left|2\Omega + \left(\beta - \sqrt{\beta^2 - 4}\sin\alpha\right)\right|^{\sqrt{\beta^2-4}+\beta} = \text{const}$$

$$\beta^2 - 4 = 0:$$
$$|2\Omega + \beta\sin\alpha| \times \exp\left\{-\frac{\beta\sin\alpha}{2\Omega + \beta\sin\alpha}\right\} = \text{const}$$

The phase portrait of system 15, 16 can be of three different types depending on the values of the parameter β.

In the expression of its first integral, also three cases are possible depending on the value of the constant β and corresponding to the existence of foci, nodes, and degenerate nodes in the phase portrait of the system.

Let us represent the parameter β as the product: $\beta = \sigma^2 n_0^2$; after that, to system 15, 16 we put in correspondence a differential equation of the form

$$\frac{d\omega}{d\tau} = \frac{-n_0^2\tau + \sigma\omega[\omega^2 - n_0^2\tau^2]}{\omega + \sigma n_0^2\tau + \sigma\tau[\omega^2 - n_0^2\tau^2]}, \quad \tau = -\sin\alpha$$

Introduce the following notation: $C_1 = 2 - \sigma n_0,\ C_2 = \sigma n_0,\ C_3 = -2 - \sigma n_0$. Performing a number of changes by the formulas $\omega - n_0\tau = u_1,\ \omega + n_0\tau = v_1,\ u_1 = v_1 t_1,\ v_1^2 = p_1$, where $v_1 \neq 0$, we obtain the Bernoulli-type equation

$$2p_1\left[C_1t_1 + C_2 + \frac{2\sigma}{n_0}t_1p_1\right] = \frac{dp_1}{dt_1}[C_3 - C_1t_1^2]$$

By the known change $p^{-1} = q_1$ for $p_1 \neq 0$, we reduce the latter equation to the form

$$\dot{q_1} = a_1(t_1)q_1 + a_2(t_1)$$

where

$$a_1(t_1) = \frac{2(C_1t_1 + C_2)}{C_1t_1^2 - C_3}, \quad a_2(t_1) = \frac{4\sigma t_1}{n_0(C_1t_1^2 - C_3)}$$

(Here, the dot denotes the derivative in t_1.)

The general solution of the linear homogeneous equation is represented in the form

$$q_{1\,HOM}(t_1) = k(C_1t_1^2 - C_3)Q(t_1), \quad k = \text{const}$$

where the function Q has the following form depending on the value of the constant C_1:

$$Q(t_1) = e^{t_1}, \quad C_1 = 0$$
$$e^{2\frac{C_2}{\sqrt{-C_1C_3}}\arctan\sqrt{-\frac{C_1}{C_3}}t_1}, \quad C_1 > 0$$
$$\left(\frac{\sqrt{-C_1}t_1 + \sqrt{-C_3}}{\sqrt{-C_1}t_1 - \sqrt{-C_3}}\right)^{\frac{C_2}{\sqrt{C_1C_3}}}, \quad C_1 < 0$$

To obtain the solution of the inhomogeneous equation, we assume that the quantity k is a function of t_1; we find it by the quadrature

$$k(t_1) = \frac{4\sigma}{n_0}\int Q^{-1}(t_1)\frac{t_1}{(C_1t_1^2 - C_3)^2}\,dt_1$$

Therefore, the transcendental first integral of system 15, 16 becomes

$$Q^{-1}(t_1)q_1(C_1t_1^2 - C_3)^{-1} - \frac{4\sigma}{n_0}\int_{t_0}^{t_1} Q^{-1}(\tau_1)\frac{\tau_1}{(C_1\tau_1^2 - C_3)^2}\,d\tau_1 = C^0$$

where $C^0 = \mathrm{const}$.

As is seen, the final form of the first integral depends on the sign of the constant C_1, and, therefore, three variants are possible. Let us examine each of them.

First variant. $C_1 = 0$. After an elementary calculation, we obtain an additional integral in the form

$$e^{-\frac{u_1}{v_1}}\left(v_1^{-2} + \frac{\sigma^2}{2}(\frac{u_1}{v_1} + 1)\right) = \mathrm{const}$$

Therefore, for $C_1 = 0$, the transcendental first integral of system 15, 16 is expressed through elementary functions.

