PAMM · Proc. Appl. Math. Mech. **10**, 63 – 64 (2010) / **DOI** 10.1002/pamm.201010024

Integrability and nonintegrability in terms of transcendental functions in dynamics of a rigid body

Maxim V. Shamolin[1,*]

[1] Institute of Mechanics, Lomonosov Moscow State University, Michurinskii Ave., 1, 119899 Moscow, Russian Federation

The results of the presented work are due to the study of the applied problem of the rigid body motion in a resisting medium. More earlier the complete lists of transcendental first integrals expressed through a finite combination of elementary functions were obtained. This circumstance allowed the author to perform a complete analysis of all phase trajectories and highlight those properties of them which exhibit the roughness and preserve for systems of a more general form.

1 Introduction

The complete integrability of those systems is related to symmetries of a latent type. Therefore, it is of interest to study sufficiently wide classes of dynamical systems having analogous latent symmetries.

As is known, the concept of integrability is sufficiently broad and undeterminate in general. In its construction, it is necessary to take into account in what sense it is understood (it is meant that a certain criterion according to which one makes a conclusion that the structure of trajectories of the dynamical system considered is especially attractive), in which function classes the first integrals are sought for, etc.

In this work, the author applies such an approach such that as first integrals, transcendental functions are elementary. Here, the transcendence is not understood in the sense of elementary functions (e.g., trigonometrical functions) but in the sense that they have essentially singular points (by the classification accepted in the theory of functions of one complex variable according to which a function has essentially singular points). In this case, it is necessary to continue them formally to the complex plane. As a rule, such systems are strongly nonconservative.

2 General characteristic of variable dissipation dynamical systems

Generally speaking, the dynamics of a rigid body interacting with a medium is just a field, where there arise either dissipative systems or systems with the so-called antidissipation (energy supporting inside the system itself). Therefore, it becomes urgent to construct a methodology precisely for those classes of systems which arise in modeling body motion the contact surface of which is a plane part, the simplest part of their exterior surface.

After certain simplifications, we can reduce the system of equations for the plane-parallel motion to the second-order pendulum systems in which there is a linear dissipative force with variable coefficient whose sign alternates for different values of the periodic phase variable in the system.

Below, we highlight the classes of essentially nonlinear systems of the second and third orders integrable in transcendental (in the sense of theory of functions of one complex variable) elementary functions. For example such systems are five-parametric dynamical systems including the majority of systems that are previously studied in the dynamics of a rigid body interacting with a medium [1, 2, 3]:

$$\begin{aligned}
\dot{\alpha} &= a\sin\alpha + b\omega + \gamma_1\sin^5\alpha + \gamma_2\omega\sin^4\alpha + \gamma_3\omega^2\sin^3\alpha + \gamma_4\omega^3\sin^2\alpha + \gamma_5\omega^4\sin\alpha,\\
\dot{\omega} &= c\sin\alpha\cos\alpha + d\omega\cos\alpha + \gamma_1\omega\sin^4\alpha\cos\alpha + \gamma_2\omega^2\sin^3\alpha\cos\alpha +\\
&\quad + \gamma_3\omega^3\sin^2\alpha\cos\alpha + \gamma_4\omega^4\sin\alpha\cos\alpha + \gamma_5\omega^5\cos\alpha.
\end{aligned}$$

Let us consider a smooth autonomous system of the $(n+1)$th order and normal form defined on the cylinder $\mathbf{R}^n\{x\}\times\mathbf{S}^1\{\alpha \bmod 2\pi\}$, where α is a periodic coordinate of period $T>0$. Denote by $\mathrm{div}(x,\alpha)$ the divergence of the right-hand side (which is a function of all phase variables in general and is not identically equal to zero) of this system. Such a system is called a system with zero (nonzero) mean variable dissipation if the function $\int_0^T \mathrm{div}(x,\alpha)\,d\alpha$ is (is not) identically equal to zero. Moreover, in some cases (for example, when at certain points of the circle $\mathbf{S}^1\{\alpha \bmod 2\pi\}$, there arise singularities), this integral is understood in the principal value sense.

It should be noted that it is sufficiently difficult to give the general definition of a system with zero (nonzero) mean variable dissipation. The definition just presented uses the concept of divergence (as is known, the divergence of the right-hand side of a normal form system characterizes the variation of the phase volume in the phase space of the system).

* Corresponding author: e-mail: `shamolin@imec.msu.ru`, Phone: +7 495 939 5143, Fax: +7 495 939 0165

3 Systems on tangent bundle of two-dimensional sphere

Let us study a system of the following form

$$\dot\alpha = -z_2 + \beta(z_1^2 + z_2^2)\sin\alpha + \beta\sin\alpha\cos^2\alpha,$$
$$\dot z_2 = \sin\alpha\cos\alpha + \beta z_2(z_1^2+z_2^2)\cos\alpha - \beta z_2\sin^2\alpha\cos\alpha - z_1^2\frac{\cos\alpha}{\sin\alpha},$$
$$\dot z_1 = \beta z_1(z_1^2+z_2^2)\cos\alpha - \beta z_1\sin^2\alpha\cos\alpha + z_1 z_2\frac{\cos\alpha}{\sin\alpha},$$

which also arises in the spatial dynamics of a rigid body interacting with a medium [1, 2, 3] and corresponds to the system with algebraic right-hand side.

In a similar way, we pass to the homogeneous coordinates u_k, $k = 1, 2$, according to the formulas $z_k = u_k\tau$.

Our system reduces to some system which, in turn, corresponds to some equation which is integrated in elementary functions since the identity $d\left((1-\beta u_2+u_2^2)/(u_1)\right) + du_1 = 0$, is integrated and has the following first integral in the coordinates (τ, z_1, z_2): $(z_1^2+z_2^2-\beta z_2\tau+\tau^2)/(z_1\tau) = \text{const}$.

Let us pose the following question: which is possibility of integrating the system

$$\frac{dz}{dx} = \frac{ax+by+cz+c_1z^2/x+c_2zy/x+c_3y^2/x}{dx+ey+fz}, \quad \frac{dy}{dx} = \frac{gx+hy+iz+i_1z^2/x+i_2zy/x+i_3y^2/x}{dx+ey+fz}$$

of a more general form in elementary functions and three-dimensional phase domains, which includes above systems and has a singularity of à form $1/x$?

As before, introducing the substitutions $y = ux$ and $z = vx$, we obtain that our system reduces to some system, we put in correspondence the equation with algebraic right-hand side. The integration of the latter equations reduces to the integration of the following equation in total differentials. In general, we have the 15-parameter family of equations of the above form. To integrate the latter identity as a homogeneous equation in elementary functions, it suffices to impose seven relations

$$g = 0, \quad i_3 = e, \quad i_1 = 0, \quad i = 0, \quad c_2 = e, \quad c = h, \quad 2c_1 = i_2 + f. \tag{1}$$

Introduce eight parameters $\beta_1,\ldots,\beta_8:\ g=0,\ h=\beta_1,\ i_1=0,\ i=0,\ i_2=\beta_2,\ i_3=\beta_3,\ d=\beta_4,\ e=\beta_3,\ f=\beta_5,\ a=\beta_6,\ b=\beta_7,\ c=\beta_1,\ c_1=(\beta_2+\beta_5)/2,\ c_2=\beta_3,\ c_3=\beta_8$ and consider them as independent parameters.

Therefore, under the group of conditions (1), our equation reduces to some form, after that, the equation under study is integrated in elementary functions.

Indeed, integrating our identity, we obtain some relation which allows us to obtain the first integral in the form in the coordinates (x, y, z):

$$\frac{(\beta_2-\beta_5)z^2/2-\beta_8y^2+(\beta_1-\beta_4)zx+\beta_6x^2}{yx} - \beta_7\ln\left|\frac{y}{x}\right| = \text{const}.$$

Therefore, we can make a conclusion on the integrability in elementary functions of the following, in general, nonconservative third-order system depending on eight parameters:

$$\frac{dz}{dx} = \frac{\beta_6x+\beta_7y+\beta_1z+(\beta_2-\beta_5)z^2/2x+\beta_3zy/x+\beta_8y^2/x}{\beta_4x+\beta_3y+\beta_5z}, \qquad \frac{dy}{dx} = \frac{\beta_1y+\beta_2zy/x+\beta_3y^2/x}{\beta_4x+\beta_3y+\beta_5z}.$$

On the set $\mathbf{S}^1\{\alpha \bmod 2\pi\}\setminus\{\alpha=0,\ \alpha=\pi\}\times\mathbf{R}^2\{z_1,z_2\}$, the third-order system

$$\begin{aligned}
\dot\alpha &= \beta_4\sin\alpha + \beta_3z_1 + \beta_5z_2,\\
\dot z_2 &= \beta_6\sin\alpha\cos\alpha + \beta_7z_1\cos\alpha + \beta_1z_2\cos\alpha + \frac{\beta_2+\beta_5}{2}z_2^2\frac{\cos\alpha}{\sin\alpha} + \beta_3z_1z_2\frac{\cos\alpha}{\sin\alpha} + \beta_8z_1^2\frac{\cos\alpha}{\sin\alpha},\\
\dot z_1 &= \beta_1z_1\cos\alpha + \beta_2z_1z_2\frac{\cos\alpha}{\sin\alpha} + \beta_3z_1^2\frac{\cos\alpha}{\sin\alpha}
\end{aligned} \tag{2}$$

depending on eight parameters has a (transcendental in general) first integral expressed through elementary functions.

In particular, for $\beta_1=\beta_3=\beta_7=0$, $\beta_2=\beta_6=1$, $\beta_5=\beta_8=-1$, and $\beta_4=\beta$, the system (2) reduces to above system.

Acknowledgements This work was supported by the Russian Foundation for Basic Research, project no. 08-01-00231-a.

References

[1] M. V. Shamolin, New two–parameter family of phase portraits in the problem of motion of a body in a medium, Dokl. Ross. Akad. Nauk **337**(5), pp. 611–614 (1994).
[2] M. V. Shamolin, (1998) On the integrability in transcendental functions, Usp. Mat. Nauk, **53**(3), pp. 209–210 (1997).
[3] M. V. Shamolin, The definition of relative structural stability and a two–parameter family of phase portraits in the dynamics of a rigid body, Usp. Mat. Nauk, **51**(1), pp. 175–176 (1996).