Second variant. $C_1 > 0$. The integration leads to the function

$$-\frac{\sigma}{4n_0}e^{-2\frac{C_2}{\sqrt{-C_1C_3}}\zeta}\left(\frac{C_2}{\sqrt{-C_1C_3}}\sin 2\zeta + \cos 2\zeta\right)$$

where

$$\zeta = \arctan\sqrt{-\frac{C_1}{C_3}}t_1$$

As is seen, in the case $C_1 > 0$, the additional first integral is expressed through elementary functions.

Third variant. $C_1 < 0$. By equivalent transformations, the integral transforms into

$$\frac{\sigma}{C_1C_2n_0}\left(2\frac{\zeta^{1-\gamma}}{\gamma-1} - 3\frac{\zeta^{-\gamma}}{\gamma} + \frac{\zeta^{-1-\gamma}}{\gamma+1}\right)$$

where

$$\gamma = \frac{C_2}{\sqrt{C_1C_3}} > 1, \quad \zeta = \frac{\sqrt{-C_1}t_1 + \sqrt{-C_3}}{\sqrt{-C_1}t_1 - \sqrt{-C_3}}$$

Therefore, in the case $C_1 < 0$, the additional first integral is also expressed through elementary functions.

And so, we study the connection between the following three properties, which are independent for the first glance, but they are sufficiently harmonically combined on systems from the rigid body dynamics:

1. The distinguished class of systems 12, 13 with the above;
2. The fact that this class of systems consists of systems with zero mean variable dissipation (in the variable α), which allows us to consider them as "almost" conservative systems;
3. In certain (although lower-dimensional) cases, these systems have first integrals, which are transcendental in general.

Let us present one more important example of a higher-order system that has the properties just listed.

To the system

$$c\dot{\alpha} = -z_2 + \beta\sin\alpha \tag{17}$$
$$\dot{z_2} = \sin\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha} \tag{18}$$
$$\dot{z_1} = z_1z_2\frac{\cos\alpha}{\sin\alpha} \tag{19}$$

which is distinguished considered in the three dimensional domain

$$\mathbf{S}^1\{\alpha \bmod 2\pi\} \setminus \{\alpha = 0, \alpha = \pi\} \times \mathbb{R}^2\{z_1, z_2\}$$

(such a system can reduce to an equivalent system on the tangent bundle of the two-dimensional sphere) and describes the spatial motion of a rigid body in a resisting medium [8], we put in correspondence the following system with algebraic right-hand side:

$$\frac{dz_2}{d\tau} = \frac{\tau - z_1^2/\tau}{-z_2 + \beta\tau}, \quad \frac{dz_1}{d\tau} = \frac{z_1 z_2/\tau}{-z_2 + \beta\tau} \tag{20}$$

In this case, it is also seen that system 17–19 is a system with zero mean variable dissipation; in order to obtain a complete correspondence with the definition, it suffices to introduce the new phase variable $z_1^* = \ln|z_1|$.

Moreover, this system has two first integrals (i.e., the full list), which are transcendental functions and are expressed through a finite combination of elementary functions; as was mentioned above, this become possible after establishing its correspondence to the (non-autonomous in general) system of equations 20 with algebraic (polynomial) right-hand side.

Therefore, the systems from the rigid body dynamics presented above not only enter the class of systems 12, 13 and have the mean zero variable dissipation, but they have a full list of transcendental first integrals expressed through a finite combination of elementary functions. In this case, the integration of systems 14 and 15, 16 reduces to the integration of the corresponding equations with algebraic right-hand side.

7 Conclusion

Generally speaking, the dynamics of a rigid body interacting with a medium is just the field where there arise either nonzero mean variable dissipation systems or systems in which the energy loss in the mean during a period can vanish. In the work, we have obtained such a methodology owing to which it becomes possible to finally and analytically study a number of plane and spatial model problems.

In qualitative describing the body interaction with a medium, because of using the experimental information about the properties of the streamline flow around, there arises a definite dispersion in modeling the force-model characteristics. This makes it natural to introduce the definitions of relative roughness (relative structural stability) and to prove such a roughness for the system studied. Moreover, many systems considered are merely (absolutely) Andronov–Pontryagin rough.

All what was said above allows one to estimate the results of the work in to totality as a new direction in qualitative theory of ordinary differential equations and the dynamics of a rigid body interacting with a medium.

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